state of the liquid where this density is larger than in any other state. But in order to obtain a clearer idea of the state of affairs at the approach of the critical state, we have to consider anew the radial distributoin in the space of relative momenta. Since $F_L^2(k, T)$ depends only on the magnitude $|\mathbf{k}|$, the distribution in k-space is isotropic. Hence, the radial distribution function in this space, that is the number of atoms per unit volume and per unit relative momentum range, at small momenta, is

$$\lim_{k\to 0}\frac{dg(k, T_c)}{dk}=\lim_{k\to 0}4\pi k^2[F_L^2(k, T_c)-1]$$
$$=\lim_{k\to 0}4\pi k^2F_L^2(k, T_c)=24\pi/r_F^2, \quad (19)$$

which is finite. But we have just shown, Eq. (14), that in all states $T<T_c$, this quantity vanishes strictly. We thus see that in the space of relative momenta the approach of the critical state, in the limit, is accompanied by an accumulation of atoms in the region of vanishing momenta. This accumulation of atoms in this vanishing momentum state is entirely similar to the Bose-Einstein condensation process in ordinary momentum space of ideal symmetric fluids. On the basis of the rigorous Eq. (19), valid only in the unique limiting critical state of normal fluids, it appears justified to describe this unique state as one of condensation in the space of relative momenta of the fluid atoms. This is the result which we set out to derive.

In concluding we note that the equivalence of these two condensation processes in the behavior of the macroscopic variables of state manifests itself, as pointed out previously,[3] in that the isothermals of these fluids reach the limiting state of condensation with a vanishing slope. Finally, it should be added that while the preceding result has been obtained in the simplest case of monatomic liquids, its extension to molecular liquids presents no difficulties, so that the condensation process in the space of relative momenta at the approach of the critical state is of general validity in liquids, subject only to the applicability of the correlation model.

PHYSICAL REVIEW VOLUME 85, NUMBER 1 JANUARY 1, 1952

Nuclear Magnetic Moment and $j-j$ Coupling Shell Model

MASATAKA MIZUSHIMA AND MINORU UMEZAWA
Physics Department, Faculty of Science, Tokyo University, Tokyo, Japan
(Received June 25, 1951)

It is shown that the magnetic moment of nuclei can be explained by a refined $j-j$ coupling shell model, where neutron and proton shells are treated simultaneously, using the isotopic spin variable. The experimental moments agree well with the calculated ones for those states which have definite isotopic spin multiplicity. It is shown that a nuclear force caused by a neutral or symmetric meson is consistent with our results, but one caused by a charged meson is excluded.

I. INTRODUCTION

THE spin-orbit coupling shell model proposed by Mayer[1] succeeded in explaining nuclear magnetic moments, β-decay and isomerism, etc.[2,3] In such a one-particle approximation, however, the nuclear magnetic moment must lie on the Schmidt line. But, as is well known, the values of nuclear magnetic moments deviate considerably to the inside region between two Schmidt lines, and this deviation has been considered to suggest that the angular momentum of the outer nucleon does not hold. It was first suggested that the nuclear state is a linear combination of two states with $l=j-\frac{1}{2}$ and $l=j+\frac{1}{2}$, which correspond to the two Schmidt limits respectively.[4] But it was soon pointed out that this is not plausible, since these two states have opposite parity and cannot combine. Next, the effect of exchange currents was examined in order to explain this deviation, but this effect was found to be not sufficiently large, and, moreover, the moment can deviate to the outside region as well as to the inside of the two Schmidt limits in that theory.[5]

On the other hand, the nuclear quadrupole moments of some nuclei are too large to be expected from a one-particle shell model, and some types of asymmetric drop model have been proposed to explain it.[6-8] Since the angular momentum of the outer nucleon is not rigorously constant in the asymmetric drop model, the model was also used to explain the deviation of the magnetic

[1] M. G. Mayer, Phys. Rev. **78**, 16 (1950).
[2] Umezawa, Nakamura, Ono, Yamaguchi, and Taketani, Prog. Theor. Phys. **6**, 408 (1951).
[3] E. Feenberg, Phys. Rev. **76**, 1275 (1949).
[4] L. W. Nordheim, Phys. Rev. **75**, 1894 (1949).
[5] R. G. Sachs, Phys. Rev. **74**, 433 (1948); R. K. Osborne and L. L. Foldy, Phys. Rev. **79**, 795 (1950); L. Spruch, Phys. Rev. **80**, 372 (1950).
[6] J. Rainwater, Phys. Rev. **79**, 432 (1950).
[7] A. Bohr, Phys. Rev. **81**, 134 (1951).
[8] L. L. Foldy and F. J. Milford, Phys. Rev. **80**, 751 (1950).

TABLE I. Irreducible representation of $\mathcal{T}(\tau_1, \tau_2, \tau_3)$ in $\mathfrak{S}_3$.

T	
3/2	χ_0
1/2	χ_2

moment from the Schmidt line.[7–9] But it was pointed out that the magnetic moments of B^{10} and N^{14} agree well with the calculated value, although they have a non-vanishing spin of 3 and 1, for which, according to the above model, the deviation from the calculated value is expected. This seems to be a contradiction to the asymmetric drop model.

On the other hand, some evidence for the $j-j$ coupling model has been obtained in the investigation of lower excited states of even-even nuclei. Horie, Yamaguchi, Yoshida, and one of the present authors[10] have shown that the ground, first, and some of the second excited states of even-even nuclei have a spin of 0, 2, and 4, respectively, except for the closed shell nuclei, and this fact can be explained easily by the $j-j$ coupling shell model, while the $L-S$ coupling model fails to explain this fact, since a lower odd spin state is also possible in the latter scheme.

In this paper we propose a refined $j-j$ coupling shell model, and calculate nuclear magnetic moments assuming some features about the hamiltonians of nuclei.[11]

II. REMARK ON THE MAGNETIC MOMENT OF NUCLEI WITH (CLOSED SHELL±ONE) NUCLEONS

It may be worth noting that by the above term, nuclei with (closed shell±one) nucleons, those nuclei are meant which have (closed shell±one) neutrons +(closed shell) protons or *vice versa*. If we neglect the contribution of the closed shells entirely, the theoretical values of the magnetic moment can be obtained straightforwardly.[12]

Comparison of theoretical and experimental values of magnetic moment for this kind of nuclei was made in Table I of our preliminary paper.[11] Among eleven nuclei cited there, disagreement is found only for B^{11}, Al^{27}, and P^{31}.

III. CALCULATION OF THE MAGNETIC MOMENT OF NUCLEI WITH (CLOSED SHELL±THREE) NUCLEONS

The following discussion can easily be extended to cases of more than three nucleons, but since they are relatively unimportant we shall limit our discussion to the case of three nucleons.

The proton and neutron are treated as a nucleon in different states, using the isotopic spin function τ. We assume that the total wave function of our nucleus can be written as

$$\Psi = \Phi(\text{inner})\mathcal{R}(r_1, r_2, r_3)\Theta(j_1, j_2, j_3)\mathcal{T}(\tau_1, \tau_2, \tau_3), \quad (1)$$

where Φ(inner) means the wave function of the nucleons in inner closed shell, $\mathcal{R}(r_1, r_2, r_3)$ is the radial wave function of the three nucleons in the outer shell, $\Theta(j_1, j_2, j_3)$ is their angular wave function in the sense of $j-j$ coupling shell model, and $\mathcal{T}(\tau_1, \tau_2, \tau_3)$ is their isotopic spin eigenfunction, which means an eigenfunction for a definite $|\mathbf{T}| = |\sum_i \boldsymbol{\tau}_i|$. The important assumption in (1) is that the angular part (including ordinary spin) and isotopic spin part can be separated from the other parts.

Since the individual isotopic spin τ is a dichotomic variable, the total spin function $\mathcal{T}(\tau_1, \tau_2, \tau_3)$, which is an irreducible representation of the rotation group in isotopic spin space, is also an irreducible representation of the permutation group of order three. The representation to which each isotopic spin state belongs is shown in Table I, where $T = |\mathbf{T}| = |\sum_i \boldsymbol{\tau}_i|$ and χ_0, χ_1, and χ_2 are the totally symmetric, antisymmetric, and two-dimensional representations of the permutation group $\mathfrak{S}_3$, respectively.

The angular part $\Theta(j_1, j_2, j_3)$ is an irreducible representation of the rotation group. The reduction of the product representation in this group can be done in the usual way; but since the spin function thus reduced is, in general, still a reducible representation of the permutation group, we must further reduce it in the latter group. The results are shown in Table II, where only the results for the case of

$$I = |\sum_i^3 \mathbf{j}_i| = j$$

with $j = 3/2$ and $5/2$ are shown, since they are the only practically important cases.[13]

The total wave function Ψ must be a representation χ_1, according to the well known Pauli principle. Thus, if T is fixed, the symmetric properties of $\Theta(j_1, j_2, j_3)$ are restricted. For a charge quartet state $(T=3/2)$ $\Theta(j_1, j_2, j_3)$ must belong to the representation χ_1, and for a charge doublet state $(T=1/2)$, it must belong to χ_2, since $\chi_1 \times \chi_0 = \chi_1$, $\chi_2 \times \chi_2 = \chi_0 + \chi_1 + \chi_2$, and no other representation can make Ψ antisymmetric. We can see from Table II that, for Li^7, Be^9, and Cl^{35}, whose configurations are $(2p_{3/2})^3$, $(2p_{3/2})^{-3}$, and $(3d_{3/2})^3$, respectively, and $I = 3/2$, the angular part of the wave func-

TABLE II. Irreducible representations of $\Theta(j_1, j_2, j_3)$ in $\mathfrak{S}_3$.

$I(=\lvert\sum_i j_i\rvert=j)$	
3/2	$\chi_0+\chi_1+\chi_2$
5/2	$\chi_0+\chi_1+2\chi_2$

[9] H. Kopfermann, Naturwiss. 38, 29 (1951).
[10] Horie, Umezawa, Yamaguchi, and Yoshida, Prog. Theor. Phys. 6, 254 (1951).
[11] A preliminary paper was published by M. Mizushima and M. Umezawa, Phys. Rev. 83, 463 (1951).
[12] L. Rosenfeld, *Nuclear Forces* (North-Holland Publishing Company, Amsterdam, 1949), Sec. II, p. 402.
[13] For other cases see, for example, M. Mizushima and T. Ito, J. Chem. Phys. 19, 739 (1951).

tion is uniquely determined for each charge multiplet state, and for Mg^{25}, whose configuration is $(3d_{5/2})^{-3}$ and spin is 5/2, there are two independent possible angular wave functions.

The magnetic moment can be easily calculated if we know $\Theta(j_1, j_2, j_3)$ and $\mathcal{T}(\tau_1, \tau_2, \tau_3)$ explicitly, by calculating the average of

$$M_z=\sum_i\{g_p(\tau_{\zeta i}-1)+g_n(\tau_{\zeta i}+1)\}j_{zi}/2$$

for $I_z=I$ state. The results are shown in Table III. In this table we see that for Li^7 and Be^9, the calculated moment for isotopic spin doublet state agrees well with experiment, while in the case of Cl^{35}, the value corresponding to the isotopic spin quartet state agrees with experiment. In the case of Mg^{25}, since there are two independent states for $I=5/2$, $T=1/2$, the calculated magnetic moment can take any value between the two limits, unless some further specification on the hamiltonian is made. It is interesting that the experimental result agrees with the lower limit for this state.

TABLE III. Magnetic moments of nuclei with (closed shell±3) nucleons.

	I	Configuration	$T\zeta$	T	μ_{calc}	μ_{exp}[a]	μ_S[b]
Li^7	3/2	$(2p_{3/2})^3$	1/2	1/2	3.07	3.26	3.79
				3/2	0		
Be^9	3/2	$(2p_{3/2})^{-3}$	1/2	1/2	−1.14	−1.18	−1.91
				3/2	1.9		
Cl^{35}	3/2	$(3d_{3/2})^3$	1/2	1/2	−0.48	0.82	0.12
				3/2	0.80		
Mg^{25}	5/2	$(3d_{5/2})^{-3}$	1/2	1/2	−1.06	−0.96	−1.91
					to 2.76		
				3/2	2.57		

[a] J. E. Mack, Revs. Modern Phys. 22, 64 (1950).
[b] Value in the Schmidt limit.

IV. DISCUSSION

Our success with the nuclei of (closed shell±one) nucleons shows that the calculation of magnetic moments on the Mayer shell model is fairly good. At this point it must be noted that our interpretation of (closed shell±one) nucleons is slightly different from hers. Mayer treated the neutron shell and the proton shell separately. Thus, for example, a nucleus with (closed shell±two) protons+(closed shell±one) neutrons is treated in the same way as a nucleus with (closed shell) proton+(closed shell±one) neutrons in Mayer's scheme. But from our standpoint, they are entirely different. The former is a nucleus with (closed shell ±three) nucleons, and is treated in Sec. III.

The good agreement between our result and experiment in the case of (closed shell±three) nucleons shows that the ground states of these nuclei have a definite isotopic spin multiplicity.

In order to be able to assign definite isotopic spin multiplicity to each stationary state, the hamiltonian H of our nuclei must satisfy a commutation relation $[|\mathbf{T}|^2, H]=0$. There are three typical kinds of charge dependence of the internuclear force in the meson theory: the forces corresponding to neutral, symmetrical, and charged mesons. In the neutral meson theory H does not contain any factor which depends on isotopic spin; thus it commutes with $|\mathbf{T}|^2$. In the symmetrical meson theory, the internuclear potential can be expressed as $f(r_{12})\boldsymbol{\tau}_1\boldsymbol{\tau}_2$ where $f(r_{12})$ is a function which depends on relative coordinate, ordinary spin, etc. By a straightforward calculation one can show that $[|\mathbf{T}|^2, \sum f(r_{ij})\boldsymbol{\tau}_i\boldsymbol{\tau}_j]=0$; thus, in this case also, each stationary state has a definite isotopic spin multiplicity. But in the charged meson theory, in which $H=\sum f(r_{ij})(\tau_{\xi i}\tau_{\xi j}+\tau_{\eta i}\tau_{\eta j})$, $|\mathbf{T}|^2$ does not commute with H. Our result that each stationary state of nuclei has a definite isotopic spin multiplicity is consistent with neutral and symmetrical meson theory, but contradicts charged meson theory.

It should be remembered that the coulomb potential $H_c=\frac{1}{4}(e^2/r_{ij})(1-\tau_{\zeta i})(1-\tau_{\zeta j})$ does not commute with $|\mathbf{T}|^2$; but this effect is usually not very large, and can be treated as a small perturbation.

That the ground states of these nuclei, except Cl^{35}, are charge doublet states is consistent with the fact that the ground states of their neighboring isobars, in which only the charge quartet state can appear, have much higher energy. Also it can be seen that the theoretical estimate of the energy by means of our wave function yields the same conclusion if an attractive internuclear potential is used. The latter conclusion is valid for both the neutral and the symmetrical meson theories. The details of the energy calculation will be published in another paper.

The reason that the ground state of Cl^{35} is charge quartet is not clear.

Our result for Mg^{25} seems to indicate that the magnetic moment can be diagonalized with H in a space diagonal in I and T. The operator of the z component of the magnetic moment is

$$M_z=\{(g_n+g_p)I_z+(g_n-g_p)\sum\tau_{\zeta i}j_{zi}\}/2$$

since the g-factors are common to nucleons in the same shell. The first term being always commutable with H, we have only to examine the commutability of the second term with H. If the range of the internuclear potential is very small compared to the dimensions of the nucleus, we can approximate it by the delta-function $\delta(r_{ij})$. In this case we can show without much difficulty that the H corresponding to the neutral and symmetrical theories commutes with M_z in our limited space, by using the commutability of $\delta(r_{ij})$ and j_{zi}. If the range of the internuclear potential is larger than the dimensions of the nucleus, we can expand the space-dependent factor of the internuclear potential as $\varphi(r_i)\varphi(r_j)+\varphi'(r_i)\varphi'(r_j)\cos\theta_{ij}+\cdots$, where $\varphi(r_i)$ is a function which depends only on the radial coordinate of the ith nucleon. Taking the average over r, we can neglect the higher terms in the above expansion under the above assumption. Thus our hamiltonian reduces to $a\sum_{i>j}1$ and $a\sum_{i>j}\boldsymbol{\tau}_i\boldsymbol{\tau}_j$ for the neutral and the symmetrical theories, respectively, confirming the commutability of H and $|\mathbf{T}|^2$.[14] Since the range of the internuclear potential is 1.5×10^{-13} cm, and the radius of Mg^{25} is 7×10^{-13} cm, the present situation seems to favor the former approximation.

It should be mentioned that the success in the case of odd-odd nuclei by Feenberg[3] is also preserved as such in our theory.

We wish to thank Professor Muneo Sasaki and Dr. Hisashi Horie for their continued help and encouragement throughout the work. We also owe thanks to Professors S. Nakamura, T. Yamanouchi, and M. Kotani, and Drs. S. Tani and H. Narumi for their discussion.

APPENDIX

The spin function which is a representation of both rotational and permutation groups can be constructed as follows:

We start from an invariant formula,

$$I=\prod_{i>j}(u_1^{(i)}u_2^{(j)}-u_2^{(i)}u_1^{(j)})^{\sigma_{ij}}\times\prod_k(u_1^{(k)}x_1+u_2^{(k)}x_2)^{2s-g^k},\quad\text{(A1)}$$

where $i,\ j,\ k=1,\ 2,\ \cdots\ \nu$, $\sum_{i>j}\sigma_{ij}=\nu s-S$ with individual spin s and total spin S, and the g_k's are so chosen as to make I a function of order $2s$ for every u_1 and u_2. Since, in general, there are n sets of $(\sigma_{ij},\ g_k)$ which satisfy the above condition, we obtain n formulas $I_1, I_2, \cdots I_n$. The representation $D(R)$ of an operator of the permutation group $\mathfrak{S}_\nu$ is then calculated for all R, taking $I_1, \cdots I_n$ as the basis. If $\mathfrak{D}_l(R)$ is an irreducible representation χ_l of the operator R and α is an arbitrary vector of order $n\times m$, where m is the order of $\mathfrak{D}_l(R)$, we can obtain m linear combinations of I_p as:

$$(I_1,\cdots,I_n)(\textstyle\sum_R D(R)\alpha\mathfrak{D}_l(R)^{-1})=(\sum_p a_p^{(1)}I_p,\ \cdots,\ \sum_p a_p^{(m)}I_p).\quad\text{(A2)}$$

Then each linear combination of I_p is expanded, and the coefficient of $x_1^{S+M}x_2^{S-M}$ is calculated as $\Theta_{S,M}^{(1)}$, $\Theta_{S,M}^{(2)}, \cdots, \Theta_{S,M}^{(m)}$. This set of $\Theta_{S,M}$ is the required spin function which is a representation of the permutation group $\mathfrak{S}_\nu$, if we put the individual spin function $U_\lambda^{(i)}$ as

$$U_\lambda^{(i)}=(u_1^{(i)})^{s+\lambda}(u_2^{(i)})^{s-\lambda}\{(s+\lambda)!(s-\lambda)!\}^{-1/2},\quad\text{(A3)}$$

where $\lambda=s_z$.

Some of the explicit spin functions thus obtained are as follows: For $\nu=3$, $s=3/2$, $S=3/2$, and the χ_2 representation,

$$\{-(3/2,-3/2,3/2)-(-3/2,3/2,3/2)+(3/2,-1/2,3/2)+(-1/2,3/2,3/2)-(3/2,1/2,-1/2)-(1/2,3/2,-1/2)+2(3/2,3/2,-3/2)\}/10^{1/2},$$

and

$$\{3(3/2,-3/2,3/2)-2(1/2,-1/2,3/2)+2(-1/2,1/2,3/2)-3(-3/2,3/2,3/2)-(3/2,-1/2,1/2)+(-1/2,3/2,1/2)+(3/2,1/2,-1/2)-(1/2,3/2,-1/2)\}/30^{1/2}.$$

For $\nu=3$, $s=5/2$, $S=5/2$ and the χ_2 representation,

$$\{-18(5/2,-5/2,5/2)+10(3/2,-3/2,5/2)-7(1/2,-1/2,5/2)+7(-1/2,1/2,5/2)-10(-3/2,3/2,5/2)+18(-5/2,5/2,5/2)+8(5/2,-3/2,3/2)-12\cdot10^{-1/2}(3/2,-1/2,3/2)+12\cdot10^{-1/2}(-1/2,3/2,3/2)-8(-3/2,5/2,3/2)-5(5/2,-1/2,1/2)+9\cdot5^{-1/2}(3/2,1/2,1/2)-9\cdot5^{-1/2}(1/2,3/2,1/2)+5(-1/2,5/2,1/2)+2(5/2,1/2,-1/2)-2(1/2,5/2,-1/2)-2(5/2,3/2,-3/2)+2(3/2,5/2,-3/2)\}\times(2\cdot3\cdot7\cdot11\cdot13/5)^{-1/2},$$

and

$$\{-6(5/2,-5/2,5/2)+2(3/2,-3/2,5/2)-(1/2,-1/2,5/2)-(-1/2,1/2,5/2)+2(-3/2,3/2,5/2)-6(-5/2,5/2,5/2)+4(5/2,-3/2,3/2)-2(2/5)^{1/2}(3/2,-1/2,3/2)+6\cdot5^{-1/2}(1/2,1/2,3/2)-2(2/5)^{1/2}(-1/2,3/2,3/2)+4(-3/2,5/2,3/2)-3(5/2,-1/2,1/2)-3\cdot5^{-1/2}(3/2,1/2,1/2)-3\cdot5^{-1/2}(1/2,3/2,1/2)-3(-1/2,5/2,1/2)+4(5/2,1/2,-1/2)+4(2/5)^{1/2}(3/2,3/2,-1/2)+4(1/2,5/2,-1/2)-6(5/2,3/2,-3/2)-6(3/2,5/2,-3/2)+12(5/2,5/2,-5/2)\}(2\cdot7\cdot11\cdot13/5)^{-1/2}.$$

The other functions needed in the present calculation can be easily obtained. In the above formulas $(\lambda_1,\lambda_2,\lambda_3)\equiv U_{\lambda_1}^{(1)}U_{\lambda_2}^{(2)}U_{\lambda_3}^{(3)}$.

Note added in proof.—Ross obtained the same results as are given in Table III.[15]

[14] It may be worth noting that in these cases M_z commutes with H in the whole space, in the neutral meson theory.

[15] E. P. Wigner (private communication).