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Internal Conversion Angular Correlations*

M. E. Rose, L. C. Biedenharn, and G. B. Arfken
Oak Ridge National Laboratory, Oak Ridge, Tennessee
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It is shown that the angular correlation between a conversion electron and any other radiation emitted in a double nuclear cascade can be obtained immediately if the corresponding correlation with a γ-ray replacing the conversion electron is known. This latter is known for all cases of practical interest. Specifically, if the correlation function for γ-rays and a radiation x is expanded in Legendre polynomials, the correlation function with a conversion electron replacing the γ-ray is obtained by multiplying the coefficients of each polynomial P_ν by a parameter b_ν. The case of conversion-conversion correlation, in all practical cases, is obtained from the $\gamma-\gamma$ correlation by inserting two factors b_ν, one for each conversion electron. The coefficients b_ν are calculated relativistically and numerical results are presented for K-shell conversion for 12 values of Z in the range $10 \leqslant Z \leqslant 96$ and transition energies from 0.3 mc^2 to 5.0 mc^2 for ten multipoles (5 electric and 5 magnetic). It is pointed out that the present results apply in γ-electron correlation if the γ is a mixed multipole but the case in which the conversion transition is mixed is not computed. The angular distribution functions for electrons in a coulomb field undergoing any type of transition are obtained in terms of the relevant matrix elements by the use of the Green function for the Dirac electron in a coulomb field. It is also shown that the angular distribution function is obtained from matrix elements based on, not the scattered wave, but on the time-space reversed scattered wave.

I. INTRODUCTION

In the investigation of the angular correlation between two radiations emitted by an excited nucleus in two successive transitions, the directional correlation involving emission of internal conversion electrons becomes important whenever either or both transitions correspond to large conversion coefficients. Thus, the effect is of primary interest in heavy nuclei,[1] low energy transitions and/or high order multipoles.[2] An essential restriction does enter from both the experimental and theoretical points of view in that it is desirable to avoid cases in which the intermediate state is long-lived.[3] However, this restriction applies only to the second transition so that, perhaps more often than not, the first transition may correspond to large internal conversion. These remarks apply, of course, only to the practical application of the results given below. So far as content of this paper is concerned, application of the results to correlations between conversion electrons and any other radiation (including another conversion electron transition) may be made and, for different radiations, the distinction between the cases in which the conversion electron is emitted first or second is trivial. We do make the assumption that the intermediate state is sufficiently short-lived so that hfs and other perturbing couplings may be ignored.[3,4]

Although a number of experimental studies of conversion correlation have been reported in the literature,[1] there had been no adequate theoretical treatment of the problem whereby the results of the experiments could be interpreted and used for the purpose of identifying the transitions involved. Previous work is confined largely to two papers by Gardner.[5] In the first of these Gardner discussed the nonrelativistic treatment of the conversion-conversion correlation. This was necessarily confined to electric multipoles and omitted the contribution to final states involving spin flip. The restriction was also made to ejection of s-electrons only but this restriction is not essential to the nonrelativistic limit. In the second paper an extension is made to the conversion-conversion correlation in which one of the transitions is a pure magnetic, the other a pure electric multipole and again the initial electronic states are s-states. In the v^2/c^2 approximation considered in this reference both final states (with and without spin flip) enter but essential interference terms between them do not appear. In any case numerical results are not given. It is of importance to note that while the error introduced by the nonrelativistic treatment of the correlation problem is, in general, much smaller than that arising in the calculation of the absolute conversion coefficients (since only ratios of matrix elements are involved in the former) the dependence of the correla-

* This paper is based on work performed for the AEC at the Oak Ridge National Laboratory.

[1] The restriction to heavy nuclei is not very stringent. Conversion correlation has been experimentally investigated in elements as light as Br, L. I. Rusenov and Ye. I. Chuykin, Doklady Akad. Nauk. U.S.S.R. **68**, 1029 (1949); also in Sm, C. M. Fowler (private communication); Hf^{181}, A. H. Ward and D. Walker, Nature **163**, 168 (1949) and A. Lundly, Phys. Rev. **76**, 993 (1949); Hg^{197}, Walter, Huber, and Zunti, Helv. Phys. Acta **23**, 697 (1950) and O. Huber and F. Humbel, Helv. Phys. Acta. **24**, 127 (1951).

[2] Rose, Goertzel, Spinrad, Harr, and Strong, Phys. Rev. **83**, 79 (1951). References to equations of this paper will be denoted with the suffix 1.

[3] H. Frauenfelder, Phys. Rev. **82**, 549 (1951); Aeppli, Bishop, Frauenfelder, Walter, and Zunti, Phys. Rev. **82**, 550 (1951).

[4] Compare, G. Goertzel, Phys. Rev. **70**, 897 (1946).

[5] J. W. Gardner, Proc. Phys. Soc. (London) **A62**, 763 (1949); **64**, 238 (1951). The result for electric multipoles, namely that the $F_L^M(\vartheta)$ functions (see Eq. (2) below) are proportional to $|Y_L^M|^2$, where (for the case considered by Gardner) L and M are the angular momentum quantum numbers of the electron, is also obtained if the spin is treated in Pauli approximation and a sum over spins is performed. The essential point here is that the orbital angular momentum of the electrons corresponds to a good quantum number. An outline of the formal aspects of the theory was discussed by M. Fierz, Helv. Phys. Acta **22**, 489 (1949).

tion functions on the multipole index L is also much less sensitive for the former (for essentially the same reason). Hence a greater accuracy is required in the correlation calculation.

A relativistic treatment of the formal aspect of the problem was given by Ling.[6] However, the pertinent matrix elements were calculated using the scattered wave function instead of the time-space reversed scattered wave function and, as the following shows (Sec. III), the latter is the correct procedure. This part of the problem was treated correctly by Lloyd.[7]

In the following we give a completely relativistic treatment of the conversion correlation problem. Since the pertinent matrix elements (Eqs. (41) and (43) below) are exactly those which arise in the conversion coefficient calculation[2] the numerical results are given for just those cases previously considered. In particular, we consider conversion from the K-shell only (without screening) although the extension to other shells is almost immediate once numerical values for the matrix elements become available. It is planned to make this extension in the near future. The form in which the numerical results are given is as follows: If the correlation between a γ-ray and any other radiation x is known and is expressed as an expansion in Legendre polynomials, the correlation with a conversion electron replacing the γ-ray is obtained by multiplying the coefficients of each Legendre polynomial $P_\nu(\cos\vartheta)$ by a coefficient $b_\nu(Z, k, \pi, L)$ (Eqs. (46) and (48)) where Z, k are as in reference 2, $\pi=e$, m refers to the parity (electric, magnetic respectively) and the multipole is a (pure) 2^L pole. For conversion-conversion correlation (Sec. V(c)) we multiply the coefficients in the $\gamma-\gamma$ correlation function[8] by $b_\nu(Z, k_1, \pi_1, L_1)b_\nu(Z, k_2, \pi_2, L_2)$. The prescription just given also applies when the x-transition is a mixed one [Sec. V(e)] involving superposition of different angular momenta. Corresponding results for the case in which the conversion transition is mixed[7] could also be obtained from the formalism given below but we do not give any numerical results for this case.

As indicated, and as would be expected, the conversion correlation is not only parity dependent but is also dependent on the physical parameters Z, k. This is in contrast to the $\gamma-\gamma$ correlation, for example, and so far as physical parameter dependence is concerned, it is in contrast to the nonrelativistic electric multipole conversion correlation.[5,6] The dependence on physical parameters enters here because of the presence of more than one final state (there are two for initial electron angular momentum $j=\frac{1}{2}$) and the fact that these final states are physically different (linearly independent radial functions). The dependence on physical parameters, and on parity as well, is a property of the angular distribution functions $F_L^M(\vartheta)$ [Eqs. (2) and (25)] which describe a Dirac electron undergoing a transition in a (coulomb) field between specified states and corresponding to a prescribed direction of motion at large distances. Up to a certain point [Eqs. (25) and (26)], the formal treatment given below applies as well to the angular distribution function for a number of other problems: emission of β-particles with coulomb effects included, ejection of photoelectrons, pair formation, etc.

[6] D. S. Ling, Ph.D. dissertation (University of Michigan, 1948).
[7] S. P. Lloyd, private communication.
[8] D. R. Hamilton, Phys. Rev. **58**, 122 (1940). See also D. S. Ling and D. L. Falkoff, Phys. Rev. **76**, 1639 (1949); W. Arnold, Phys. Rev. **80**, 34 (1950).

II. GENERAL FORMALISM

We consider a double cascade in which a nucleus makes a transition from a state of angular momentum J_1 to an intermediate state J emitting radiation 1 of angular momentum L_1 and then emits radiation 2 of angular momentum L_2 in going to a final state J_2. This cascade is designated by $J_1(L_1)J(L_2)J_2$. The components of J_1, L_1, J, L_2, J_2 on an arbitrary quantization axis are m_1, M_1, m, M_2, m_2 respectively with the usual conservation rules applying. Selecting the direction of the second radiation as the quantization axis, and with ϑ denoting the angle between radiations, the angular correlation function can be written in the form[9]

$$W(\vartheta)=\sum_{m_1mm_2}[C_{m_1M_1}{}^{J_1L_1J}]^2F_{L_1}{}^{M_1}(\vartheta)\times[C_{mM_2}{}^{JL_2J_2}]^2F_{L_2}{}^{M_2}(0),\quad (1)$$

where $C_{m_1M_1}{}^{J_1L_1J}$ is the vector addition coefficient[10] and $F_{L_i}{}^{M_i}(\vartheta)$ is the angular distribution function for radiation i. It will be noted that the dependence on J_1, J, J_2 is entirely contained in the C-coefficients. Since we are interested only in the angular dependence of W, multiplicative factors in the $F_L{}^M$ independent of the magnetic quantum numbers are irrelevant and will be discarded, from time to time, in the sequel.

For an electron in a final state described by a wave function $\Psi(\mathbf{r}, t)$ the $F_L{}^M$ is obtained from the radial Dirac current per unit solid angle:

$$F_L{}^M(\vartheta)=r^2J_r=-r^2(\Psi^\dagger\alpha_r\Psi)\quad (2)$$

with $r\rightarrow\infty$, and with a summation over final spins and an average over initial spins implied. The dagger in Eq. (2) means hermitian conjugate. For a transition induced by a time-dependent perturbation $\mathcal{H}_1e^{-ikt}+\mathcal{H}_1{}^*e^{ikt}$ we have[11]

$$(\mathcal{H}_0+\mathcal{H}_1e^{-ikt})\Psi=i\partial\Psi/\partial t\quad (3)$$

wherein $\mathcal{H}_0$ is the Dirac hamiltonian for an electron in a coulomb field and we have emitted the conjugate term, as usual, since it will make no final contribution. The current in (2) is then calculated from the perturbation solution of (3).

[9] D. L. Falkoff and G. E. Uhlenbeck, Phys. Rev. **79**, 323 (1950).
[10] E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, London, 1935), Chapter III. There denoted by $(J_1L_1m_1M_1|J_1L_1Jm)$. See Appendix C.
[11] Throughout we use relativistic units: $\hbar=m=c=1$. In the following kmc^2 will be the energy released in the nuclear transition.

An alternative, and completely equivalent procedure is the following: We introduce the fourier transform $\psi(E, r)$ according to

$$\Psi(\mathbf{r}, t)=\int_{-\infty}^{\infty} \psi(E, \mathbf{r})e^{-iEt}dE \tag{4}$$

then (3) becomes

$$(\mathcal{H}_0-E)\psi(E, \mathbf{r})=-\mathcal{H}_1\psi(E-k, \mathbf{r})\rightarrow-\mathcal{H}_1\psi_i(E_i, \mathbf{r}) \tag{3a}$$

where the replacement indicated by the arrow introduces the initial state wave function ψ_i with total energy E_i, consistent with the perturbation solution to solution to first order in $\mathcal{H}_1$. Then if the Green function solution of (3a) is used to obtain the current, the result is the same as given by Eq. (2).[12] That is,

$$F_L{}^M(\vartheta)=-r^2(\psi^\dagger\alpha_r\psi) \tag{2a}$$

with $r\rightarrow\infty$.

A. The Green Function[13]

We wish to solve Eq. (3a), that is

$$(E-V+\boldsymbol{\alpha}\cdot\mathbf{p}+\beta)\psi=\mathcal{H}_1\psi_i, \tag{5}$$

where V is a central field $(V=-\alpha Z/r)$ and the initial state ψ_i is specified, by the use of the Green function for the operator on the left-hand side of (5). The most convenient procedure is to obtain the free-particle Green function and then make the appropriate modifications in the radial functions. With

$$(E+\boldsymbol{\alpha}\cdot\mathbf{p}+\beta)G^{(0)}(\mathbf{r}, \mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}')I,$$

where I is the (4×4) unit matrix, we have

$$G^{(0)}(\mathbf{r}, \mathbf{r}')=-(E-\beta+i\boldsymbol{\alpha}\cdot\nabla)\frac{e^{ip|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{6}$$

In (6) the ∇ operator acts on $\mathbf{r}$ and $p=(E^2-1)^{\frac{1}{2}}$. In (6) we use the familiar expansion

$$\frac{e^{ip|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}=ip\sum_{lm} h_l(pr_>)j_l(pr_<)Y_l{}^m(\mathbf{r})Y_l{}^{m*}(\mathbf{r}'), \tag{7}$$

where the order of the arguments in spherical harmonics is immaterial, and h_l, j_l are the spherical hankel functions of the first kind and spherical bessel function respectively.

$$\boldsymbol{\alpha}\cdot\nabla=\rho_1\sigma_r\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)+\rho_1\sigma_r\rho_3K/r \tag{8}$$

and ρ_1, ρ_3 are the matrices operating in Dirac (ρ) space,

$$\rho_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \rho_3=\beta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

while $\boldsymbol{\sigma}$-matrices operate in Pauli space. Of course, products of ρ- and $\boldsymbol{\sigma}$-operators imply direct products. In (8)

$$K=-\rho_3(\boldsymbol{\sigma}\cdot\mathbf{L}+1),\quad \mathbf{L}=-i\mathbf{r}\times\nabla. \tag{9}$$

Noting that we can write the unit matrix in $\boldsymbol{\sigma}$-space in the form (Appendix A),

$$I_\sigma=\sum_\tau \chi_{\frac{1}{2}}{}^\tau\chi_{\frac{1}{2}}{}^{\tau\dagger} \tag{10}$$

we have the result

$$\sum_m Y_l{}^m(\mathbf{r})Y_l{}^{m*}(\mathbf{r}')I_\sigma=\sum_{\kappa\mu} \chi_\kappa{}^\mu(\mathbf{r})\chi_\kappa{}^{\mu\dagger}(\mathbf{r}') \tag{11}$$

where $\chi_\kappa{}^\mu$ is the Pauli central field spinor (Appendix A) and the sum over κ includes only l, $-l-1$. From (6), (7), (11), and (A11) we obtain for $r>r'$:

$$\begin{aligned}G_{11}{}^{(0)}/(E-1)&=-ip\sum_{\kappa\mu} h_{l(\kappa)}(pr)j_{l(\kappa)}(pr')\chi_\kappa{}^\mu(\mathbf{r})\chi_\kappa{}^{\mu\dagger}(\mathbf{r}')\\ &=G_{22}{}^{(0)}/(E+1)\\ G_{12}{}^{(0)}=G_{21}{}^{(0)}&=-p^2\sum_{\kappa\mu} S(\kappa)h_{l(-\kappa)}(pr)\\ &\quad\times j_{l(\kappa)}(pr')\chi_{-\kappa}{}^\mu(\mathbf{r})\chi_\kappa{}^{\mu\dagger}(\mathbf{r}').\end{aligned} \tag{12}$$

It is very convenient to write Eq. (12) in a form such that each term (κ, μ) is expressed in terms of field-free Dirac eigenfunctions in the angular momentum representation. These are

$$\phi_\kappa{}^\mu(\mathbf{r})=\begin{pmatrix}-iS(\kappa)(E-1/E+1)^{\frac{1}{2}}z_{l(-\kappa)}(pr)\chi_{-\kappa}{}^\mu(\mathbf{r})\\ z_{l(\kappa)}(pr)\chi_\kappa{}^\mu(\mathbf{r})\end{pmatrix}. \tag{13}$$

Retaining the notation $\phi_\kappa{}^\mu$ for standing waves (spherical cylinder function $z_l=j_l$) and using $\bar{\phi}_\kappa{}^\mu$ for outgoing waves $(z_l=h_l)$ we find by changing κ to $-\kappa$ in the first column of $G^{(0)}$, thus simply reordering the terms, the result for $r>r'$

$$G^{(0)}(r, r')=-ip(E+1)\sum_{\kappa\mu} \bar{\phi}_\kappa{}^\mu(\mathbf{r})\phi_\kappa{}^{\mu\dagger}(\mathbf{r}'). \tag{14}$$

To obtain the Green function for the coulomb field, we replace $\bar{\phi}_\kappa{}^\mu$ and $\phi_\kappa{}^\mu$ by coulomb field solutions which differ from (13) only in that z_l is replaced by Dirac radial functions with exactly the same normalization.[14,15] It is convenient for comparison purposes and later identification of radial matrix elements to use the same radial functions used before.[2,15] These latter differ by a factor $-S(\kappa)(p(E+1)/\pi)^{\frac{1}{2}}$ from the former. Thus, we write, in accord with previous notation,

$$\Phi_\kappa{}^\mu(\mathbf{r})=\begin{pmatrix}-if_\kappa\chi_{-\kappa}{}^\mu\\ g_\kappa\chi_\kappa{}^\mu\end{pmatrix} \tag{15a}$$

for the standing wave solution of the Dirac coulomb

[12] The proof of this equivalence in the nonrelativistic case is given by H. A. Bethe, Ann. Physik 4, 443 (1930). The generalization to the relativistic case is almost immediate. See E. Greuling and M. L. Meeks, Phys. Rev. 82, 531 (1951).

[13] Appendix A contains a discussion of the sign convention and representation used throughout this and subsequent sections.

[14] M. E. Rose, Phys. Rev. 82, 389 (1951).

[15] M. E. Rose, Phys. Rev. 51, 484 (1937).

field and

$$\bar{\Phi}_\kappa{}^\mu(\mathbf{r})=\begin{pmatrix}-i\bar{f}_\kappa\chi_{-\kappa}{}^\mu\\ \bar{g}_\kappa\chi_\kappa{}^\mu\end{pmatrix}\tag{15b}$$

for the outgoing wave solution. Specifically, the normalization is such that $r\rightarrow\infty$

$$\begin{aligned}f_\kappa(\infty)&=-\left(\frac{p(E-1)}{\pi}\right)^{\frac{1}{2}}\lim_{r\rightarrow\infty}j_{l(-\kappa)}(pr+\Delta_\kappa),\\ g_\kappa(\infty)&=-\left(\frac{p(E+1)}{\pi}\right)^{\frac{1}{2}}S(\kappa)\lim_{r\rightarrow\infty}j_{l(\kappa)}(pr+\Delta_\kappa),\end{aligned}\tag{16}$$

where

$$\Delta_\kappa=\delta_\kappa(Z)-\delta_\kappa(Z=0)\tag{16a}$$

and $\delta_\kappa(Z)$ is the phase defined in reference 15. The irregular coulomb field solutions are defined so that $\bar{f}_\kappa$ and $\bar{g}_\kappa$ have the asymptotic behavior (16) with $j_{l(\pm\kappa)}$ replaced by $h_{l(\pm\kappa)}$. The definition of Δ_κ is unchanged.

Finally, the desired Green function is, for $r>r'$

$$G(\mathbf{r},\mathbf{r}')=-\pi i\sum_{\kappa\mu}\bar{\Phi}_\kappa{}^\mu(\mathbf{r})\Phi_\kappa{}^{\mu\dagger}(\mathbf{r}').\tag{17}$$

The Green function for $r<r'$ is

$$G(\mathbf{r},\mathbf{r}')=-\pi i\sum_{\kappa\mu}\Phi_\kappa{}^\mu(\mathbf{r})\bar{\Phi}_\kappa{}^{\mu\dagger}(\mathbf{r}')\tag{18}$$

but the hermitian conjugate operation applies only to the spin-angular spinors $\chi_{\pm\kappa}{}^\mu$.

B. The Angular Distribution

The solution of (5) which has the required asymptotic behavior, that of an outgoing wave, is

$$\psi(\mathbf{r})=\int d\mathbf{r}G(\mathbf{r},\mathbf{r}')\mathcal{H}_1(\mathbf{r}')\psi_i(\mathbf{r}').\tag{19}$$

Using Eq. (17) the asymptotic form of ψ is

$$\psi_\infty=-\pi i\sum_{\kappa\mu}\bar{\Phi}_\kappa{}^\mu(\mathbf{r})\langle\Phi_\kappa{}^\mu(\mathbf{r}')|\mathcal{H}_1|\psi_i\rangle.\tag{20}$$

Here, and in the following angular brackets designate scalar product formation over spin and configuration space. More explicitly, Eq. (20) may be written in the form

$$\psi_\infty=i\pi^{\frac{1}{2}}\frac{e^{ipr}}{r}\sum_{\kappa\mu}e^{i\Delta_\kappa}S(\kappa)\langle\Phi_\kappa{}^\mu(\mathbf{r}')|\mathcal{H}_1|\psi_i\rangle$$
$$\times\begin{pmatrix}-iS(\kappa)(E-1)^{\frac{1}{2}}p^{-\frac{1}{2}}\chi_{-\kappa}{}^\mu\exp[-\frac{1}{2}\pi i(l(-\kappa)+1)]\\ (E+1)^{\frac{1}{2}}p^{-\frac{1}{2}}\chi_\kappa{}^\mu\exp[-\frac{1}{2}\pi i(l(\kappa)+1)]\end{pmatrix}.\tag{21}$$

The angular distribution function, which is the current per unit solid angle in the radially outward direction for a specific initial state and for large r, is

$$\begin{aligned}F_L{}^M(\vartheta)&=-r^2(\psi|\rho_1\sigma_r|\psi)\\ &=i\sum_{\kappa\kappa'}\sum_{\mu\mu'}e^{i(\Delta_\kappa-\Delta_{\kappa'})}S(\kappa)S(\kappa')\\ &\quad\times\langle\kappa'\mu'|\mathcal{H}_1|\psi_i\rangle^*\langle\kappa\mu|\mathcal{H}_1|\psi_i\rangle\\ &\quad\times\{S(\kappa)\exp\{\tfrac{1}{2}\pi i[l(\kappa')-l(-\kappa)]\}\\ &\quad\times(\chi_{\kappa'}{}^{\mu'}|\sigma_r|\chi_{-\kappa}{}^\mu)\\ &\quad-S(\kappa')\exp\{\tfrac{1}{2}\pi i[l(-\kappa')-l(\kappa)]\}\\ &\quad\times(\chi_{-\kappa'}{}^{\mu'}|\sigma_r|\chi_\kappa{}^\mu)\}.\end{aligned}\tag{22}$$

This represents the angular distribution function for an initial state of specific magnetic quantum number, τ, say. What is eventually required is the average of (22) over τ. However, in order to avoid confusion of notation we retain the symbol $F_L{}^M$ for both angular distribution functions and rely on the context to make the meaning clear. Also, in Eq. (22) and in the following, the logarithmic term in the phases Δ_κ and δ_κ can be dropped since it is clear that it cancels out in the expression for $F_L{}^M$.

From Appendix A we find

$$\begin{aligned}(\chi_{-\kappa'}{}^{\mu'}|\sigma_r|\chi_\kappa{}^\mu)&=-(\chi_{-\kappa'}{}^{\mu'}|\chi_{-\kappa}{}^\mu)\\ &=-(\chi_{\kappa'}{}^{\mu'}|\chi_\kappa{}^\mu)=(\chi_{\kappa'}{}^{\mu'}|\sigma_r|\chi_{-\kappa}{}^\mu)\end{aligned}\tag{23}$$

and we note that

$$\begin{aligned}&S(\kappa')\exp\{\tfrac{1}{2}\pi i[l(-\kappa')-l(\kappa)]\}\\ &\qquad=-S(\kappa)\exp\{\tfrac{1}{2}\pi i[l(\kappa')-l(-\kappa)]\}.\end{aligned}\tag{24}$$

Further, we consider a definite magnetic quantum number, τ, for the initial state ψ_i so that $\mu=\mu'=M+\tau$. Then, averaging over τ,

$$\begin{aligned}F_L{}^M(\vartheta)&=-i\sum_{\kappa\kappa'\tau}e^{i(\Delta_\kappa-\Delta_{\kappa'})}S(\kappa)\\ &\quad\times\exp\{\tfrac{1}{2}\pi i[l(-\kappa')-l(\kappa)]\langle\kappa',M+\tau|\mathcal{H}_1|\psi_i\rangle^*\\ &\quad\times\langle\kappa,M+\tau|\mathcal{H}_1|\psi_i\rangle(\chi_{\kappa'}{}^{M+\tau}|\chi_\kappa{}^{M+\tau}).\end{aligned}\tag{25}$$

Using the definition (A5), (A1), and Appendix B we find

$$\begin{aligned}(\chi_{\kappa'}{}^\mu|\chi_\kappa{}^\mu)&=\textstyle\sum_\nu[(2l(\kappa)+1)(2l(\kappa')+1)|\kappa\kappa'|/\pi(2\nu+1)]^{\frac{1}{2}}\\ &\quad\times C_0{}^{l(\kappa)}{}_0{}^{l(\kappa')\nu}C_\mu{}^j{}_{-\mu}{}^{j'\nu}(-)^{\nu+\frac{1}{2}-\mu+\kappa+\kappa'}\\ &\quad\times W(l(\kappa),l(\kappa'),j,j';\nu\tfrac{1}{2})Y_\nu{}^0(\cos\vartheta),\end{aligned}\tag{26}$$

where $j=|\kappa|-\frac{1}{2}$, $j'=|\kappa'|-\frac{1}{2}$, and W is a Racah coefficient.[16]

Combining the results (25) and (26) one obtains the starting point for calculating the angular distribution functions $F_L{}^M$ for any interaction involving an electron in a coulomb field with the direction of motion specified

[16] G. Racah, Phys. Rev. **62**, 438 (1942); **63**, 367 (1943).

asymptotically with reference to an arbitrary axis of quantization.

III. RELATION TO SCATTERED WAVE

The result (21) for the wave function describing the outgoing wave has a rather interesting physical interpretation. In fact, we show in this section that the radial current, summed over spins of initial and final states, can be calculated from the square modulus of matrix elements of the perturbation $\mathcal{H}_1$ between the initial state and a "final" state ψ_f where ψ_f is *not* the scattered wave but is the time and space reversed scattered wave. Thus, ψ_f has the asymptotic behavior of a plane wave plus an *incoming* wave rather than a plane wave plus an outgoing wave.[17] This is the point which was overlooked in previous work.[6,18]

In order to demonstrate the result just stated we carry out the sum over μ in Eq. (21). Since, as the Green function (17) shows, the sum in question is of the form $\sum_\mu \chi_\kappa^\mu(\mathbf{r})\chi_\kappa^{\mu\dagger}(\mathbf{r}')$, which is invariant under rotations in spin and configuration space, we can choose the direction of $\mathbf{r}$ as the quantization axis for the present purpose. Then since

$$\begin{aligned}\chi_\kappa^\mu(0)&=(|\kappa|/4\pi)^{\frac{1}{2}}[-S(\kappa)]^{\mu+\frac{1}{2}}\chi_{\frac{1}{2}}^\mu; \quad \mu=\pm\tfrac{1}{2}\\ &=0; \quad \mu\neq\pm\tfrac{1}{2}\end{aligned} \tag{27}$$

we obtain from (21)

$$\psi_\infty=\frac{ie^{ipr}}{r}\sum_\kappa(|\kappa|/4\pi)^{\frac{1}{2}}e^{i\Delta_\kappa}\times\{\langle\Phi_\kappa^{\frac{1}{2}}|\mathcal{H}_1|\psi_i\rangle\Lambda_+ +\langle\Phi_\kappa^{-\frac{1}{2}}|\mathcal{H}_1|\psi_i\rangle\Lambda_-\}, \tag{28}$$

where the spinors $\Lambda_\pm$ are

$$\Lambda_+=\begin{pmatrix}-\dfrac{ip}{E+1}\chi_{\frac{1}{2}}^{\frac{1}{2}}\exp[-\frac{1}{2}\pi i(l(-\kappa)+1)]\\ -\chi_{\frac{1}{2}}^{\frac{1}{2}}S(\kappa)\exp[-\frac{1}{2}\pi i(l(\kappa)+1)]\end{pmatrix}, \tag{29}$$

and Λ_- is obtained by changing $\chi_{\frac{1}{2}}^{\frac{1}{2}}$ to $\chi_{\frac{1}{2}}^{-\frac{1}{2}}$ and multiplying by $S(\kappa)\rho_3$. Since

$$-i\exp[-\tfrac{1}{2}\pi i(l(-\kappa)+1]=S(\kappa)\exp[-\tfrac{1}{2}\pi i/2(l(\kappa)+1)],$$

and $l(\kappa)=l(-\kappa)+S(\kappa)$, we can write (28) in the form $\psi_\infty=\psi_++\psi_-$

$$\begin{aligned}\psi_+&=\frac{ie^{ipr}}{4\pi r}\sum_\kappa(4\pi|\kappa|)^{\frac{1}{2}}e^{i\Delta_\kappa}\exp[-\tfrac{1}{2}\pi il(-\kappa)]\times\langle\Phi_\kappa^{\frac{1}{2}}|\mathcal{H}_1|\psi_i\rangle D_+,\\ \psi_-&=\frac{ie^{ipr}}{4\pi r}\sum_\kappa(4\pi|\kappa|)^{\frac{1}{2}}e^{i\Delta_\kappa}\exp[-\tfrac{1}{2}\pi i(l(\kappa)+1)]\times\langle\Phi_\kappa^{-\frac{1}{2}}|\mathcal{H}_1|\psi_i\rangle D_-\end{aligned} \tag{30}$$

and $D_\pm$ are the Dirac plane wave spinors for the momentum along the quantization axis with + and − corresponding to spin "up," spin "down," respectively:

$$D_+=\begin{pmatrix}-p/(E+1)\\0\\1\\0\end{pmatrix}\qquad D_-=\begin{pmatrix}0\\p/(E+1)\\0\\1\end{pmatrix}. \tag{31}$$

If in (30) we write the sum of matrix elements as a single matrix element, that is

$$\psi_\pm=\langle\Phi_\pm|\mathcal{H}_1|\psi_i\rangle\frac{e^{ipr}}{4\pi r}D_\pm \tag{32}$$

then

$$\begin{aligned}\Phi_+&=-i\textstyle\sum_\kappa(4\pi|\kappa|)^{\frac{1}{2}}e^{-i\Delta_\kappa}\exp[\frac{1}{2}\pi il(-\kappa)]\Phi_\kappa^{\frac{1}{2}},\\ \Phi_-&=-i\textstyle\sum_\kappa(4\pi|\kappa|)^{\frac{1}{2}}e^{-i\Delta_\kappa}\exp[\frac{1}{2}\pi i(l(\kappa)+1)]\Phi_\kappa^{-\frac{1}{2}}.\end{aligned} \tag{33}$$

We now compare with the scattered wave which has the asymptotic behavior: plane wave plus outgoing spherical wave. For a fixed spin direction specified by σ

$$\psi_{sc}^\sigma=\sum_\kappa(4\pi|\kappa|)^{\frac{1}{2}}e^{i\Delta_\kappa}\times\exp[\tfrac{1}{2}\pi i(l(\kappa)-(\sigma+\tfrac{1}{2})(1+S(\kappa))]\Phi_\kappa^\sigma,\quad \sigma=\pm\tfrac{1}{2}. \tag{34}$$

To this we apply the space and time reversal operators I_S (β times space inversion) and I_T ($i\sigma_2$ times complex conjugation). Then

$$I_TI_S\Phi_\kappa^\sigma=(-)^{\sigma+\frac{1}{2}-\kappa}\Phi_\kappa^{-\sigma}.$$

Noting that $|\kappa|=l(\kappa)-\frac{1}{2}(S(\kappa)-1)$ we obtain

$$I_TI_S\psi_{sc}^{\frac{1}{2}}=\sum_\kappa(4\pi|\kappa|)^{\frac{1}{2}}e^{-i\Delta_\kappa}\exp[\tfrac{1}{2}\pi il(\kappa)]\Phi_\kappa^{-\frac{1}{2}}, \tag{35a}$$

$$I_TI_S\psi_{sc}^{-\frac{1}{2}}=\sum_\kappa(4\pi|\kappa|)^{\frac{1}{2}}e^{-i\Delta_\kappa}\times\exp[\tfrac{1}{2}\pi i(l(-\kappa)+1)]\Phi_\kappa^{\frac{1}{2}}. \tag{35b}$$

Comparing with (33) we see that

$$\Phi_+=-I_TI_S\psi_{sc}^{-\frac{1}{2}},\quad \Phi_-=I_TI_S\psi_{sc}^{\frac{1}{2}}. \tag{36}$$

Thus, except for a spin reversal, the wave functions $\psi_f(\equiv\Phi_\pm)$ which are used in calculating the current are the time and space reversed scattered wave. For the current summed over final spins the spin reversal is irrelevant. However, the effect of time and space reversal is to change the sign of the coulomb phase shifts δ_κ (Eq. 16(a)) in the final results and this is not an irrelevant change.

IV. APPLICATION TO K-SHELL INTERNAL CONVERSION

For the K-shell (or L_I shell, for example) we have $\psi_i\equiv\psi_{-1}^\tau$ and $\tau=\pm\frac{1}{2}$. The radiation fields in a convenient gauge have been given elsewhere.[2] Discarding an irrelevant multiplicative factor $([2/\pi L(L+1)])$ throughout, we discuss the magnetic and electric conversion separately.

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[17] This result is well known for the nonrelativistic problem (see N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, New York, 1933), p. 258). For noncentral potentials a proof is given by W. Rarita and J. Schwinger, Phys. Rev. **59**, 556 (1941).

[18] M. Fuchs, Ph.D. dissertation (University of Michigan, 1951).

(A) Magnetic Conversion

For a magnetic 2^L pole we have

$$\langle\kappa\mu|\mathcal{H}_1{}^m|-1\tau\rangle=\langle\kappa\mu|\rho_1(\boldsymbol{\sigma}\cdot\mathbf{L}Y_L{}^M)|-1\tau\rangle$$
$$=i[R_3'\langle\chi_{-\kappa}{}^\mu|\boldsymbol{\sigma}\cdot\mathbf{L}Y_L{}^M|\chi_{-1}{}^\tau\rangle$$
$$-R_4'\langle\chi_\kappa{}^\mu|\boldsymbol{\sigma}\cdot\mathbf{L}Y_L{}^M|\chi_1{}^\tau\rangle]_\kappa \quad (37)$$

where in the notation of reference 2, Eq. (17_1), has been introduced for the radial matrix elements and the final state in the R' integrals is described by the single index κ. In (37) $\mathbf{L}$ acts on $Y_L{}^M$ only. This latter fact enables us to conclude that

$$\langle\chi_\kappa{}^\mu|\boldsymbol{\sigma}\cdot\mathbf{L}Y_L{}^M|\chi_1{}^\tau\rangle=-\langle\chi_{-\kappa}{}^\mu|\boldsymbol{\sigma}\cdot\mathbf{L}Y_L{}^M|\chi_{-1}{}^\tau\rangle. \quad (38)$$

Equation (38) together with (A.9) yields

$$\langle\chi_{-\kappa}{}^\mu|\boldsymbol{\sigma}\cdot\mathbf{L}Y_L{}^M|\chi_1{}^\tau\rangle=(\kappa-1)\langle\chi_\kappa{}^\mu|Y_L{}^M\chi_{-1}{}^\tau\rangle. \quad (39)$$

However,

$$Y_L{}^M\chi_{-1}{}^\tau=(4\pi)^{-\frac{1}{2}}Y_L{}^M\chi_{\frac{1}{2}}{}^\tau=(4\pi)^{-\frac{1}{2}}\textstyle\sum_{\bar\kappa}C_{M\tau}{}^{L\frac{1}{2}\bar{j}}\chi_{\bar\kappa}{}^{M+\tau}, \quad (40)$$

where $\bar\kappa=L,\ -L-1$, and $\bar{j}=|\bar\kappa|-\frac{1}{2}$. Finally,

$$\langle\kappa\mu|\mathcal{H}_1{}^m|-1\tau\rangle=i(4\pi)^{-\frac{1}{2}}C_{M\tau}{}^{L\frac{1}{2}j}(\kappa-1)$$
$$\times(R_3'+R_4')_\kappa(\delta_{\kappa,-L}+\delta_{\kappa,L+1})$$
$$\equiv C_{M\tau}{}^{L\frac{1}{2}j}Q(\kappa,L,m), \quad (41)$$

where the last factor in the first equality, contains Kronecker deltas.

(B) Electric Conversion

Here we evaluate the matrix element

$$\langle\kappa\mu|\mathcal{H}_1{}^e|-1\tau\rangle=\langle\kappa\mu|\rho_1h_{L-1}(r\boldsymbol{\sigma}\cdot\nabla+L\sigma_r)Y_L{}^M$$
$$+iLY_L{}^Mh_L|-1\tau\rangle \quad (42)$$

and the spherical hankel functions, which appear explicitly, have argument kr. Using

$$r\boldsymbol{\sigma}\cdot\nabla=\sigma_r(\mathbf{r}\cdot\nabla+i\boldsymbol{\sigma}\cdot\mathbf{r}\times\boldsymbol{\nabla})=\sigma_r(\partial/\partial r-\boldsymbol{\sigma}\cdot\mathbf{L})$$

Eq. (42) becomes

$$\langle\kappa\mu|\mathcal{H}_1{}^e|-1\tau\rangle$$
$$=iL[R_1+R_2-R_3+R_4]_\kappa\langle\chi_\kappa{}^\mu|Y_L{}^M|\chi_{-1}{}^\tau\rangle$$
$$-i(\kappa+1)(R_3+R_4)_\kappa\langle\chi_\kappa{}^\mu|Y_L{}^M|\chi_{-1}{}^\tau\rangle$$
$$=i(4\pi)^{-\frac{1}{2}}C_{M\tau}{}^{L\frac{1}{2}j}[L(R_1+R_2-R_3+R_4)$$
$$-(\kappa+1)(R_3+R_4)]_\kappa(\delta_{\kappa,L}+\delta_{\kappa,-L-1})$$
$$\equiv C_{M\tau}{}^{L\frac{1}{2}j}Q(\kappa,L,e), \quad (43)$$

where the notation (13_1) has been introduced.

(C) The Angular Distribution for K-Shell Conversion

In terms of the Q-coefficients defined by (41) and (43) the $F_L{}^M$ may be written formally for either parity. From (25) and (26)

$$F_L{}^M(\vartheta)=-i\sum_{\kappa\kappa'\tau\nu}(-)^{\nu+\frac{1}{2}-M-\tau+\kappa+\kappa'}S(\kappa)$$
$$\times e^{i(\Delta_\kappa-\Delta_{\kappa'})}\exp\{\tfrac{1}{2}\pi i[l(-\kappa')-l(\kappa)]\}$$
$$\times[(2l(\kappa)+1)(2l(\kappa')+1)|\kappa\kappa'|/\pi(2\nu+1)]^{\frac{1}{2}}$$
$$\times C_{M\tau}{}^{L\frac{1}{2}j}C_{M\tau}{}^{L\frac{1}{2}j'}C_0{}^{l(\kappa)}{}_0{}^{l(\kappa')\nu}$$
$$\times W(l(\kappa)l(\kappa')jj';\nu\tfrac{1}{2})Q^*(\kappa',L,\pi)$$
$$\times Q(\kappa,L,\pi)Y_\nu{}^0(\cos\vartheta). \quad (44)$$

Using (B.3) the sum over τ may be performed and with $\delta_\kappa(Z=0)=\frac{1}{2}\pi[\frac{1}{2}(1+S(\kappa))-|\kappa|]$ we obtain for electric multipoles

$$F_L{}^M(\vartheta)=\sum_{\nu=0}^{2L}(-)^MC_{1-1}{}^{LL\nu}C_{M-M}{}^{LL\nu}$$
$$\times(4\pi(2\nu+1))^{-\frac{1}{2}}b_\nu'Y_\nu{}^0(\cos\vartheta) \quad (45)$$

where ν takes on only even values ($C_{00}{}^{LL\nu}=0$ for ν odd) and

$$\frac{\nu(\nu+1)-2L(L+1)}{2L(L+1)(2L+1)}b_\nu'$$
$$=L^2|Q(L,L,e)|^2W^2(L,L,L-\tfrac{1}{2},L-\tfrac{1}{2};\nu\tfrac{1}{2})$$
$$+(L+1)^2|Q(-L-1,L,e)|^2$$
$$\times W^2(L,L,L+\tfrac{1}{2},L+\tfrac{1}{2};\nu\tfrac{1}{2})$$
$$-2L(L+1)Re[(e^{i\Delta_L}Q(L,L,e))^*e^{i\Delta_{-L-1}}$$
$$\times Q(-L-1,L,e)]W^2(L,L,L+\tfrac{1}{2},L-\tfrac{1}{2};\nu\tfrac{1}{2}). \quad (45a)$$

The angular distribution function given by (45) differs from that for emission of electromagnetic radiation[8,19] only in the factor $b_\nu'(e)$ and the two become identical (aside from a trivial normalization factor) when $b_\nu'=1$. We normalize to $b_0'(e)=1$ and using (B.6–8) we obtain for $b_\nu(e)=b_\nu'(e)/b_0'(e)$

$$b_\nu(e)=1+\frac{\nu(\nu+1)}{2L(L+1)-\nu(\nu+1)}$$
$$\times\frac{L}{2L+1}\frac{|L+1+T_e|^2}{L(L+1)+|T_e|^2}, \quad (46)$$

where

$$T_e=\frac{e^{i\delta_L}}{e^{i\delta_{-L-1}}}\frac{[L(R_1+R_2)-(2L+1)R_3-R_4]_{\kappa=L}}{[R_1+R_2+2R_4]_{\kappa=-L-1}}. \quad (46a)$$

[19] Biedenharn, Arfken, and Rose, Phys. Rev. **83**, 586 (1951).

For magnetic multipoles the allowed κ-values are $-L$, $L+1$. Using (B.4, 5) we find

$$F_L{}^M(\vartheta)=\sum_0^{2L}(-)^M C_{1-1}{}^{LL\nu}C_{M-M}{}^{LL\nu}\times(4\pi(2\nu+1))^{-\frac{1}{2}}b_\nu(m)Y_\nu{}^0(\cos\vartheta) \quad (47)$$

and for the b_ν normalized to $b_0=1$ we have

$$b_\nu(m)=1+\frac{\nu(\nu+1)}{2L(L+1)-\nu(\nu+1)}\times\frac{L(L+1)}{2L+1}\frac{|1-T_m|^2}{L+1+L|T_m|^2}, \quad (48)$$

where

$$T_m=\frac{e^{i\delta_{L+1}}}{e^{i\delta_{-L}}}\frac{(R_3'+R_4')_{\kappa=L+1}}{(R_3'+R_4')_{\kappa=-L}}. \quad (48a)$$

It is of interest to recognize that for γ-rays, spin $\frac{1}{2}$ particles and for spinless particles as well, the M-dependence of the angular distribution functions, when expanded in Legendre polynomials, is entirely contained in the vector addition coefficient $(-)^M C_{M-M}{}^{LL\nu}$. It is for this reason that the simple parametrization given by (45) and (47) is possible. This would not have been possible if any other expansion, powers of $\cos^2\vartheta$ say, had been employed. It is evident, Sec. V, that these remarks also apply to the angular correlation function $W(\vartheta)$.[20]

V. THE ANGULAR CORRELATION

A. Special Cases of Conversion Correlation

It is clear from the foregoing that the prescription given in Sec. I applies; namely, if the Legendre polynomial expansion of the correlation between a γ-ray and a radiation x is

$$W_{\gamma-x}(\vartheta)=\sum_{\nu=0}^{\nu_m}A_\nu(\gamma-x)P_\nu(\cos\vartheta) \quad (49a)$$

then the correlation function for a conversion electron and radiation x is [see (c) below],

$$W_{e-x}(\vartheta)=\sum_{\nu=0}^{\nu_m}b_\nu A_\nu(\gamma-x)P_\nu(\cos\vartheta). \quad (49b)$$

In (49) $\frac{1}{2}\nu_m$ is the smallest integer of the set L_1, L_2, J (or $J-\frac{1}{2}$) if J is integer (or half-integer). The order in which the transitions occur is contained solely in the coefficients A_ν and we observe that the correlation $J_1(L_1)J(L_2)J_2$ is the same as the correlation $J_2(L_2)J(L_1)J_1$. In (49a, b) it is to be understood that the γ-ray is a pure 2^L multipole. Then b_ν is obtained from (46) for electric or from (48) for magnetic conversion transitions. See, however, (e) below.

[20] This result is implicitly contained in S. P. Lloyd, Ph.D. dissertation (University of Illinois, 1951).

We consider the following correlations:

(a) Conversion-Gamma Correlation

Then in (49b) $A_\nu(\gamma-x)\equiv A_\nu(\gamma-\gamma)$. The $\gamma-\gamma$ correlation has been quite extensively studied. For the most practical case that the γ-rays are emitted with the lowest angular momentum allowed by angular momentum selection rules, that is

$$L_i=|J-J_i|\quad(J_i\neq J);\quad L_i=1\quad(J_i=J) \quad (50)$$

Lloyd[21] has given a tabulation of $A_\nu(\gamma-\gamma)$ for all transitions up to 2^4 pole–2^6 pole. In Lloyd's tabulation the normalization is $A_0=1$. For cases not covered by (50) the $A_\nu(\gamma-\gamma)$ may be obtained for $L_1\leqslant 2$, $L_2\leqslant 2$ from the tabulation of Falkoff and Uhlenbeck[9] and of Hamilton.[8] A useful form for $W_{\gamma\gamma}$ is (see Appendix B and reference 19)

$$W_{\gamma-\gamma}(\vartheta)=N\sum_{\nu=0}^{\nu_m}C_{1-1}{}^{L_1L_1\nu}C_{1-1}{}^{L_2L_2\nu}\times W(JJL_1L_1;\nu J_1)W(JJL_2L_2;\nu J_2)P_\nu(\cos\vartheta) \quad (51)$$

where, with the normalization factor

$$N=(-)^{J_1-J_2}(2L_1+1)(2L_2+1)(2J+1),$$

the coefficient of P_0 is equal to unity. The tabulation of the coefficients A_ν for cases not covered by (50) can then be extended considerably using Racah function tabulation of Biedenharn.[22]

(b) β-Conversion Correlation

The coefficients $A_\nu(\beta-\gamma)$ have been given[23] for the five pure β-invariants up through second forbidden transitions in the approximation $Z=0$. The case $Z\neq 0$ has been considered in the approximation $Z/137v\ll 1$ by Fuchs.[18] A calculation without this approximation could be carried out using the results of Sec. II but this has not yet been done.

(c) Conversion-Conversion Correlation

After the first conversion transition has taken place the half-empty K-shell will be filled very rapidly by x-ray emission or emission of Auger electrons. In fact the measured widths of x-ray lines shows that this filling-up time is much smaller than the intermediate state lifetime in any practical correlation experiment.[24]

[21] S. P. Lloyd, Phys. Rev. **83**, 716 (1951).

[22] L. C. Biedenharn and M. E. Rose, Oak Ridge National Laboratory Report No. 1098 (1951).

[23] D. L. Falkoff and G. E. Uhlenbeck, Phys. Rev. **79**, 334 (1950).

[24] A. H. Compton and S. K. Allison, *X-Rays in Theory and Experiment* (D. Van Nostrand Company, Inc., New York, 1935); also R. M. Steffen, Helv. Phys. Acta **22**, 167 (1949). If this were not so the initial electronic states would be different for the first and second conversion transitions and the probability amplitude for the double transition would be composed of two parts corresponding to two choices for the order of emission of the K-electrons. Then (reference 6) there would be a cross term between the two modes by which the cascade could take place. The evidence cited shows that this cross term is exceedingly negligible.

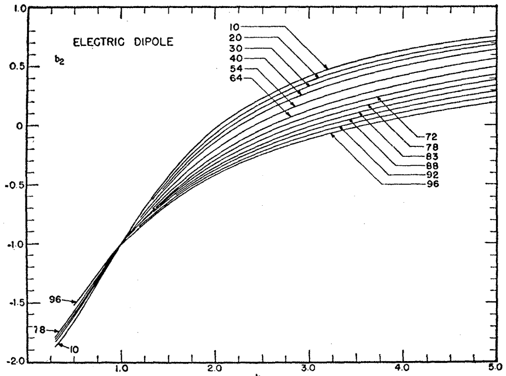

FIG. 1. b_2 coefficients for electric 2^l-pole transitions *versus* k (transition energy in mc^2 units). The attached numbers refer to the value of Z.

Also, the filling-up time is much shorter than the precession period corresponding to the hfs interaction of the nucleus and a single K-electron. Therefore it follows that the conversion-conversion correlation function is

$$W_{e-e}(\vartheta)=\sum_{\nu=0}^{\nu m} b_\nu(1)b_\nu(2)A_\nu(\gamma-\gamma)P_\nu(\cos\vartheta) \qquad (52)$$

where $b_\nu(1)$ refers to the first conversion electron emitted, etc.

(d) *Conversion-Alpha Correlation*

Such a cascade would be of interest in the case of a few heavy elements. The correlation function is

$$W_{e-\alpha}(\vartheta)=\sum_{\nu=0}^{\nu m} b_\nu(e)b_\nu(\alpha)A_\nu(\gamma-\gamma)P_\nu(\cos\vartheta), \qquad (53)$$

where

$$b_\nu(\alpha)=-C_{00}{}^{LL\nu}/C_{1-1}{}^{LL\nu}$$
$$=2L(L+1)/[2L(L+1)-\nu(\nu+1)]. \qquad (53a)$$

This applies when the α-particles are emitted with essentially a single orbital angular momentum.

(e) *Mixed Transitions*

The case in which the conversion electron transition is not pure could be calculated along exactly the same lines as the pure transitions discussed, (Sec. IV). These would involve new coefficients, different from the b_ν. However, if the nonconversion transition is a mixed one, the conversion transition pure, then (49b) still applies. However the $A_\nu(\gamma-\gamma)$ must take into account the fact that a superposition of different angular momenta are emitted in one of the transitions. For correlation with both transitions mixed the coefficients have been given by Lloyd[21] to first order in ϵ, where ϵ^2 is the ratio of intensities of electric 2^{L+1} pole to magnetic 2^L pole. For one pure γ and one mixed γ the coefficients (with no neglections) are given by Ling and Falkoff[8] for a mixture of electric quadrupole and magnetic dipole and a pure dipole or quadrupole. For a correlation with a pure conversion transition of multipole index L_1 and either parity and a mixed γ-transition in which the radiation field is electric 2^{L_2+1} and magnetic 2^{L_2} the correlation function is

$$W_{e-\gamma}(\vartheta)=1+(1+\epsilon^2)^{-1}\sum_{\nu=2}^{\nu m} b_\nu(L_1,\pi)\{\epsilon^2A_\nu^{(e)}(\gamma-\gamma)$$
$$+A_\nu^{(m)}(\gamma-\gamma)\pm\epsilon xy_\nu(L_2)A_\nu^{(m)}\}P_\nu(\cos\vartheta). \qquad (54)$$

The choice of $\pm$ sign in the cross term depends on whether the mixed transition occurs first or second[21] but note that, in the absence of a reliable nuclear model, the sign of ϵ is undetermined. However ϵ is real.[25] The $A_\nu^{(e,m)}$ are the coefficients for the pure 2^{L_2+1} electric,

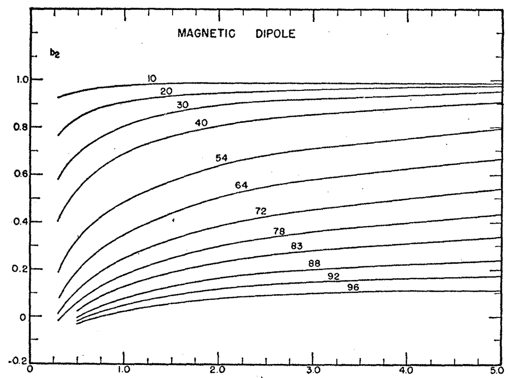

FIG. 2. Same as Fig. 1 for magnetic 2^l-pole transitions.

2^{L_2} magnetic radiations correlated with a pure $\gamma-2^{L_1}$ pole. The normalization is always $A_0=1$. Of course, for pure radiations the A_ν are parity independent and the indices e or m merely serve to indicate whether one uses L_2+1 or L_2 (respectively) for the multipole index for one transition along with L_1 for the other. The quantities x and $y_\nu(L)$ are defined by Lloyd.[21] The coefficients $A_\nu^{(m)}$ are given in Lloyd's tables[21] but $A_\nu^{(e)}$ corresponds to the correlation $J_1(L_1)J(L_2+1)J\pm L_2$ and these are not included in reference 21. The required coefficients $(A_\nu^{(e)}(\gamma-\gamma))$ can readily be obtained from the tabulated ones. In the transition to $J+L_2$

$$A_2=[J(L_2-2)-3(L_2+1)]A_2'/J(L_2+1). \qquad (55a)$$

The superscript e has been dropped and A_2' refers to the correlation involving $J_2=J+L_2+1$ as given in reference 21. For the transition to $J-L_2$

$$A_2=[J(L_2-2)+4L_2+1]A_2''/(J+1)(L_2+1) \qquad (55b)$$

with A_2'' referring to a transition to $J-L_2-1$. If $L_2=1$, (dipole-quadrupole mixture),

$$A_4=-2(2J+5)A_4'/J;\quad J_2=J+1 \qquad (55c)$$

or

$$A_4=-2(2J-3)A_4''/J;\quad J_2=J-1. \qquad (55d)$$

[25] S. P. Lloyd, Phys. Rev. 81, 161 (1951).

The above correction factors cover the vast majority of experimentally realized cases.[26] For correction factors for correlations involving quadrupole-octupole mixtures or higher multipoles in cascade with quadrupoles or higher ($L_1 \geqslant 2$), reference should be made to the Racah function tables.[22]

(f) *Triple Correlations*

For a triple cascade[19] in which three radiations are emitted successively the procedure for converting from a standard correlation, say $\gamma-\gamma-\gamma$, to one in which one or more γ-rays are replaced by conversion electrons is just the same. One inserts a factor b_ν for each conversion electron. Consider, for example, a triple cascade in which the correlation between only the first and third radiations is observed. The correlation function with conversion electrons replacing either the first or third or both first and third γ-rays is obtained by inserting $b_\nu(1)$ or $b_\nu(3)$ or $b_\nu(1)b_\nu(3)$, respectively, in the $\gamma_1-\gamma_3$ correlation:

$$W_{\gamma_1-\gamma_3}(\vartheta)=\sum_{\nu=0}^{\nu_m} \bar{A}_\nu P_\nu(\cos\vartheta).$$

Designating the cascade by $J_1(L_1)J(L_2)J'(L_3)J_2$, where L_2 refers to a pure 2^{L_2} pole γ-ray, we have $\frac{1}{2}\nu_m=$ minimum of $[L_1, L_3, J$ (or $J-\frac{1}{2}$), J' (or $J'-\frac{1}{2}$)]. The coefficients $\bar{A}_\nu$ are given elsewhere.[27]

B. Results

From Eqs. (46) and (48) it is obvious that for both electric and magnetic transitions all the coefficients

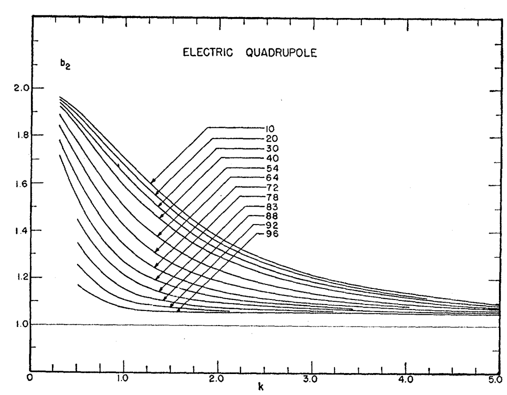

FIG. 3. Same as Fig. 1 for electric 2^2-pole transitions.

b_ν ($\nu>2$) are readily obtained from b_2 by

$$b_\nu(\pi)-1=\frac{\nu(\nu+1)[L(L+1)-3]}{3[2L(L+1)-\nu(\nu+1)]}(b_2(\pi)-1) \quad (56)$$

($\pi=e$ or m), so that only b_2 need be tabulated for each multipole.

In Figs. 1–10 the coefficients b_2 are given as functions of k for 12 values of Z in the range 10–96 for ten multipoles: $L=1-5$ electric and $L=1-5$ magnetic.[28] Figures 5, 7, and 9 give the b_2 coefficients for electric 2^3, 2^4, and 2^5 poles for the smaller Z-values and these coefficients for the same multipoles and for the larger Z-values are given in Figs. 5a, 7a, and 9a on an enlarged scale. The lowest k-value for which computations could be made is $k=0.3(Z\leqslant 78)$ and $k=0.5(Z>78)$.

For smaller k values, in the electric case, extrapolation toward the threshold can be made with the aid of the nonrelativistic limit:

$$b_2(e)=L(L+1)/[L(L+1)-3], \quad (\alpha Z\ll 1, k\ll 1). \quad (57)$$

For the magnetic transitions the "nonrelativistic" limit is Z, k dependent.[5] Experience with the corresponding limits for the conversion coefficients[2] indicates that this limit may be somewhat less reliable in the magnetic case. As an additional guide in extrapolation we can use the fact that $W(\vartheta)\geqslant 0$ by definition. From this it is possible to conclude that for both electric and magnetic transitions

$$\begin{aligned} -2\leqslant b_2\leqslant 1;\quad & L=1 \\ 1\leqslant b_2\leqslant b_2(\alpha);\quad & L>1. \end{aligned} \quad (57a)$$

Compare Eqs. (46) and (48). The fact that the nonrelativistic limit (for electric multipole) is an upper ($L>1$) or a lower ($L=1$) bound is a more useful extrapolation aid for the electric than for the magnetic transitions.

The high energy limit can be obtained by using the Casimir approximation[6] (asymptotic forms of the radial Dirac functions are used). Then one finds immediately that $b_2(\pi)=1(\pi=e$ or $m)$ and thus all $b_\nu(\pi)=1$. This implies that at high energies the conversion electrons give the same correlation function as the corresponding cascade with a photon replacing the conversion electron. The fact that the two limits, nonrelativistic and high

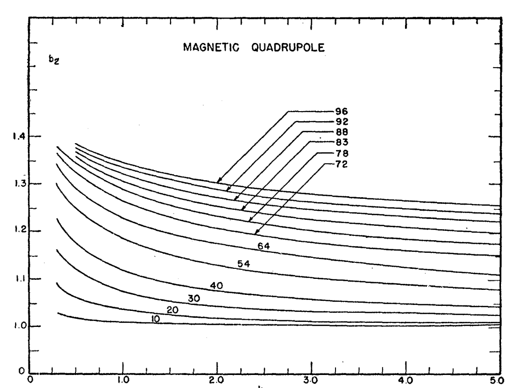

FIG. 4. Same as Fig. 1 for magnetic 2^2-pole transitions.

[26] According to M. Goldhaber and A. W. Sunyar, Phys. Rev. 83, 906 (1951), multipole mixtures should occur only for $L_2=1$ (i.e., magnetic-dipole plus electric quadrupole). If $L_1=1$ and/or $J=1$ or $3/2$, $\nu_m=2$ and only the coefficient A_2 enters so that (55a, b) give the complete correction factor to Lloyd's tables in this case.

[27] Arfken, Biedenharn, and Rose, Oak Ridge National Laboratory Report No. 1103 (1951).

[28] The numerical results from which the curves of Fig. 1–10 were obtained are given in Rose, Biedenharn, and Arfken, Oak Ridge National Laboratory Report No. 1097 (1951).

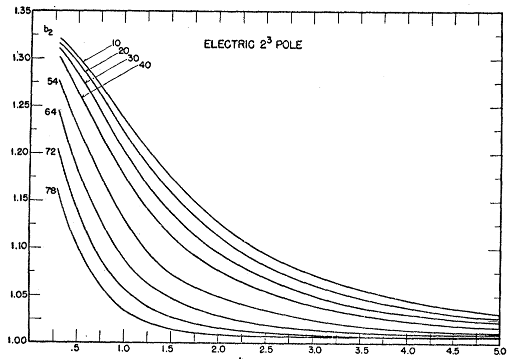

FIG. 5. Same as Fig. 1 for electric 2^3-pole transitions. Only lower Z values shown.

energy approach a common value with increasing L is clearly evident from the results which show a decreasing sensitivity, so far as energy dependence is concerned, with increasing L. The limit $Z=0$, for magnetic multipoles only, also gives a photon correlation; that is, $b_\nu(m)=1$ for all ν. The electric multipoles have the

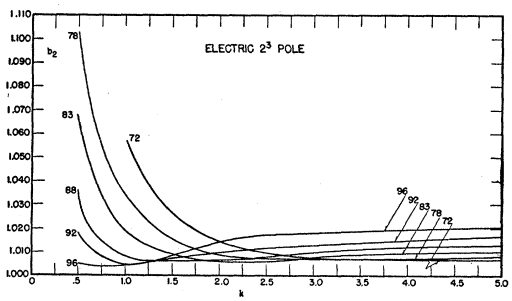

FIG. 5a. Same as Fig. 5 for the higher Z values and on an enlarged scale.

opposite Z-dependence; that is, they increase with decreasing Z (except for $L=1$ and $k\leqslant 1$) and for $L\geqslant 2$ the values of the b_2 coefficients as given lie between unity and the $Z=0$ limit:

$$b_2(Z=0)=1+\frac{12L}{[L(L+1)-3][(L+1)k^2+4L]}.$$

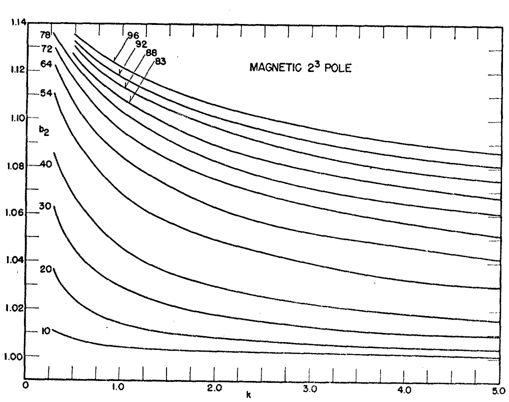

FIG. 6. Same as Fig. 1 for magnetic 2^3-pole transitions.

It will be noted that for cascades in which $\nu_m=2$ the sign of the anisotropy, $W(\vartheta)-1$, is reversed as compared to the γ-ray case for the low energy electric dipole conversion electrons but is the same for other cases.

The authors are indebted to Dr. C. L. Perry, Mr. Carl Perhacs, and Mrs. N. Dismuke of the Mathematics Panel, Oak Ridge National Laboratory, for performing the computations.

APPENDIX A. SIGN CONVENTIONS, PROPERTIES OF THE SPINORS

Throughout we use the Condon-Shortley[10] definition of the spherical harmonics Y_l^m. This differs by a phase

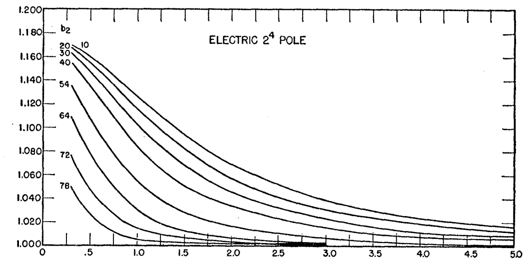

FIG. 7. Same as Fig. 5 for electric 2^4-pole transitions.

$(-)^m$ from the definition adopted in reference 15. We also use the well-known result that

$$Y_L{}^M Y_{L'}{}^{M'}=\sum_{\nu=|L-L'|}^{L+L'}[(2L+1)(2L'+1)/4\pi(2\nu+1)]^{\frac{1}{2}} \times C_{00}{}^{LL'\nu}C_{M-M'}{}^{LL'\nu}Y_\nu{}^{M+M'}. \quad (A.1)$$

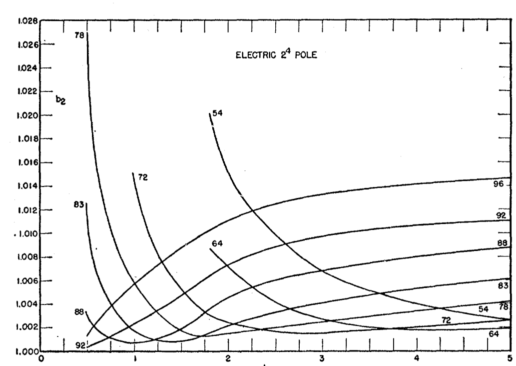

FIG. 7a. Same as Fig. 5a for electric 2^4-pole transitions.

The representation of the Dirac matrices is defined by (5) and $\boldsymbol{\alpha}=\rho_1\boldsymbol{\sigma}$ where

$$\boldsymbol{\sigma}=\begin{pmatrix}\mathbf{k} & \mathbf{i}-i\mathbf{j}\\ \mathbf{i}+i\mathbf{j} & -\mathbf{k}\end{pmatrix}$$

in terms of the cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. The Pauli spinors introduced in (10) are such that

$$\sigma_z\chi_{\frac{1}{2}}{}^{\tau}=2\tau\chi_{\frac{1}{2}}{}^{\tau}. \quad (A.2)$$

The discussion of the Dirac spinors (types (*a*) and (*b*)[15]) is unified by the introduction of κ the eigenvalue of K [Eq. (9)]:

$$j=|\kappa|-\tfrac{1}{2}, \tag{A.3}$$

$$l(\kappa)=\kappa, \quad \kappa>0$$
$$=|\kappa|-1, \quad \kappa<0 \tag{A.4}$$

so that κ, which is a nonvanishing integer, specifies both j and l. Then the Pauli central field spinors are

$$\chi_\kappa{}^\mu=\sum_\tau C_{\mu-\tau\ \tau}{}^{l(\kappa)\frac{1}{2}j}\chi_{\frac{1}{2}}{}^\tau Y_{l(\kappa)}{}^{\mu-\tau} \tag{A.5}$$

with the properties (capital letters in (A.6–10) designate operators)

$$\mathbf{J}^2\chi_\kappa{}^\mu=(|\kappa|-\tfrac{1}{2})(|\kappa|+\tfrac{1}{2})\chi_\kappa{}^\mu, \tag{A.6}$$

$$J_z\chi_\kappa{}^\mu=\mu\chi_\kappa{}^\mu, \tag{A.7}$$

$$\mathbf{L}^2\chi_\kappa{}^\mu=l(\kappa)(l(\kappa)+1)\chi_\kappa{}^\mu=\kappa(\kappa+1)\chi_\kappa{}^\mu, \tag{A.8}$$

$$(\boldsymbol{\sigma}\cdot\mathbf{L}+1)\chi_\kappa{}^\mu=-\kappa\chi_\kappa{}^\mu, \tag{A.9}$$

$$(\mathbf{r}/r)\cdot\boldsymbol{\sigma}\chi_\kappa{}^\mu\equiv\sigma_r\chi_\kappa{}^\mu=-\chi_{-\kappa}{}^\mu. \tag{A.10}$$

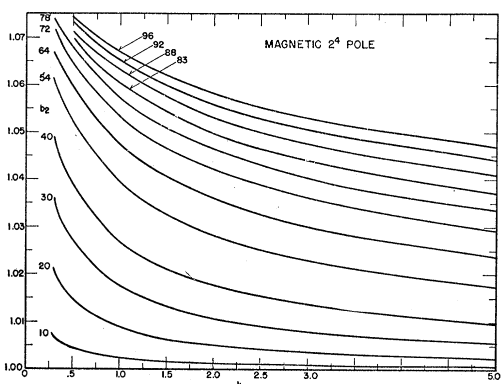

FIG. 8. Same as Fig. 1 for magnetic 2^4-pole transitions.

The relation between $\chi_\kappa{}^\mu$ and the $\Omega_{lj}{}^\mu$ introduced in Eq. (4_1) is

$$\chi_\kappa{}^\mu=(-)^{\mu-\frac{1}{2}}S(-\kappa)\Omega_{lj}{}^\mu,$$

where $S(x)\equiv$sign of x.

For the spherical cylinder functions (Eq. (7)),

$$\left(\frac{d}{dr}+\frac{1+\kappa}{r}\right)Z_{l(\kappa)}(pr)=pS(\kappa)Z_{l(-\kappa)}(pr). \tag{A.11}$$

APPENDIX B. SOME PROPERTIES OF THE VECTOR ADDITION AND RACAH COEFFICIENTS[16]

Repeated use is made in Sec. II and III of various symmetry relations for the vector addition and Racah coefficients. For convenience these are listed below. We consider the vector addition coefficient $C_{m_1m_2m_3}{}^{j_1j_2j_3}$. In the notation used we omit m_3 (the z-component of j_3) or, in general, the third subscript since in all cases it is the sum of the first two. Then,

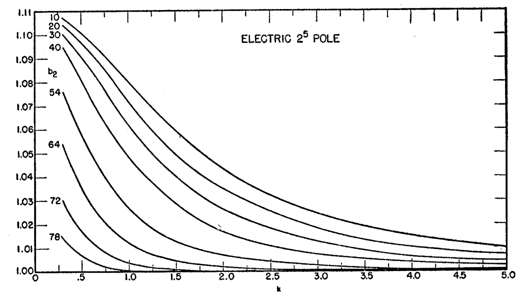

FIG. 9. Same as Fig. 5 for electric 2^5-pole transitions.

$$\begin{aligned}C_{m_1m_2}{}^{j_1j_2j_3}&=(-)^{j_1+j_2-j_3}C_{-m_1-m_2}{}^{j_1j_2j_3}\\&=(-)^{j_1+j_2-j_3}C_{m_2m_1}{}^{j_2j_1j_3}\\&=(-)^{j_1-m_1}[(2j_3+1)/(2j_2+1)]^{\frac{1}{2}}C_{m_1-m_3}{}^{j_1j_3}\\&=(-)^{j_2+m_2}[(2j_3+1)/(2j_1+1)]^{\frac{1}{2}}C_{-m_3m_2}{}^{j_3j_2j_1}\\&=(-)^{j_1-j_3+m_2}[(2j_3+1)/(2j_1+1)]^{\frac{1}{2}}C_{m_2-m_3}{}^{j_2j_3j_1}.\end{aligned} \tag{B.1}$$

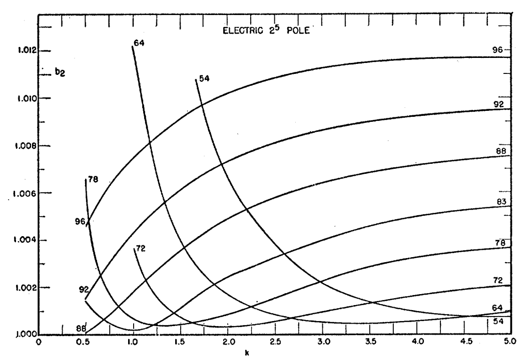

FIG. 9a. Same as Fig. 5a for electric 2^5-pole transitions.

The Racah coefficients obey the following symmetry relations:

$$\begin{aligned}W(abcd;ef)&=W(badc;ef)=W(cdab;ef)\\&=W(acbd;fe)=(-)^{e+f-a-d}W(ebcf;ad)\\&=(-)^{e+f-b-c}W(aefd;bc)\end{aligned} \tag{B.2}$$

and all the other relations which can be obtained by

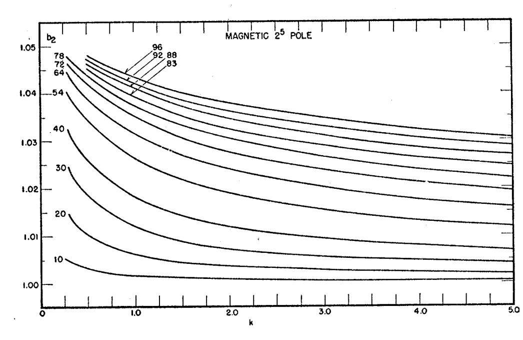

FIG. 10. Same as Fig. 1 for magnetic 2^5-pole transitions.

combining two or more of the operations indicated in (B.2). The Racah coefficients are introduced in the foregoing by the following:

$$C_{m_1m_2}{}^{j_1j_2j}C_{m_1+m_2m_3}{}^{jj_3j_4}$$
$$=\sum_J(2J+1)^{\frac{1}{2}}(2j+1)^{\frac{1}{2}}C_{m_2m_3}{}^{j_2j_3J}C_{m_1m_2+m_3}{}^{j_1Jj_4}$$
$$\times W(j_1j_2j_4j_3;jJ). \quad \text{(B.3)}$$

We also use

$$(2L+1)C_{00}{}^{LL\nu}W(LLL-\tfrac{1}{2}L-\tfrac{1}{2};\nu\tfrac{1}{2})$$
$$=(2L-1)C_0{}^{L-1}{}_0{}^{L-1\nu}$$
$$\times W(L-1\;L-1\;L-\tfrac{1}{2}\;L-\tfrac{1}{2};\nu\tfrac{1}{2}), \quad \text{(B.4)}$$

$$(2L+1)C_{00}{}^{LL\nu}W(LLL-\tfrac{1}{2}\;L+\tfrac{1}{2};\nu\tfrac{1}{2})$$
$$=[(2L-1)(2L+3)]^{\frac{1}{2}}C_0{}^{L-1}{}_0{}^{L+1\nu}$$
$$\times W(L-1\;L+1\;L-\tfrac{1}{2}\;L+\tfrac{1}{2};\nu\tfrac{1}{2}) \quad \text{(B.5)}$$

and

$$W^2(LLL-\tfrac{1}{2}\;L-\tfrac{1}{2};\nu\tfrac{1}{2})=\frac{(2L-\nu)(2L+\nu+1)}{4L^2(2L+1)^2}, \quad \text{(B.6)}$$

$$W^2(LLL+\tfrac{1}{2}\;L+\tfrac{1}{2};\nu\tfrac{1}{2})=\frac{(2L+1-\nu)(2L+2+\nu)}{4(L+1)^2(2L+1)^2}, \quad \text{(B.7)}$$

$$W^2(LLL+\tfrac{1}{2}\;L-\tfrac{1}{2};\nu\tfrac{1}{2})=\frac{\nu(\nu+1)}{4L(L+1)(2L+1)^2}. \quad \text{(B.8)}$$

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A Note on the Adiabatic Thermomagnetic Effects

HERBERT B. CALLEN
Randal Morgan Laboratory of Physics, University of Pennsylvania, Philadelphia, Pennsylvania

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The thermodynamic theory of the transverse adiabatic thermomagnetic effects is extended; and the final complete set of thermodynamic relations among all thermomagnetic effects is summarized.

1. INTRODUCTION

IN a previous paper[1] various thermomagnetic[2] effects were analyzed by the methods of irreversible thermodynamics, which provides a framework whereby any thermomagnetic coefficient, when once defined, may be expressed in terms of a set of six independent constants of the material, called the "kinetic coefficients." It was shown that, in addition to the Bridgeman relation, there exist further relations among the thermomagnetic coefficients. Because of the complexity of the formulas for the thermomagnetic coefficients in terms of the kinetic coefficients, these predicted relations could only be given implicitly in (I), and only the limiting forms for small fields were given explicitly. In a recent paper,[3] however, Mazur and Prigogine have shown that a particular inversion of the fundamental equations greatly simplifies the resultant formulas and allows the new relations to be obtained in explicit form. Moreover, the adiabatic Hall and Nernst effects were specifically excluded from the analysis in (I) because, as will be seen, the analysis of these effects requires an extension of the methods there employed. Here again Mazur and Prigogine[3] have considered the problem and have given a treatment of these effects, which is valid for metals. The general theory of these effects will be given, and the final complete set of thermodynamic relations among all thermomagnetic effects will be summarized.

I wish to acknowledge here my conversations with Dr. Mazur and Professor Prigogine, in which the clarification of these problems was jointly evolved.

2. THE DEFINITION OF THE ADIABATIC EFFECTS

For definiteness we shall first consider the adiabatic Hall effect. This effect concerns the appearance of a transverse potential gradient when a longitudinal electric current flows perpendicularly to a magnetic field. The appropriate boundary conditions are that no longitudinal temperature gradient nor any transverse currents of either heat or electricity may exist. The adiabatic Hall coefficient R_a is then

$$R_a=(\text{potential gradient})_y/H_z(eJ_x), \quad \text{(2.1)}$$

with the conditions that

$$dT/dX=Q_y=J_y=0. \quad \text{(2.2)}$$

Here H_z is the magnetic field, taken in the Z-direction; eJ_x is the longitudinal electric current (e being the electronic charge); eJ_y is the transverse electric current; Q_y is the transverse heat current, and T is the temperature.

Although Eqs. (2.1) and (2.2) seem at first to provide a quantitative definition of the adiabatic Hall coeffi-

[1] H. B. Callen, Phys. Rev. **73**, 1349 (1948), hereafter referred to as (I).

[2] We abandon here the rather cumbersome division of the effects into thermomagnetic and galvanomagnetic effects, and we simply use the first term to imply all such effects in a magnetic field.

[3] P. Mazur and I. Prigogine, J. phys. et radium **12**, 616 (1951).