A Collective Description of Electron Interactions: II. Collective *vs* Individual Particle Aspects of the Interactions

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The behavior of the electrons in a dense electron gas is analyzed in terms of their density fluctuations. These density fluctuations may be split into two components. One component is associated with the organized oscillation of the system as a whole, the so-called "plasma" oscillation. The other is associated with the random thermal motion of the individual electrons and shows no collective behavior. It represents a collection of individual electrons surrounded by comoving clouds of charge which screen the electron fields within a distance of the order of magnitude of the Debye length. This split up of the density fluctuations corresponds to an effective separation of the Coulomb interaction into long-range and short-range parts; the separation occurs at roughly the Debye length.

The relation between the individual and collective aspects of the electron gas is discussed in detail, and a general physical picture of the behavior of the system is given. It is shown that for phenomena involving distances greater than the Debye length, the system behaves collectively; for distances shorter than this length, it may be treated as a collection of approximately free individual particles, whose interactions may be described in terms of two-body collisions.

This approach is used to study the interaction of a specified electron with the remainder of the electron gas. It is shown that the collective part of the response of this remainder to the field of the specified particle screens this field within a distance of the order of the Debye length; this furnishes a detailed description of the screening process. Moreover, if the specified particle moves with greater than the mean thermal speed, it excites collective oscillations in the form of a wake trailing the particle. The frequency of these collective oscillations and the energy emitted by the particle are calculated. A correspondence theoretical method is used to treat this phenomenon for the electrons in a metal. The results are in good agreement with the experiments of Ruthemann and Lang on the energy loss of kilovolt electrons in this metallic films.

The generalization of these methods to an arbitrary interparticle force is carried out, and a criterion is obtained for the validity of a collective description of the particle interactions. It is shown that strong forces and high particle density favor collective behavior, while high random thermal velocities oppose it.

I. INTRODUCTION

IN this paper we wish to develop a detailed physical picture of the behavior of the electrons in a dense electron gas. We do this with the aid of a collective description of the particle motion. In a previous paper,[1] hereafter referred to as I, we used a collective description in treating the organized behavior of the electrons resulting from the transverse electromagnetic interactions. This was done by means of a canonical transformation to a set of collective coordinates which were appropriate for a description of the organized behavior. In the present paper, we are concerned with the organization produced by the Coulomb interactions, which are far more important quantitatively than the transverse electromagnetic interactions. We stress the physical picture of the electron behavior here, because it is essential for the proper development and understanding of the necessary mathematical formulation. In a subsequent paper we shall extend our results to the quantum theory by developing the canonical transformation for the Coulomb case in a manner similar to that given in I.

In a dense electron gas, the particles interact strongly because of the long range of the Coulomb force; in fact, each particle interacts simultaneously with all the other particles. As a result the equations of motion become extremely difficult to solve. The usual perturbation theory solution based on the assumption of a small interaction between pairs of particles breaks down. As was the case in I, a collective description provides a far better starting point for a solution than a description in terms of the individual particles. For the collective description makes possible a simple method of treating the simultaneous interaction of many electrons (as opposed to the individual particles approach which gives a simple method of treating two-body collisions).

Certain examples of collective behavior in an electron gas are well known from the study of gaseous discharges. These are the organized oscillations of the system as a whole, the "plasma" oscillations.[2,3] These oscillations have been studied theoretically with the simplifying assumption that the gas is composed of a distribution of beams of charge, each beam having a well-defined velocity at each point in space.[4,5] This approach, although it gives many useful and instructive results concerning the oscillations, represents an excessive abstraction which is not capable of describing many other important aspects of the organized behavior in the gas.

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[1] D. Bohm and D. Pines, Phys. Rev. **82**, 625 (1951).

[2] L. Tonks and I. Langmuir, Phys. Rev. **33**, 195 (1929).

[3] H. J. Merrill and H. W. Webb, Phys. Rev. **55**, 1191 (1939).

[4] A. Vlasov, J. Phys. (U.S.S.R.) **9**, 25, 130 (1945).

[5] D. Bohm and E. P. Gross, Phys. Rev. **75**, 1851 and 1864 (1949); Paper A discusses the origin of medium-like behavior, and gives many references to the earlier work on plasma oscillations; Paper B deals with the excitation and damping of oscillations.

In this paper we start with a gas of interacting point electrons. Instead of following the motion of the individual particles, we describe the gas in terms of the Fourier components of the electron density at each point in space. These Fourier components are proportional to the density fluctuations in the electron gas. We find that the density fluctuations can be split into two parts. One part represents an organized oscillation with the characteristic "plasma" frequency, and is clearly associated with the collective behavior of the system. The other part is associated with the random thermal motion of the individual particles, and shows no collective behavior. For wavelengths greater than a certain critical length λ_D (the Debye length), the fluctuations are primarily collective. When this is the case a description of the behavior of the electrons solely in terms of their collective motion is a good approximation. For wavelengths smaller than λ_D, however, the fluctuations are primarily associated with individual particle motion, and in this case the system is best described by following the individual particles.

We use the above split-up of the density fluctuations to study the collective response of the electron gas to the field of an individual charged particle moving with a specified velocity, $\mathbf{v}_0$. When $\mathbf{v}_0$ is less than the mean thermal speed of the gas, we find that the collective response is just such as to screen out the field of the specified particle within a distance of order of λ_D. When $\mathbf{v}_0$ is greater than the mean thermal speed, a similar screening occurs, but in addition a new phenomenon appears: *viz.*, the excitation of collective oscillations in the form of a wake trailing the particle. The phenomenon resembles the Čerenkov radiation produced by fast electrons in dielectric materials. Experiments by Ruthemann and Lang, on the bombardment of thin metallic films by fast electrons tend to verify our theoretical predictions concerning this type of excitation of collective oscillations.[6,7]

On the basis of the above results we are led to the following physical picture of the screening process. As any electron moves through the assembly, the other electrons are pushed away from it by the Coulomb repulsion. Each particle is thus surrounded by a cloud of extent λ_D, in which there is a deficiency of electrons, which is responsible for screening the field of the particle in question. As a result of this screening, the cross section for interparticle collision is so greatly reduced that the mean-free path of an electron is considerably greater than the interparticle spacing. Thus for many purposes the electron plus its associated cloud may be regarded as an effective free particle.

The splitup of the density fluctuation into collective and individual particles components may be viewed in the following way. The collective part includes the effects of the long range of the Coulomb force which leads to the simultaneous interaction of many particles. The individual particles component represents the density fluctuations arising from the randomly moving individual particles plus their comoving electron clouds, and thus includes the effects of the residual short-range screened Coulomb force, which leads only to two-body collisions.

Our main conclusion is that neither the collective description nor the individual particles description of the electron gas is by itself entirely adequate. For not only in each description needed in its appropriate region, but also the interaction between collective and individual aspects determines many important properties of the system. It is just this synthesis of individual and collective aspects that makes the electron gas such an exceptionally interesting medium.

There are a wide variety of systems to which the methods developed in this paper may be applied. Some of these are: (a) Ion gases of high density (plasmas), which occur in gaseous discharges, interstellar nebulae, atmosphere of sun and stars, ionosphere of earth, etc. (b) The electrons and ions in a metal. The ions in a metal are also susceptible to a collective description, and, in interaction with electrons, they give rise to sound waves, whose properties can be calculated with the collective method. In this way, one can obtain an improved treatment of the so-called "lattice-electron" interaction, which is important in the theory of electrical conductivity, and probably in superconductivity.[8]

In Appendix II, we discuss the generalization of these methods to an arbitrary interparticle force. Preliminary considerations indicate that the collective description may be applicable to the particles in an atomic nucleus.[9]

II. COLLECTIVE OSCILLATION

We begin by a study of the way in which the interactions in an assembly of electrons bring about organized behavior and collective oscillation. We shall consider an aggregate of approximately free electrons embedded in a medium of fixed positive charges whose average density is equal to that of the electrons. For most purposes, this distribution of charge can be regarded as uniformly smeared out throughout the entire system. Hence, it merely serves to neutralize the mean electron charge. The previous simplifications are adequate for the treatment of electron behavior in gaseous discharges, as well as in those metals in which the effects of the lattice periodicity are not very important. The approximations used here will also provide a good starting point for the investigation of other metals.

Each electron in the assembly is acted on by the sum of the forces arising from all of the other electrons plus

[6] G. Ruthemann, Ann phys. **2**, 113 (1948).
[7] W. Lang, Optik **3**, 233 (1948).

[8] This problem is now under investigation by one of us (D.B.) and T. Staver. Preliminary results are given in Phys. Rev. **84**, 836 (1951).
[9] One of us (D.P.) and M. I. Ferentz are currently investigating a nuclear model based on a collective description of nucleon interactions.

that resulting from the smeared-out positive charge. The potential energy of interaction between the ith and jth electrons, $e^2/|\mathbf{x}_i-\mathbf{x}_j|$, may be expanded as a Fourier series in a box of unit volume with periodic boundary conditions, and is

$$(e^2/|\mathbf{x}_i-\mathbf{x}_j|)=4\pi e^2\sum_k(1/k^2)e^{i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}. \quad (1)$$

The equation of motion of the ith electron is given by

$$\ddot{\mathbf{x}}_i=-(4\pi e^2 i/m)\sum'_{j,k}(\mathbf{k}/k^2)e^{i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}, \quad (2)$$

where the prime denotes a sum in which $\mathbf{k}=0$ is excluded.[10] (In the previous sum, the term with $j=i$ should also be excluded. However, since $1/k^2$ is a spherically symmetric function of $\mathbf{k}$, this term averages out to zero and therefore does not have to be explicitly excluded.)

Equation (2) is in general extremely difficult to solve by following the motions of the individual particles, especially when the assembly is so dense that many-body collisions become important. Moreover, the range of the Coulomb potential is so great that many-body collisions are important even in an electron gas of low density. Under these conditions, the electrons move together in organized fashion, and one finds the well-known phenomenon of "plasma" oscillations of the system as a whole.[2] In these cases the usual method of approximation in solving (2) which involves the assumption that the interaction forces produce small perturbations on uniform straight line particle motion, fails. Our approach to the equations of motion is aimed at making use of the simplicity of the collective behavior as a starting point for a tractable solution. We shall see that this method leads to a far better description of the motion than one based solely on following the individual particles.

We shall be primarily concerned with two questions: the description of the motion of the particle assembly in terms of its collective behavior, and the limits on the applicability of such a collective description. As a first step, we study the fluctuations in the particle density, because, as we shall see, their behavior provides a good measure of the applicability of a collective description. We assume, for all practical purposes, that we are dealing with point particles, so that the particle density in our box of unit volume is given by,

$$\rho(\mathbf{x})=\sum_i\delta(\mathbf{x}-\mathbf{x}_i). \quad (3)$$

We shall find it more convenient to work with the Fourier components, ρ_k, of the density. These are given by,

$$\rho_k=\int d\mathbf{x}\rho(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}=\sum_i e^{-i\mathbf{k}\cdot\mathbf{x}_i}, \quad (4)$$

and

$$\rho(\mathbf{x})=\sum_{i,k}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}_i)}. \quad (5)$$

We note that ρ_0 represents the mean electron density, n, and the ρ_k with $k\neq 0$ describe fluctuations about that mean density. It is readily verified that the equations of motion (2), may be re-expressed as

$$\ddot{x}_i=-(4\pi e^2 i/m)\sum'_k(\mathbf{k}/k^2)\rho_k e^{i\mathbf{k}\cdot\mathbf{x}_i}. \quad (6)$$

The ρ_k thus determine the force acting on each particle.

We now obtain the equations describing the time behavior of the ρ_k. On differentiating (4), we have

$$\dot{\rho}_k=-i\sum_i(\mathbf{k}\cdot\mathbf{v}_i)e^{-i\mathbf{k}\cdot\mathbf{x}_i}, \quad (7a)$$

$$d^2\rho_k/dt^2=-\sum_i[(\mathbf{k}\cdot\mathbf{v}_i)^2+i\mathbf{k}\cdot\dot{\mathbf{v}}_i]e^{-i\mathbf{k}\cdot\mathbf{x}_i}. \quad (7b)$$

We obtain $\dot{\mathbf{v}}_i$ from the equations of motion (2), and $d^2\rho_k/dt^2$ becomes

$$d^2\rho_k/dt^2=-\sum_i(\mathbf{k}\cdot\mathbf{v}_i)^2e^{-i\mathbf{k}\cdot\mathbf{x}_i}-\sum_{\substack{k'ij\\k'\neq 0}}[4\pi e^2/m(k')^2]\mathbf{k}\cdot\mathbf{k}'\{\exp[i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}_i]\}\exp(-i\mathbf{k}'\cdot\mathbf{x}_j). \quad (8)$$

It is in the treatment of the second term on the right-hand side of the previous equation that we encounter one of the central approximations used in the development of a collective description. For we split the sum over $\mathbf{k}'$ into two parts. The first part, with $\mathbf{k}'=\mathbf{k}$, is independent of the coordinate, $\mathbf{x}_i$, so that the sum over i yields n, the total number of particles.[11] The second part, (those terms with $\mathbf{k}'\neq\mathbf{k}$) contains phase factors $\exp i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}_i$, which depend on the positions of the particles. These terms tend to average out to zero, since there are a very large number of particles distributed very nearly in random positions. As a first approximation, we neglect such terms. This procedure we call the random phase approximation. The validity of this approximation will be demonstrated in detail in Sec. VI. Using the random phase approximation, we then obtain

$$d^2\rho_k/dt^2=-\sum_i(\mathbf{k}\cdot\mathbf{v}_i)^2e^{-i\mathbf{k}\cdot\mathbf{x}_i}-(4\pi ne^2/m)\sum_i e^{-i\mathbf{k}\cdot\mathbf{x}_i}. \quad (9)$$

The first term on the right-hand side of (9) is one that would be present even in the absence of particle interaction, and arises simply from the random thermal motion of the individual particles. The second term represents the effects of particle interactions. For sufficiently small $\mathbf{k}$ it is clear that the first term can be neglected in comparison with the second. Under these conditions, (9) becomes

$$d^2\rho_k/dt^2+(4\pi ne^2/m)\rho_k=0. \quad (10)$$

[10] The exclusion of the term with $k=0$ takes into account the uniform background of positive charge, and hence the over-all charge neutrality of the system. This may readily be seen by Fourier-analyzing the electron charge density and noting that Fourier-component corresponding to $k=0$ describes the mean density of electrons which is canceled by the equal density of uniformly distributed positive charge. The Fourier-expansion for the potential at a given point there fore does not contain the term with $k=0$.

[11] Since we are working with a unit volume, n is numerically equal to the mean density.

Thus, as a result of the Coulomb interaction, the electron density oscillates with the well-known "plasma" frequency,

$$\omega_P=(4\pi ne^2/m)^{\frac{1}{2}}. \tag{11}$$

The excitation of a particular ρ_k corresponds to a wave-like density fluctuation, analogous to a sound wave.

Let us now consider the physical significance of the terms which determine the time variation of the ρ_k. In the absence of interaction, each particle moves in a straight line with a constant velocity, $\mathbf{v}_{0i}$. In this case $\mathbf{x}_i=\mathbf{x}_{0i}+\mathbf{v}_{0i}t$, and ρ_k takes the form

$$\rho_k{}^0=\sum_i e^{i\mathbf{k}\cdot(\mathbf{x}_{0i}+\mathbf{v}_{0i}t)}. \tag{12}$$

In order that $\rho_k{}^0$ be large at a particular time, say $t=0$, it is necessary that the $\mathbf{x}_{0i}$ be so distributed that the various terms $e^{i\mathbf{k}\cdot\mathbf{x}_{0i}}$ arising from each particle tend to be in phase. However, each of these terms oscillates with an angular frequency $\mathbf{k}\cdot\mathbf{v}_{0i}$, which is, in general, different from that of any other. Consequently, even if they initially have definite phase relations, the contribution of different particles to ρ_k will soon tend to get out of phase and cancel each other. This means that in a gas of free particles a particular fluctuation cannot persist for an appreciable length of time. The tendency of the random motion of each particle to contribute a term to the density variation having an angular frequency, $\omega=\mathbf{k}\cdot\mathbf{v}_i$, is reflected in the first term on the right-hand side of (9), which by itself would result in a similar type of time variation of the ρ_k. We conclude then that a collection of free particles shows no organized behavior, but that instead its characteristic property is that a disturbance tends to die out as a result of the random diffusion of the particles. On the other hand, the effect of the Coulomb force of interaction, appearing in the second term on the right-hand side of (9), is to cause each particle to make a contribution to $d^2\rho_k/dt^2$ which oscillates with the same angular frequency as that of every other particle. Thus, if the random thermal motions were not present the Coulomb forces would produce perfectly organized behavior of the ρ_k. Actually, of course, both the Coulomb forces and the random thermal motions are present simultaneously, so that the net behavior of the electron gas will show some collective aspects and some of the aspects of an aggregate of randomly moving individual particles. To the extent that the effects of the Coulomb forces predominate in determining $d^2\rho/dt^2$, the gas will display primarily its collective aspect because the ρ_k determine the force on each particle and therefore the behavior of the entire system. On the other hand, when the random individual particle motions predominate in determining $d^2\rho_k/dt^2$, the collective description will be inappropriate, and a description of the system in terms of the motions of the individual particles becomes a better starting point. A rough criterion for the applicability of a collective description is therefore that for most particles the collective term in (9) be much greater than the term arising from the random thermal individual particle motions, or that

$$(4\pi ne^2/m)\gg\langle(\mathbf{k}\cdot\mathbf{v}_i)^2\rangle_{Av}. \tag{13}$$

Thus we see that a high particle density favors organized oscillation, while high random velocities oppose it. A strong force of interaction (as measured by the e^2 factor) also favors organized behavior.

For an electron gas with an isotropic Maxwellian velocity distribution at a temperature T, our criterion becomes

$$k^2\ll(12\pi ne^2/m\langle\mathbf{v}_i{}^2\rangle_{Av})=(4\pi ne^2/\kappa T)=\lambda_D{}^{-2}, \tag{14}$$

where κ is Boltzmann's constant, and λ_D is the well-known Debye length,[12] defined by

$$\lambda_D=(\kappa T/4\pi ne^2)=(1/3)[(\langle v_i{}^2\rangle_{Av})/\omega_P{}^2]. \tag{15}$$

This means that in a collection of electrons, organized behavior is most important in phenomena connected with distances greater than λ_D, while for phenomena connected with shorter distances the individual particles point of view provides the best starting point. This result has already been obtained by other investigators using different methods.[13] It is because of the long range of the Coulomb force (as expressed in terms of the $1/k^2$ in its Fourier component) that the collective approximation is sure to apply if one goes to large enough wavelengths ($\lambda>\lambda_D$). A long-range force favors the collective approximation at long wavelengths, because a large number of particles can be made to contribute cumulatively to a collective oscillation in the charge density.

It is of interest to evaluate λ_D for a few examples. Thus in a typical gaseous discharge of density $n=10^{12}$, and a mean electron kinetic energy of 3 ev, we obtain $\omega_P=5\times10^{10}$ and $\lambda_D\cong10^{-3}$ cm. For an electron gas density $\sim10^{23}$, such as one finds in a typical metal, $\omega_P\cong2\times10^{16}$, and $\lambda_D\cong0.4$A, as compared with an interparticle spacing of ~2A.[14]

We have seen that the time variation of the ρ_k reflects both the collective and the individual particle aspects of the assembly, so that the ρ_k do not oscillate harmonically for appreciable values of k, and therefore do not consistitue a satisfactory set of collective coordinates. We might expect that somewhat different functions of the particle coordinates and velocities exist which do oscillate harmonically for nonzero k, despite the effects of random thermal motion. For small

[12] The Debye length was first introduced in connection with screening processes in highly ionized electrolytes. P. Debye and E. Huckel, Physik Z. 24, 185 (1923).

[13] See D. Bohm and E. P. Gross, Paper A, reference 5.

[14] For this case λ_D clearly cannot constitute the minimum distance at which organization becomes important, since it is less than the interparticle spacing. However, this indicates that the organization is very important for the electrons in a metal, since it does extend down to regions of the order of the interparticle spacing. The appropriate minimum distance for a degenerate electron gas will be discussed in our subsequent paper on the quantum theory.

k, these functions should approach ρ_k. We now demonstrate that such functions exist and are in fact proportional to

$$q_k=\sum_i\{1/[\omega^2-(\mathbf{k}\cdot\mathbf{v}_i)^2]\}e^{-i\mathbf{k}\cdot\mathbf{x}_i}. \quad (16)$$

We shall find it convenient to introduce the quantities

$$\xi_{k,\omega}=\sum_i[1/(\omega-\mathbf{k}\cdot\mathbf{v}_i)]e^{-i\mathbf{k}\cdot\mathbf{x}_i}, \quad (17)$$

in terms of which the q_k may be expressed as

$$q_k=(1/2)[\xi_{k,\omega}-\xi_{k,-\omega})/\omega]. \quad (18)$$

We shall show that the q_k satisfy the harmonic oscillator equation

$$\ddot{q}_k+\omega^2 q_k=0, \quad (19)$$

by proving that

$$\dot{\xi}_{k,\omega}+i\omega\xi_{k,\omega}=0. \quad (20)$$

That (19) is a consequence of (20) may be readily verified by differentiating (18).

To prove that the $\xi_{k,\omega}$ satisfy (20), we differentiate (17), obtaining

$$\dot{\xi}_{k,\omega}=-i\sum_i\{[(\mathbf{k}\cdot\mathbf{v}_i)/(\omega-\mathbf{k}\cdot\mathbf{v}_i)] +(\mathbf{k}\cdot\dot{\mathbf{v}}_i)/(\omega-\mathbf{k}\cdot\mathbf{v}_i)^2\}e^{-i\mathbf{k}\cdot\mathbf{x}_i}. \quad (21)$$

Using the equations of motion (2), we find:

$$\dot{\xi}_{k,\omega}+i\omega\xi_{k,\omega}=i\sum_i e^{-i\mathbf{k}\cdot\mathbf{x}_i}-i\sum_{ik'}\frac{(\mathbf{k}\cdot\mathbf{k}')4\pi e^2}{mk^2(\omega-\mathbf{k}\cdot\mathbf{v}_i)^2} \times[\exp i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}_i]\exp(-i\mathbf{k}'\cdot\mathbf{x}_j). \quad (22)$$

As in Eq. (8), the terms with $\mathbf{k}'\neq\mathbf{k}$ in the sum on the right hand side will be neglected in the random phase approximation. Retaining only the terms with $\mathbf{k}'=\mathbf{k}$, we then obtain for the right-hand side, after interchanging i and j.

$$i\sum e^{-i\mathbf{k}\cdot\mathbf{x}_i}-i\sum_{ij}\frac{4\pi e^2}{m(\omega-\mathbf{k}\cdot\mathbf{v}_j)^2}e^{-i\mathbf{k}\cdot\mathbf{x}_i}.$$

In order that $\xi_{k,\omega}$ shall oscillate harmonically, the previous expression must vanish for abitrary $\mathbf{x}_i$. Thus we find that the $\xi_{k,\omega}$, and hence the q_k, oscillate harmonically, provided ω satisfies the following dispersion relation

$$1=(4\pi e^2/m)\sum_j 1/(\omega-\mathbf{k}\cdot\mathbf{v}_j)^2. \quad (23)$$

This same dispersion relation has already been obtained by other investigators using the beams treatment.[15] It is an integral equation which is difficult to solve exactly. However, for sufficiently small k, the denominator in (23) may be expanded in a series of powers of $(\mathbf{k}\cdot\mathbf{v}_j)/\omega$, and we obtain the approximate dispersion relation

$$\omega^2=\omega_P^2+k^2\langle v^2\rangle_{Av}, \quad (24)$$

provided we assume an isotropic distribution of velocities.[16] We note that for small k this reduced to our earlier dispersion relation, (11). The condition for the validity of (24) is essentially our criterion for the applicability of the collective description, (14). We also note that for $[(\mathbf{k}\cdot\mathbf{v})/\omega]\ll 1$, q_k approaches proportionality to ρ_k.

Where the described expansion is not very accurate, ω can be obtained by a numerical solution of (23). Other investigators have shown that no solution exists for k larger than a critical value of the order of λ_D^{-1}.[15] This result confirms our general physical picture in which the collective description is to be used only for distances $>\lambda_D$. Furthermore, over almost all values of k for which (23) has a solution, the approximate dispersion relation (24) is valid.

III. SEPARATION BETWEEN COLLECTIVE AND INDIVIDUAL COMPONENTS OF DENSITY FLUCTUATIONS

We have seen that the density fluctuations, ρ_k, can, for sufficiently small k, be described almost entirely in terms of the collective coordinate, q_k. On the other hand, since there are no collective coordinates for $k\gtrsim\lambda_D^{-1}$, the density fluctuations here must be associated primarily with the random thermal motions of the individual particles. For intermediate values of k, both types of fluctuations may be expected to be significant. In this section, we shall treat the problem of separating individual and collective aspects of a general fluctuation by splitting ρ_k into two parts as follows,

$$\rho_k=a_k q_k+\eta_k, \quad (25)$$

where a_k is a suitable constant, which will be chosen by methods to be developed presently. The first part, $a_k q_k$, is clearly a collectively describable part of ρ_k. We shall see that a choice of a_k is possible such that η_k describes fluctuations associated only with the random thermal motions of the individual particles. Moreover, the η_k will be shown to be the Fourier coefficients of a density distribution in which each electron is surrounded by a comoving cloud containing a deficiency of electrons, which screens the field of the given electron within a distance of the order of λ_D.

We begin with a consideration of (25) for $k\lesssim k_D$, where $k_D\cong(1/\lambda_D)$. Using our definition (16) of q_k, we may express η_k as

$$\eta_k=\sum_i\{1-a_k/[\omega^2-(\mathbf{k}\cdot\mathbf{v}_i)^2]\}e^{-i\mathbf{k}\cdot\mathbf{x}_i}. \quad (26)$$

[15] For a more detailed discussion of the dispersion relation (Eq. (23)] see Bohm and Gross, Paper A, reference 5.

[16] The mean kinetic energy will depend both on the temperature, which determines the random thermal motion, and on the amplitude of organized oscillation. Normally, the thermal energy is much greater than the organized oscillation energy, so that for all practical purposes, the frequency does not depend appreciably on the amplitude of organized oscillation. The slight dependence on amplitude resembles a similar phenomenon obtained in connection with sound waves, which likewise increase in frequency as their amplitude increases. In any case, the entire effect is in the domain of the nonlinear aspects of the problem, and therefore can be neglected in a linear approximation. Within the linear approximation, the $\langle v^2\rangle_{Av}$ appearing in the dispersion relation should be the value existing in the absence of organized oscillation.

With an arbitrary choice of a_k, $\ddot{\eta}_k$, like $d^2\rho_k/dt^2$ in Eq. (9), may be expected to have two kinds of terms. The first kind are those we obtain in the absence of interparticle forces, (in which case $\dot{v}_i=0$), and are

$$\ddot{\eta}_k^{(1)}=\sum_i -(\mathbf{k}\cdot\mathbf{v}_i)^2\{1-[a_k/[\omega^2-(\mathbf{k}\cdot\mathbf{v}_i)^2]]\}e^{-i\mathbf{k}\cdot\mathbf{x}_i}. \quad (27)$$

These arise from the random thermal motion of the particles, and can in no way possess an organized component. The second kind arise because of the interparticle forces, and, like the second term on the right-hand side of (9), represent the organized behavior of the system. To insure that η_k has no collective component, we must choose a_k in such a way that terms of this second kind vanish and $\ddot{\eta}_k$ is given by (27).

Using Eqs. (9) and (19), we obtain

$$\begin{aligned} d^2\eta_k/dt^2&=d^2\rho_k/dt^2-a_k d^2q_k/dt^2 \\ &=\sum_i\{-(\mathbf{k}\cdot\mathbf{v}_i)^2-\omega_P^2 \\ &\qquad +a_k\{\omega^2/[\omega^2-(\mathbf{k}\cdot\mathbf{v}_i)^2]\}\}e^{-i\mathbf{k}\cdot\mathbf{x}_i}, \end{aligned} \quad (28)$$

where we have continued to use the random phase approximation. This may be rearranged as

$$\ddot{\eta}_k=\ddot{\eta}_k^{(1)}+\sum_i\{(a_k-\omega_P^2)/[\omega^2-(\mathbf{k}\cdot\mathbf{v}_i)^2]\}e^{-i\mathbf{k}\cdot\mathbf{x}_i}. \quad (29)$$

Hence, if we choose $a_k=\omega_P^2$, we obtain the desired separation with

$$\eta_k=\sum_i \frac{[\omega^2-\omega_P^2-(\mathbf{k}\cdot\mathbf{v}_i)^2]}{\omega^2-(\mathbf{k}\cdot\mathbf{v}_i)^2}e^{-i\mathbf{k}\cdot\mathbf{x}_i}. \quad (30)$$

It will be convenient to include the coefficient ω_P^2 in our basic definition of q_k so that we have

$$q_k=\sum_i\{\omega_P^2/[\omega^2-(\mathbf{k}\cdot\mathbf{v}_i)^2]\}e^{-i\mathbf{k}\cdot\mathbf{x}_i}, \quad (31)$$

and

$$\rho_k=q_k+\eta_k \quad (k<k_D). \quad (32)$$

With these definitions, we see that the charge density has split into two independent parts. One of these, q_k, oscillates harmonically with frequency ω, and can be given an arbitrary amplitude of oscillation dependent on the boundary conditions. The other part, η_k, has no collective behavior, and is therefore best regarded simply as the sum of independent contributions from individual particles. This part is present even in the absence of organized oscillation.

When $k\gtrsim k_D$, there are no collective coordinates, so that $\rho_k=\eta_k$. However, ρ_k, (or η_k), in this region does not satisfy an equation of motion like (27) since we still have the second term on the right-hand side of (9), *viz.*, $\omega_P^2\rho_k$, in the expression for $\ddot{\eta}_k$. However, for values of k appreciably greater than k_D this term is negligible, and hence, except for a region of $k\sim k_D$, which we shall see is of no great importance in most problems, we may regard η_k as satisfying (27) for all values of k.

We now wish to investigate in detail the meaning of the density fluctuations. To do this, we consider the contribution to the spatial distribution of electron density arising from that part of η_k associated with the rth particle,

$$\begin{aligned} \eta_r(\mathbf{x})&=\sum_k \eta_{kr}e^{-i(\mathbf{k}\cdot\mathbf{x})} \\ &=\sum_{k<k_D}\left\{\frac{\omega^2-\omega_P^2-(\mathbf{k}\cdot\mathbf{v}_r)^2}{\omega^2-(\mathbf{k}\cdot\mathbf{v}_r)^2}\right\}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}_r)} \\ &\qquad +\sum_{k>k_D} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}_r)}. \end{aligned} \quad (33)$$

In the first term on the right-hand side, since we are limited to $k<k_D$, we may expand ω^2 according to (24), obtaining,

$$\begin{aligned} \eta_r(\mathbf{x})&=\sum_{k<k_D}\left\{\frac{k^2\langle v^2\rangle_{\text{Av}}-(\mathbf{k}\cdot\mathbf{v}_r)^2}{\omega_P^2+k^2\langle v^2\rangle_{\text{Av}}-(\mathbf{k}\cdot\mathbf{v}_r)^2}\right\}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}_r)} \\ &\qquad +\sum_{k>k_D} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}_r)}. \end{aligned} \quad (34)$$

We note that for $k\gg k_D$,

$$k^2\langle v^2\rangle_{\text{Av}}-(\mathbf{k}\cdot\mathbf{v}_r)^2\cong\omega_P^2+k^2\langle v^2\rangle_{\text{Av}}-(\mathbf{k}\cdot\mathbf{v}_r)^2.$$

Hence, as an approximation, we can take

$$\eta_r(\mathbf{x})\cong\sum_{\text{all }k}\left\{\frac{k^2\langle v^2\rangle_{\text{Av}}-(\mathbf{k}\cdot\mathbf{v}_r)^2}{\omega_P^2+k^2\langle v^2\rangle_{\text{Av}}-(\mathbf{k}\cdot\mathbf{v}_r)^2}\right\}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}_r)}. \quad (35)$$

This is an accurate description of $\eta_r(x)$ for $k\gg k_D$ and $k\ll k_D$. It is not far off in the region $k\cong k_D$.

To evaluate the sum over $\mathbf{k}$ we choose the direction of $\mathbf{v}_r$ as the z axis in $\mathbf{k}$ space. We also make the following change of variables in $\mathbf{k}$ and $\mathbf{x}$ space

$$\begin{aligned} (\langle v^2\rangle_{\text{Av}}-v_r^2)k_z^2&=\langle v^2\rangle_{\text{Av}}(k_z')^2, \\ \langle v^2\rangle_{\text{Av}}(z-z_r)^2&=(\langle v^2\rangle_{\text{Av}}-v_r^2)(z'-z_r')^2, \end{aligned}$$

together with $k_x=k_x'$, $k_y=k_y'$, $x=x'$, and $y=y'$. We then obtain, on changing the sum over $\mathbf{k}$ to an integral,

$$\begin{aligned} \eta_r(\mathbf{x})=\frac{1}{(2\pi)^3}\left\{\frac{\langle v^2\rangle_{\text{Av}}}{\langle v^2\rangle_{\text{Av}}-v_r^2}\right\}^{\frac{1}{2}}\int d\mathbf{k}'\left\{\frac{(k')^2\langle v^2\rangle_{\text{Av}}}{\omega_P^2+(k')^2\langle v^2\rangle_{\text{Av}}}\right\} \\ \times\exp[i\mathbf{k}'\cdot(\mathbf{x}'-\mathbf{x}_r')]. \end{aligned} \quad (36)$$

This may be re-expressed as

$$\begin{aligned} \eta_r(\mathbf{x})=-\left(\frac{1}{2\pi}\right)^3\left\{\frac{\langle v^2\rangle_{\text{Av}}}{\langle v^2\rangle_{\text{Av}}-v_r^2}\right\}^{\frac{1}{2}}\nabla_{x'}{}^2 \\ \times\int d\mathbf{k}'\left\{\frac{\exp[i\mathbf{k}'\cdot(\mathbf{x}'-\mathbf{x}_r')]}{(k')^2+(\omega_P^2/\langle v^2\rangle_{\text{Av}})}\right\}. \end{aligned} \quad (37)$$

The integral over $\mathbf{k}'$ is readily evaluated, and we obtain

$$\begin{aligned} \eta_r(\mathbf{x})=-\left\{\frac{\langle v^2\rangle_{\text{Av}}}{\langle v^2\rangle_{\text{Av}}-v_r^2}\right\}^{\frac{1}{2}}\nabla_{x'}{}^2 \\ \times\left[\frac{\exp\{-(\omega_P^2/\langle v^2\rangle_{\text{Av}})^{\frac{1}{2}}|\mathbf{x}'-\mathbf{x}_r'|\}}{4\pi|\mathbf{x}'-\mathbf{x}_r'|}\right]. \end{aligned} \quad (38)$$

Using the well-known result, $(\nabla^2-k^2)\{e^{-kr}/r\}=-4\pi\delta(\mathbf{r})$ we get

$$\eta_r(\mathbf{x})=\frac{\langle v^2\rangle_{Av}^{\frac{1}{2}}}{(\langle v^2\rangle_{Av}-v_r^2)^{\frac{1}{2}}}\left\{\delta(\mathbf{x}'-\mathbf{x}_r')-\frac{\omega_P^2\exp\{-(\omega_P^2/\langle v^2\rangle_{Av})^{\frac{1}{2}}|\mathbf{x}'-\mathbf{x}_r'|\}}{4\pi\langle v^2\rangle_{Av}|\mathbf{x}'-\mathbf{x}_r'|}\right\}. \tag{39}$$

Returning to our original coordinate system, we have:

$$\eta_r(\mathbf{x})=\delta(\mathbf{x}-\mathbf{x}_r)-\frac{\omega_P^2\alpha_r}{4\pi\langle v^2\rangle_{Av}}\left\{\frac{\exp\{-(\omega_P^2/\langle v^2\rangle_{Av})^{\frac{1}{2}}[(x-x_r)^2+(y-y_r)^2+\alpha_r^2(z-z_r)^2]^{\frac{1}{2}}\}}{[(x-x_r)^2+(y-y_r)^2+\alpha_r^2(z-z_r)^2]^{\frac{1}{2}}}\right\}, \tag{40}$$

where $\alpha_r^2=\langle v^2\rangle_{Av}/[\langle v^2\rangle_{Av}-v_r^2]$.

This represents a particle at $\mathbf{x}=\mathbf{x}_r$ surrounded by a comoving cloud in which the electron density is reduced below the average. The cloud is elliptical in shape, being shortened in the direction of particle motion by the ratio $(\alpha_r)^{-1}$. It should be noted that for those particles with $v_r^2>\langle v^2\rangle_{Av}$ the present treatment fails. This case will be discussed in Sec. IV. Here we restrict ourselves to those particles moving with less than the mean thermal speed, $[\langle v^2\rangle_{Av}]^{\frac{1}{2}}$.

The computation of the potential arising from the charge density $-e\eta_r$ is complicated by the elliptical shape of the electron cloud. To get a rough estimate of this potential, we consider the special case of a particle at rest. For this case the cloud is spherical and the potential is found to be

$$-e\frac{\exp\{-|\mathbf{x}-\mathbf{x}_r|(\omega_P^2/\langle v^2\rangle_{Av})^{\frac{1}{2}}\}}{|\mathbf{x}-\mathbf{x}_r|}.$$

This is a screened Coulomb potential with a screening radius of the order of λ_D.

More generally we can see that screening still takes place in a distance of the order of λ_D when $\mathbf{v}_r$ takes a nonzero value. We recall that the electrons are embedded in a uniform distribution of positive charge. If the negative charge distribution were perfectly uniform, the electric field would vanish. However, the field does not vanish since we have a collection of negative point charges which produce local fluctuations whose Fourier components are the ρ_k. Let us fix our attention on the individual particles component $\eta_r(\mathbf{x})$ of the density associated with the rth particle as given by (40). Without the comoving cloud described by the second term on the right-hand side of (40), the density is a δ-function, which gives rise to a simple Coulomb potential $-e/|\mathbf{x}-\mathbf{x}_r|$. The comoving cloud represents a region from which electrons have been displaced by the repulsive Coulomb potential of the rth electron. As a result this region contains a net positive charge. In fact the total net positive charge may be obtained by integrating our expression for the electron deficiency [as obtained from (40)] over all space, and the result is readily seen to be unity. Since most of this charge is clearly within a region $\sim\lambda_D$ surrounding the electron, we may deduce from Gauss' theorem that the electric field associated with this electron becomes negligible at distances greater than λ_D from the electron. This constitutes a striking demonstration that the individual particles do not produce important effects at distances greater than the Debye length, so that in this region, the collective coordinates q_k provide an adequate description of all important properties of the electron gas. Although this result has thus far obtained only for these particles moving with less than the mean thermal speed, it will be shown in Sec. IV that similar conclusions apply to the remaining particles.

IV. EXCITATION OF COLLECTIVE OSCILLATIONS AND SCREENING

In this section, we study the collective response of the electron gas to an individual particle moving through the system with velocity $\mathbf{v}_0$. We first show that if $\mathbf{v}_0$ is less than approximately the mean thermal speed, the particles of the electron gas respond in such a way that when a steady state is finally established, the field of the particle is screened out within a distance of the order of λ_D. In this way, we obtain a more detailed understanding of how the screening, discussed in Sec. III, is brought about. We then show that if v_0 is more than approximately the mean thermal speed, the field of the particle continues to be screened, but a new phenomenon appears, *viz.*, the excitation of a wake consisting of collective oscillations that carry energy away from the particle. This wake resembles the Čerenkov radiation obtained when fast electrons go through a dielectric. We find that the energy loss to the collective oscillations is of the same order of magnitude as the loss caused by short-range Coulomb collisions with the individual particles.

We begin by Fourier-analyzing the charge density of the specified particle, moving with a constant velocity, $\mathbf{v}_0$

$$-e\rho_s=-e\delta(\mathbf{x}-\mathbf{v}_0t)=-e\sum_k e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{v}_0t)}. \tag{41}$$

Our problem is to calculate the response of the q_k to the field produced by ρ_s. We recall that

$$q_k=(\omega_P^2/2\omega)(\xi_{k,\omega}-\xi_{k,-\omega}). \tag{42}$$

We first find the response of the $\xi_{k,\omega}$ to ρ_s. We have

$$\dot{\xi}_{k,\omega}=\sum_i\left\{\frac{-i(\mathbf{k}\cdot\mathbf{v}_i)}{\omega-\mathbf{k}\cdot\mathbf{v}_i}+\frac{\mathbf{k}\cdot\dot{\mathbf{v}}_i}{[\omega-(\mathbf{k}\cdot\mathbf{v}_i)]^2}\right\}e^{-i\mathbf{k}\cdot\mathbf{x}_i}. \tag{42a}$$

The force on the ith particle arises from two sources: the other particles in the electron gas, and the specified

particle. When the latter is not present, we have, according to (20),

$$\dot{\xi}_{k,\omega}=-i\omega\xi_{k,\omega}.$$

The external particle changes $\dot{\mathbf{v}}_i$, according to (6) and (41), by

$$\dot{v}_i^{(s)}=-(4\pi e^2/m)i\sum_{k'}[\mathbf{k}'/(k')^2]\times\exp[i\mathbf{k}'\cdot(\mathbf{x}_i-\mathbf{v}_0t)]. \quad (43)$$

Thus we obtain, using the random phase approximation,

$$\dot{\xi}_{k,\omega}=-i\omega\xi_{k,\omega}-(4\pi e^2/m)i\sum_i[1/(\omega-\mathbf{k}\cdot\mathbf{v}_i)^2]e^{-i\mathbf{k}\cdot\mathbf{v}_0t}. \quad (44)$$

Applying the dispersion relation, (23), we have

$$\dot{\xi}_{k,\omega}=-i\omega\xi_{k,\omega}-ie^{-i\mathbf{k}\cdot\mathbf{v}_0t}. \quad (45)$$

From (42) we then get

$$\dot{q}_k=-i(\omega_P^2/2)(\xi_{k,\omega}+\xi_{k,-\omega}).$$

To obtain $\ddot{q}_k$, we differentiate again, obtaining

$$\ddot{q}_k=-\omega(\omega^2{}_P/2)(\xi_{k,\omega}-\xi_{+k,-\omega})-\omega_P^2e^{-i\mathbf{k}\cdot\mathbf{v}_0t},$$

and hence

$$\ddot{q}_k+\omega^2q_k=-\omega_P^2e^{-i\mathbf{k}\cdot\mathbf{v}_0t}. \quad (46)$$

Equation (46) describes a forced harmonic oscillation. A particular solution which describes the steady-state response of the Fourier component of the collective part of the charge density to the field of the specified particle is given by[17]

$$q_k=-[\omega_P^2/\{\omega^2-(\mathbf{k}\cdot\mathbf{v}_0)^2\}]e^{-i\mathbf{k}\cdot\mathbf{v}_0t}. \quad (47)$$

We exclude for the time being the case in which the denominator of (47) vanishes. To obtain the Fourier component of the total charge density associated with the specified particle, we must add the density (45) coming from the charge itself. We get

$$\rho_{ks}=\{1-[\omega_P^2/\{\omega^2-(\mathbf{k}\cdot\mathbf{v}_0)^2\}]\}e^{-i\mathbf{k}\cdot\mathbf{v}_0t}. \quad (48)$$

The charge density as a function of position is

$$\rho_s(\mathbf{x})=\sum_k\frac{[\omega^2-\omega_P^2-(\mathbf{k}\cdot\mathbf{v}_0)^2]}{\omega^2-(\mathbf{k}\cdot\mathbf{v}_0)^2}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{v}_0t)}. \quad (49)$$

However, Eq. (49) is formally equivalent to $\eta_s(\mathbf{x})$ (see Eq. (35)). Thus we can conclude that when a particle moves through the electron gas with less than the mean thermal speed, the collective response to its electric field is just such as to screen out that field within a distance of the order of magnitude of λ_D. This is the origin of the screening cloud discussed in the previous section.

[17] In obtaining this solution we have taken $\mathbf{v}_0$ as constant. This will be a good approximation as long as the change in velocity during the period of an oscillation is small compared to $\mathbf{v}_0$, which is the case in most electron gases. See Section V.

We now consider a particle moving with a velocity $\mathbf{v}_0$ such that the denominator of (47) can vanish for some $k<k_D$, i.e., such that

$$(\mathbf{k}\cdot\mathbf{v}_0)^2=\omega^2\cong\omega_P^2+k^2\langle v^2\rangle_{\text{Av}}. \quad (50)$$

For those values of $\mathbf{k}$ such that (50) is satisfied, a steady-state solution for q_k is impossible, since as can be seen from (46), the correct solution corresponds to a resonant excitation of the appropriate q_k. Let us now find the value of $\mathbf{k}$ and $\mathbf{v}_0$ such that (50) can be satisfied. Taking k_z in the direction of $\mathbf{v}_0$, we obtain

$$k_z^2(v_0^2-\langle v^2\rangle_{\text{Av}})=\omega_P^2+(k_x^2+k_y^2)\langle v^2\rangle_{\text{Av}}. \quad (51)$$

It is immediately evident that real solutions of (51) cannot exist when v_0^2 is less than $\langle v^2\rangle_{\text{Av}}$, so that particles moving with less than the mean thermal speed will not excite collective oscillations. The minimum value of v_0 for which collective oscillations can be excited may be found by setting $k_x=k_y=0$, and k_z equal to its maximum value k_D. The criterion for excitation becomes

$$v_0^2\geqslant(4/3)\langle v^2\rangle_{\text{Av}}, \quad (52)$$

where we have used (15).

We must now solve (46) with the correct boundary conditions for the resonant case. The proper choice of boundary conditions can be seen from the following considerations. The group velocity of the collective oscillations, which measures the speed with which a disturbance is propagated, is given by

$$v_g=(d\omega/dk)\cong[k\langle v^2\rangle_{\text{Av}}/\omega].$$

The maximum value of v_g, obtained by setting $k=k_D$, is $[\frac{3}{4}\langle v^2\rangle_{\text{Av}}]^{\frac{1}{2}}$. A particle moving with a speed high enough to excite oscillation therefore runs ahead of the disturbance that it creates. Hence, as is well known, the disturbance will take the form of a wake trailing behind the particle. The correct boundary condition for excitation is, therefore, that no disturbance exist ahead of the particle. This situation can be contrasted to that existing when $v_0^2<\frac{3}{4}\langle v^2\rangle_{\text{Av}}$. In the latter case the disturbance can propagate ahead of the particle, so that the field around the particle ultimately reaches a steady state. (Actually a steady state is reached as long as the condition (50) cannot be satisfied, but the use of the expansion (24) for ω^2 is not permissible near the Debye length. This is the origin of the discrepancy between our estimated maximum speed of transmission of a disturbance $[\frac{3}{4}\langle v^2\rangle_{\text{Av}}]^{\frac{1}{2}}$ and minimum speed, $(v_0=[(4/3)\langle v^2\rangle_{\text{Av}}]^{\frac{1}{2}})$, at which excitation of collective oscillation can occur.)

To solve the boundary condition problem, we first consider $q(\mathbf{x})$ obtained from the q_k given in (46) by

$$q(\mathbf{x})=\sum_{k<k_D}q_ke^{i\mathbf{k}\cdot\mathbf{x}}=\sum_{k<k_D}\{-\omega_P^2/[\omega^2-(\mathbf{k}\cdot\mathbf{v}_0)^2]\}e^{i(\mathbf{k}\cdot\mathbf{x}-\mathbf{v}_0)}. \quad (52a)$$

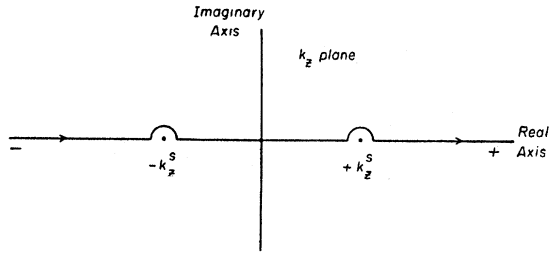

FIG. 1. Contour for the evaluation of the integral in Eq. (54).

When we choose k_z in the direction of $\mathbf{v}_0$, we obtain:

$$q(\mathbf{x})=\sum_{k<k_D}\frac{-\omega_P^2\exp\{i[k_x x+k_y y+k_z(z-v_0t)]\}}{\omega_P^2+(k_x^2+k_y^2)\langle v^2\rangle_{Av}+k_z^2[\langle v^2\rangle_{Av}-v_0^2]}, \quad (52b)$$

where we have expanded ω^2 according to (24). The denominator in (52b) vanishes for $k_z=k_z^s$, where

$$k_z^s=\pm[(\omega_P^2+(k_x^2+k_y^2)\langle v^2\rangle_{Av})/(v_0^2-\langle v^2\rangle_{Av})]^{\frac{1}{2}}. \quad (53)$$

In the summation over k_z it will be seen to be mathematically convenient to extend our limits from $\pm(k_D^2-k_x^2-k_y^2)^{\frac{1}{2}}$ to $\pm\infty$. This will certainly be justified for v_0 sufficiently larger than $\langle v^2\rangle_{Av}^{\frac{1}{2}}$ and will provide an order of magnitude estimate for all values of v_0. For with v_0 sufficiently large, almost all values of k_z^s will lie within the limits $\pm(k_D^2-k_x^2-k_y^2)^{\frac{1}{2}}$, and the denominator of (52b) will be large for the values of k_z outside these limits. Hence, in this case a negligible error is introduced by extending the range of integration to $\pm\infty$. (For v_0 comparable with $\langle v^2\rangle_{Av}^{\frac{1}{2}}$, the error may become appreciable, and a more detailed calculation is required.) Thus we obtain, on changing our sum to an integral,

$$q(\mathbf{x})=-(1/2\pi)^3\int dk_x\int dk_y\int_{-\infty}^{\infty}dk_z \times\left\{\frac{\omega_P^2\exp[i(k_x x+k_y y+k_z(z-v_0t)]}{\omega_P^2+(k_x^2+k_y^2)\langle v^2\rangle_{Av}+k_z^2(\langle v^2\rangle_{Av}-v_0^2)}\right\}. \quad (54)$$

The boundary conditions may be introduced, according to well-known techniques, by integrating over the complex k_z plane and choosing a suitable path of integration. It has been shown that $q(\mathbf{x})$ should correspond to a wave moving behind the particle. To insure that there is no disturbance ahead of the particle, we require that $q(\mathbf{x})$ vanish for positive $(z-v_0t)$. It is readily verified that the proper contour is that given in Fig. 1. For positive $z-v_0t$ we close the contour by integrating over a large semicircle in the upper half-plane, and thus obtain zero in accordance with our boundary condition. For negative $z-v_0t$ we close the contour with a large semicircle in the lower half-plane. The contour can be shrunk down to the two singularities, and the residues yield,

$$q(\mathbf{x})=-\frac{\omega_P^2}{4\pi^2[v_0^2-\langle v^2\rangle_{Av}]^{\frac{1}{2}}}\int\int dk_x dk_y \times\left\{\frac{\{\sin[k_z^s(z-v_0t)]\}e^{i(k_x x+k_y y)}}{\omega_P^2+(k_x^2+k_y^2)\langle v^2\rangle_{Av}}\right\}. \quad (55)$$

The precise form of $q(x)$ is of no great interest to us here, as we shall be mainly interested in computing the energy given up by the incident particle to the wake. This is determined by the reaction of the electric field of the wake on the emitting particle located at $\mathbf{x}=\mathbf{v}_0t$ or $x=0$, $y=0$, $z=v_0t$.[18] We need only consider the z component $\mathcal{E}_z$, since by symmetry the x and y components vanish. $\mathcal{E}_z$ vanishes for $z>v_0t$ and is finite for $z<v_0t$. There is a discontinuity at $z=v_0t$, and hence, according to a well-known property of Fourier series, the correct value at $z=v_0t$ is the sum taken by approaching this point from both sides. Thus we take $(\frac{1}{2})\mathcal{E}_z(v_0t)$ as approached from the left. A little algebra shows that

$$\mathcal{E}_z(v_0t)=-\frac{e\omega_P^2}{2\pi}\int\int dk_x dk_y\left\{\frac{1}{\omega_P^2+(k_x^2+k_y^2)v_0^2}\right\}. \quad (56)$$

The force on the particle caused by the wake is thus

$$F_z=-e\mathcal{E}_z(v_0t). \quad (57)$$

On performing the integration, using polar coordinates, and taking

$$(k_x^2+k_y^2)_{\max}\cong(\tfrac{2}{3})k_D^2=2\omega_P^2/\langle v^2\rangle_{Av},$$

we get:

$$F_z=(e^2\omega_P^2/2v_0^2)\ln(1+2v_0^2/\langle v^2\rangle_{Av}). \quad (58)$$

The limits of integration of (56) were loosely defined, but since the integral has a logarithmic dependence on the actual limits the result is insensitive to the exact limits taken. Thus the rate of energy loss per unit distance dE/dz which is equal to F_z is

$$(dE/dz)^{(1)}=(\pi ne^4/E_0)\ln(1+2v_0^2/\langle v^2\rangle_{Av}), \quad (59a)$$

where E_0 is the energy of the incident particle.

Thus far we have only considered the energy given up by the specified particle to the collective oscillations. It can also transfer energy to the individual particles in short-range collisions. The local electric field around each particle in the electron gas, as we have seen, corresponds to a screened Coulomb potential of range λ_D arising from the η_k contribution to the charge density.

[18] We here are calculating the effects of the $q(\mathbf{x})$ on the particle at a distance for which the concept of a $q(\mathbf{x})$ is clearly not applicable, since we are certainly less than a Debye length removed from the particle. Nevertheless the reacting force may be obtained correctly in this way. For the energy is carried away by the radiation at long distances, where the $q(\mathbf{x})$ can correctly be applied. Because energy is conserved, this energy can come only from the incident particle. Therefore, by formally extrapolating the theory into distances below those for which its physical application is correct, we are able to obtain the correct energy loss.

When the specified particle enters the range of this force it can transfer energy by means of a collision. This transfer is essentially independent of the excitation of collective oscillations, and can best be calculated with aid of the usual collision theory applied to the individual particles. One finds that the energy loss resulting from these short-range collisions is approximately

$$(dE/dz)^{(2)}=(2\pi ne^4/E_0)\ln(\lambda_D E_0/e^2). \quad (59b)$$

It is clear that the two modes of energy loss, described by (59a) and (59b), will generally be of the same order of magnitude.

Experiments by Ruthemann[6] and Lang[7] on energy loss of electrons of energies of the order of 1 to 10 kev in thin films of metal tend to confirm our theoretical prediction of Čerenkov-like radiation. Although the quantum theory should be used in a rigorous treatment of a metal (and will be used in a subsequent paper), the right results for this problem can be obtained by using a correspondence principle argument. We first note that electrons must emit energy in the form of quanta with $E=\hbar\omega$. The probability of emission of a quantum may be estimated from the requirement that the mean energy radiated be equal to the classically calculated value. We then obtain for the number of quanta emitted per unit path length,

$$(dN/dx)=(1/\hbar\omega)(dE/dx).$$

The mean free path for emission of a quantum is the reciprocal of this expression, and is thus given by

$$\lambda=\frac{E_0\hbar\omega}{\pi ne^4\ln(1+2v_0^2/\langle v^2\rangle_{Av})}.$$

We might expect that the fundamental quantum of energy loss should be quite close to $\hbar\omega_P$. To see this, we note first, using Eq. (53), that for large v_0, k_z must be close to ω_P/v_0, which is small. From Eq. (56) we see that the major contribution to the decelerating field comes when $(k_x^2+k_y^2)\lesssim(\omega_P^2/v_0^2)$. Thus we conclude that $\omega^2=\omega_P^2+k^2\langle v^2\rangle_{Av}$ must be close to $\omega_P^2[1+(\langle v^2\rangle_{Av}/v_0^2)]\cong\omega_P^2$. The experiments quoted previously indicate that the electrons actually do lose energy in integral multiples of a fairly sharply defined basic unit. For Al this unit is 14.7 ev, and for Be 19.0 ev. Our calculated $\hbar\omega_P$ for these metals, under the assumption that the valence electrons are all free, is 15.9 ev for Al and 18.8 ev for Be. These results are in remarkably good agreement with experiment, since the effective number of free electrons probably differs somewhat from the number of valence electrons.

Lang has given data on the thickness of his films, from which one can conclude that the mean free path for emission of a quantum is somewhat less than 185A. This compares favorably with our theoretical value of ~150A for this case. A more detailed study of these problems will be given in a subsequent paper.

In conclusion, we now show, without giving a detailed calculation, that the collective response $q(\mathbf{x})$ screens the field of a specified particle of any velocity. To do this, we show that the total responding charge $-eq(\mathbf{x})$ integrated over a small sphere surrounding the incident particle is nearly equal to, and of opposite sign, to that of the incident particle.

From (52a), we obtain for such an integral over a sphere of radius R surrounding $\mathbf{x}=\mathbf{v}_0t$

$$I=-\int_{\text{sphere}}d\mathbf{x}q(\mathbf{x})e=\sum_{k<k_D}\left\{\frac{-\omega_P^2e}{\omega^2-(\mathbf{k}\cdot\mathbf{v}_0)^2}\right\}\int_0^R r^2dr\times\int_0^\pi d(\cos\vartheta)\int_0^{2\pi}d\varphi e^{ikr\cos\vartheta}. \quad (60)$$

Carrying out the angular integrations, we have

$$I=\sum_{k<k_D}-e[\omega_P^2/\{\omega^2-(\mathbf{k}\cdot\mathbf{v}_0)^2\}]\times\int_0^R dr4\pi\{r[\sin(kr)]/k\}, \quad (61)$$

and hence,

$$I=\sum_{k<k_D}e[\omega_P^2/\{\omega^2-(\mathbf{k}\cdot\mathbf{v}_0)^2\}]\times\{4\pi R^2\{[\cos(kR)]/(kR)\}\}. \quad (62)$$

The expression inside the bracket is a sharply peaked function of k which approaches a Dirac δ function in k space at large R, and is negligible for $kR\gtrsim1$. For sufficiently large R, the sum over k yields just the value of $\omega_P^2/[\omega^2-(k\cdot v_0)^2]$ at $k=0$, which is unity. Thus for a sufficiently large R, which we denote by R_c, the total charge associated with the collective response is equal in magnitude and opposite in sign to the incident charge. Thus the incident charge is screened out within a distance R_c. The order of magnitude of R_c, the effective screening radius, may be seen from the following argument. In order that the charge be screened, the half-width $k\cong(1/R_c)$ must be sufficiently narrow so that the ratio $\omega_P^2/[\omega^2-(\mathbf{k}\cdot\mathbf{v}_0)^2]$ shall be close to unity. If there is no singularity in the denominator, this ratio will be close to unity for k up to $\sim k_D$, so that $R_c\cong\lambda_D$. If there is a singularity, then the ratio will be unity only up to the neighborhood of $k=k_s$ or k_D, whichever is the smaller. Thus $R_c=\lambda_D$ or $1/k_s$, whichever is the larger. Since high velocity incident particles may have a small k_s, in these cases the screening will not be as good as for a low velocity particle.

We conclude that the field of each particle in the electron gas is screened as the result of the collective response of all the other particles. For most particles, the field is effectively screened within a distance of approximately λ_D. For those few particles with much higher than mean thermal speed, the screening radius may be somewhat greater than λ_D. These particles also excite collective oscillation, and in fact this excitation

is one of the means of bringing such particles into thermal equilibrium with the rest of the assembly. The excitation of collective oscillations is a consequence of the long-range part of the Coulomb interactions; one may regard it as a description of the effect of the simultaneous many-body collisions brought by the long range of the force. The remaining screened short-range Coulomb interactions can, as we have seen, be described in terms of the usual two-body collision theory. Thus we see that the use of an individual particle model for an electron gas is justified in the treatment of phenomena involving distances less than the Debye length.

V. THE ROLE OF INDIVIDUAL PARTICLES IN ORGANIZED BEHAVIOR

In this section we wish to develop a physical picture of the motion of the individual particles as they take part in the collective motion. We first consider the state of the electron gas in the absence of collective oscillation, i.e., when the q_k and $\dot{q}_k$ are zero for all k. In this state, we have

$$\sum_i [\omega_P^2/\{\omega^2-(\mathbf{k}\cdot\mathbf{v}_{0i})^2\}]e^{-i\mathbf{k}\cdot\mathbf{x}_{0i}}=0, \quad (63a)$$

$$\sum_i [(\mathbf{k}\cdot\mathbf{v}_{0i})\omega_P^2/\{\omega^2-(\mathbf{k}\cdot\mathbf{v}_{0i})^2\}]e^{-i\mathbf{k}\cdot\mathbf{x}_{0i}}=0, \quad (63b)$$

where $\mathbf{x}_{0i}$ and $\mathbf{v}_{0i}$ are the position and velocity of the ith particle in the absence of organized oscillation. The state of no collective oscillation described by (63a) and (63b) differs from a state in which the particles move in a completely random way, as would exist in the absence of interaction.

In the absence of interaction the particles move in straight lines with constant velocities. This motion produces random fluctuations in the particle density which make the most likely absolute value of q_k different from zero. To see this, consider $\langle|q_k|^2\rangle_{Av}$ which is, according to (31),

$$\langle|q_k|^2\rangle_{Av}=\left\langle\sum_{ij}\frac{\omega_P^4\exp[i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)]}{[\omega^2-(\mathbf{k}\cdot\mathbf{v}_i)^2][\omega^2-(\mathbf{k}\cdot\mathbf{v}_j)^2]}\right\rangle. \quad (64)$$

Since the particles are distributed at random, only those terms with $i=j$ contribute to the average, and for small k one then obtains approximately.

$$\langle|q_k|^2\rangle_{Av}=n. \quad (65)$$

Thus we see that the particles in the state of lowest collective energy cannot move in straight lines and possess a random distribution of positions. Instead there must be fluctuations in particle velocities resulting from the Coulomb interaction which lead to just such correlations of particles positions as to make $q_k=0$. For this state, the charge density ρ_k reduces to η_k, the individual particles part, which for small k is much less than ρ_k. Thus these correlations produce the screening of the field of each individual particle by the electron gas.

Since for small k, q_k is very nearly equal to ρ_k, we see that the state with $q_k=0$ corresponds to one in which the density fluctuations are greatly reduced below the values to be expected in a random distribution. The state $q_k=0$ corresponds to the state of lowest possible collective energy, but does not correspond to a state of thermal equilibrium. It will be shown in Appendix I, however, that the state of thermal equilibrium is not far from the state of lowest collective energy. There we evaluate the probability of fluctuation in the charge density for a state of thermal equilibrium, and show that as a consequence of the Coulomb repulsion, this probability is greatly reduced below that present in the absence of interaction. Thus we verify in an independent way the reduction in density fluctuations implied by $q_k=0$.

It is difficult to solve in detail for the motion of the particle in the state of zero oscillation. One of the advantages of the collective description is that detailed solutions for this motion are not required. Since each particle moves in the screened Coulomb field of the other particles, its motion will usually not differ markedly from straight line motion. An estimate of the validity of the uniform straight line motion assumption may be obtained by comparing the mean-free path for collision (because of the screened Coulomb force) with the interparticle distance. It should be emphasized here that the assumption that the particles in an electron gas are approximately independent (free) is valid only because the organized behavior acts to screen out the long-range part of the Coulomb interparticle force.

Let us now consider the way in which the motion of the individual particles changes as a result of collective oscillation. We solve for the motion of the particle under the assumption that its motion in the absence of oscillation can be approximated as uniform, rectilinear motion. This approximation will be suitable when the fractional fluctuations of the particle velocity resulting from the short-range screened Coulomb interparticle forces are small during the period of an oscillation. These fluctuations will be small provided the mean-free path for short-range collisions is considerably greater than the distance covered by the particle during this period, which is approximately λ_D. This criterion is satisfied in all electron gases of interest. Under these circumstances the short-range collisions give rise only to a small damping of the organized oscillation, with a damping time $\tau=\tau_0(l/\lambda_D)$, where τ_0 is the period of an oscillation and l is the mean free path.[19] When l is of the order of magnitude of λ_D, the whole concept of organized oscillations becomes doubtful.

In the straight-line approximation we have, in the absence of oscillation,

$$\dot{\mathbf{x}}_{0i}=\mathbf{v}_{0i}, \quad \dot{\mathbf{v}}_{0i}=0. \quad (66)$$

For small amplitudes of collective oscillation, there is a correspondingly small disturbance of the motion of each

[19] For a discussion of damping, see D. Bohm and E. P. Gross, Paper B, reference 5.

particle. Thus we may write

$$\mathbf{x}_i=\mathbf{x}_{0i}+\delta\mathbf{x}_i, \quad \mathbf{v}_i=\mathbf{v}_{0i}+\delta\mathbf{v}_i. \tag{67}$$

We obtain the equations of motion for $\delta\mathbf{x}_i$ and $\delta\mathbf{v}_i$ by substituting (67) into the equations of motion (6), using $\rho_k=q_k+\eta_k$. However, the use of the straight-line approximation for the motion of the particles in the absence of oscillation is equivalent to neglecting the force arising from the individual particles part of the charge density, η_k. Thus we may write

$$\delta\dot{\mathbf{x}}_i=\delta\mathbf{v}_i, \tag{68}$$

$$\delta\dot{\mathbf{v}}_i=-(4\pi e^2/m)i\sum_k(\mathbf{k}/k^2)q_k e^{i\mathbf{k}\cdot(\mathbf{x}_{0i}+\delta\mathbf{x}_i)}. \tag{69}$$

We may expand the exponential in (69), and, in the linear approximation, neglect terms like $q_k(\mathbf{k}\cdot\delta\mathbf{x}_i)$ and $(\mathbf{k}\cdot\delta\mathbf{x}_i)^2$. Thus (69) becomes

$$\delta\dot{\mathbf{v}}_i=-(4\pi e^2/m)i\sum_k(\mathbf{k}/k^2)q_k e^{i\mathbf{k}\cdot\mathbf{x}_{0i}}. \tag{70}$$

Using $q_k=(\omega_P^2/2\omega)(\xi_{k,\omega}-\xi_{k,-\omega})$ one can then easily verify that the following represent an approximate steady-state solution of (68) and (69),

$$\delta\mathbf{x}_i\cong i\frac{2\pi e^2\omega_P^2}{m}\sum_k\left\{\frac{\xi_{k,\omega}}{(\omega-\mathbf{k}\cdot\mathbf{v}_{0i})^2}-\frac{\xi_{k,-\omega}}{(\omega+\mathbf{k}\cdot\mathbf{v}_{0i})^2}\right\}\frac{\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{x}_{0i}}}{k^2\omega}. \tag{71}$$

$$\delta\mathbf{v}_i=\frac{2\pi e^2\omega_P^2}{m\omega}\sum_k\frac{\mathbf{k}}{\omega k^2}\left\{\frac{\xi_{k,\omega}}{\omega-\mathbf{k}\cdot\mathbf{v}_{0i}}+\frac{\xi_{k,-\omega}}{\omega+\mathbf{k}\cdot\mathbf{v}_{0i}}\right\}e^{i\mathbf{k}\cdot\mathbf{x}_{0i}}. \tag{72}$$

From (71) and (72), we see that in a state of collective oscillation there is a small wave-like perturbation in the position and velocity of each particle, possessing a definite phase relation to the collective coordinates $\xi_{k,\omega}$ and $\xi_{k,-\omega}$ (i.e., q_k and $\dot{q}_k$). We may calculate the additional contribution to the charge density resulting from this perturbation. This is

$$\delta\rho_k=\sum_i\{e^{-i\mathbf{k}\cdot(\mathbf{x}_{0i}+\delta\mathbf{x}_i)}-e^{-i\mathbf{k}\cdot\mathbf{x}_{0i}}\}$$

$$\cong\sum_i e^{-i\mathbf{k}\cdot\mathbf{x}_{0i}}(-i\mathbf{k}\cdot\delta\mathbf{x}_i) \tag{73}$$

in the linear approximation. Substituting for $\delta\mathbf{x}_i$ from (71) and using the random phase approximation, we obtain

$$\delta\rho_k\cong\frac{2\pi e^2\omega_P^2}{m\omega}\sum_i\left\{\frac{\xi_{k,\omega}}{(\omega-\mathbf{k}\cdot\mathbf{v}_{0i})^2}-\frac{\xi_{k,-\omega}}{(\omega+\mathbf{k}\cdot\mathbf{v}_{0i})^2}\right\}. \tag{74}$$

In order that the part of the charge density arising from the response of the individual particles to the collective part of the charge density q_k should be equal to this same collective charge density, it is necessary that the dispersion relation,

$$1=(4\pi e^2/m)\sum_i\{1/(\omega-\mathbf{k}\cdot\mathbf{v}_{0i})^2\}, \tag{75}$$

be satisfied for both $\pm\omega$, as may be verified using (42). But within the linear approximation the dispersion relation (75) is identical with the dispersion relation (23) derived earlier. Thus our dispersion relation guarantees the self-consistency of our assumption of the collective oscillation arising from the cumulative small response of the individual particles to the collective field.

The consistency of the separation of the charge density into q_k and η_k further requires that η_k, in the linear approximation, be independent of q_k. Thus the change in η_k resulting from the $\delta\mathbf{x}_i$ and the $\delta\mathbf{v}_i$ should vanish. This change is, by our definition of η_k, (30),

$$\delta\eta_k=\sum_i\left[\left\{1-\frac{\omega_P^2}{\omega^2-(\mathbf{k}\cdot\mathbf{v}_{0i})^2}\right\}\{-i\mathbf{k}\cdot\delta\mathbf{x}_i\}-\left\{\frac{2\omega_P^2\mathbf{k}\cdot\mathbf{v}_{0i}\mathbf{k}\cdot\delta\mathbf{v}_i}{[\omega^2-(\mathbf{k}\cdot\mathbf{v}_{0i})^2]^2}\right\}\right]e^{-i\mathbf{k}\cdot\mathbf{x}_{0i}}. \tag{76}$$

On substituting for $\delta\mathbf{x}_i$ and $\delta\mathbf{v}_i$ from (71) and (72), and using the random phase approximation it is readily established that $\delta\eta_k=0$.

VI. RANDOM PHASE APPROXIMATION

In this section we justify our use of the random phase approximation, which is perhaps the central approximation in our treatment of the electron gas. We justify this approximation in detail for a specific case. The justification of the approximation as used elsewhere in the paper can be carried out using similar methods.

We first encounter the random phase approximation in Eq. (8), where we neglect the term

$$R=\sum_{\substack{ijk'\\k'\neq k}}\frac{4\pi e^2}{m}\frac{\mathbf{k}\cdot\mathbf{k}'}{(k')^2}\exp[i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}_i]\exp[-i\mathbf{k}'\cdot\mathbf{x}_j]$$

$$=\frac{4\pi e^2}{m}\sum_{\substack{k'\\k'\neq k}}\frac{\mathbf{k}\cdot\mathbf{k}'}{(k')^2}\rho_{k'-k}\rho_{k'}, \tag{77}$$

in comparison with the term

$$S=(4\pi e^2/m)\sum_j e^{-i\mathbf{k}\cdot\mathbf{x}_j}. \tag{78}$$

If the particles were distributed at random, the average of R would be zero. However, the value of R would tend to undergo random fluctuations as a result of the random motions of the particles. These fluctuations would produce a corresponding random wavering of the frequency of plasma oscillations, and would lead to coupling between oscillations of different wavelengths, (as can be seen from (77) and (8)). The random phase approximation will be justified provided we can show that the effects of these fluctuations can be neglected.

Before we attempt to estimate the size of R, it will be convenient to rewrite it in a form which groups together certain pairs of terms having definite phase relations. To do this, we interchange i and j in (77) and replace $\mathbf{k}'$ by $\mathbf{k}'-\mathbf{k}$. Since the substitutions do not change the value of the sum, we can re-express R as the

mean of the two expressions. We obtain

$$R=(2\pi e^2/m)\sum_{\substack{ijk'\\k'\neq k}}\left\{\frac{k^2}{(\mathbf{k}-\mathbf{k}')^2}+\frac{\mathbf{k}\cdot\mathbf{k}'(k^2-2\mathbf{k}\cdot\mathbf{k}')}{(k')^2(\mathbf{k}-\mathbf{k}')^2}\right\}$$
$$\times\exp[i(\mathbf{k}'-\mathbf{k})-(\mathbf{x}_i-\mathbf{x}_j)]\exp(-i\mathbf{k}\cdot\mathbf{k}_j). \quad (79)$$

Let us begin with first term in the above expression, which we rewrite with the aid of the substitution, $\mathbf{k}'-\mathbf{k}=\mathbf{k}''$, as

$$R'=(2\pi e^2k^2/m)\sum_{\substack{ijk''\\k''\neq 0}}[1/(k'')^2]$$
$$\times\exp[i\mathbf{k}''\cdot(\mathbf{x}_i-\mathbf{x}_j)]\exp(-i\mathbf{k}\cdot\mathbf{x}_j). \quad (80)$$

The expression

$$\sum_{\substack{ijk''\\k''\neq 0}}[1/(k'')^2]\exp[i\mathbf{k}''\cdot(\mathbf{x}_i-\mathbf{x}_j)]$$

is proportional to the potential of the jth electron in the field of all of the other electrons plus that of the "smeared out" distribution of positive charge. This potential is

$$\varphi(\mathbf{x}_j)=4\pi e\sum_{\substack{k\\k\neq 0}}(\rho_k/k^2)e^{i\mathbf{k}\cdot\mathbf{x}_j}. \quad (81)$$

Now we wish to calculate the mean fluctuation in this quantity. To do this, we first obtain $[\varphi(\mathbf{x}_j)]^2$, and then average over-all values of ρ_k. We have, from (81),

$$[\varphi(\mathbf{x}_j)]^2=(4\pi e)^2\sum_{kk'}[\rho_k\rho_{k'}/k^2(k')^2]\exp[i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}_j].$$

We shall average ρ_k under the assumption that the system is in thermodynamic equilibrium. This is close to the situation, usually present in practice, in which there are organized oscillations of small amplitude superposed on a background of random thermal motion.[20] From Appendix I [Eqs. (A10) and (A11)] we find that the probability of a given set of ρ_k is

$$W(\cdots\rho_k\cdots)\sim\exp\{-\textstyle\sum_k|\rho_k|^2[(4\pi e^2/k^2\kappa T)+(1/n)]\}$$
$$\times\{\cdots d(r_k^2/1)\cdots\}\{\cdots d\varphi_k\cdots\},$$

where $\rho_k=r_ke^{i\varphi_k}$. As shown in Appendix I, this expression includes the effects of the correlations in particle positions brought about by the Coulomb force. Since W does not depend on φ_k products like $\rho_k\rho_{k'}=r_kr_{k'}e^{i(\varphi_k+\varphi_{k'})}$ average out to zero unless $k=-k'$. Using this fact, we find

$$\langle[\varphi(\mathbf{x}_j)]^2\rangle_{Av}=(4\pi e)^2\sum_{k\neq 0}\langle r_k^2\rangle_{Av}/k^4. \quad (82)$$

[20] The assumption of thermal equilibrium in the estimation of $[\varphi(\mathbf{x}_j)]^2$ necessarily neglects those irreversible effects associated with the damping of the collective oscillation. Actually, the effects of damping must be contained in terms like R, since these effects are not included in the terms we retain in the random phase approximation. However, as was pointed out in Sec. V, for electron gases of interest this damping will be small.

On substituting for $\langle r_k^2\rangle_{Av}$ its value $nk^2/[k^2+(4\pi ne^2/\kappa T)]$ as obtained in (A12), we obtain

$$\langle[\varphi(\mathbf{x}_j)]^2\rangle_{Av}=(4\pi e)^2\sum_{k\neq 0}\frac{n}{k^2[k^2+(4\pi e^2/\kappa T)n]}. \quad (83)$$

We transform the sum over k to an integral, and obtain, after integration,

$$\langle[\varphi(\mathbf{x}_j)]^2\rangle_{Av}=(4\pi e)^2(\lambda_D/4\pi)n. \quad (84)$$

From (80) and (84) we see that the fluctuation in R', $\delta R'$ is thus given by

$$\delta R'=(\pi n\lambda_D)^{\frac{1}{2}}e^2\rho_k(k^2/m). \quad (85)$$

This must be compared with the term S given in (78). The ratio is

$$\frac{\delta R'}{S}=\frac{\delta R'}{\omega_P{}^2\rho_k}=\frac{k^2a^{\frac{3}{2}}\lambda_D{}^{\frac{1}{2}}}{4\pi^{\frac{1}{2}}}=\frac{k^2\langle v^2\rangle_{Av}}{\omega_P{}^2}\left(\frac{a}{\lambda_D}\right)^{\frac{3}{2}}\frac{1}{12\pi^{\frac{1}{2}}}, \quad (86)$$

where $n=1/a^3$.

We see that the modification in the equation of motion (8) resulting from the fluctuation in R' is of order $[k^2\langle v^2\rangle_{Av}/\omega_P{}^2]$, but is multiplied by a term of order $\sim(1/20)(a/\lambda_D)^{\frac{3}{2}}$, which is very small in most electron gases.

For a typical density of 10^{12} per cm^3, we have $\lambda_D\sim10^{-3}$ cm and $a=10^{-4}$ cm. Thus not only does the R' term introduce a very small correction to the net frequency, but this correction is much less than the $[k^2\langle v^2\rangle_{Av}/\omega_P{}^2]$ correction which arises when we consider the effect of thermal motions. In the case described, the fluctuations produce a fractional correction of about 10^{-3} in the $[k^2\langle v^2\rangle_{Av}/\omega_P{}^2]$ term.

The correction $(1/20)(a/\lambda_D)^{\frac{3}{2}}$ would only become appreciable for $\lambda_D\ll a$. Such cases occur in practice only for degenerate electron gases. In these cases, however, Fermi statistics reduce the effects of the fluctuations relative to the Boltzmann estimate, so that the random phase approximation will still be justified. This problem will be considered in our paper extending these results to the quantum theory.

It can be shown by arguments similar to the foregoing that the remaining terms in R are of the same order of magnitude as R' or smaller. A similar justification can be applied for the use of the random phase approximation elsewhere in this paper.

VII. CONCLUSION

In conclusion we give a brief summary of our results in terms of a physical picture of the behavior of the electron gas. As we have seen, the density fluctuations can be split into two approximately independent components, associated, respectively, with the collective and individual particle aspects of the assembly. The collective component, which is present only for wavelengths $>\lambda_D$, represents organized oscillation brought about by the long-range part of the Coulomb interaction. When

such an oscillation is excited, each individual particle suffers a small perturbation of its velocity and position arising from the combined potential of all the other particles. The contribution to the density fluctuations resulting from these perturbations is in phase with the potential producing it, so that in an oscillation we find a small organized wave-like perturbation superposed on the much larger random thermal motion of the particle. The cumulative potential of all the particles may, however, be considerable because the long range of the force permits a very large number of particles to contribute to the potential at a given point.

The individual particles component of the density fluctuation is associated with the random thermal motion of the particles and shows no collective behavior. It represents the individual particles surrounded by comoving clouds which screen their fields within a distance $\sim\lambda_D$. Thus it describes an assembly of effectively free particles interacting only through the short-range part of the Coulomb force. The screening of the field of a given particle is actually brought about by the Coulomb repulsion which leads to a deficiency of electrons in the immediate neighborhood of the particle. This same process also leads to a large reduction in the random fluctuations of the density in the electron gas for wavelength larger than λ_D.

When we fix our attention on a specific individual electron in the assembly, we may study its interaction with the other electrons by applying our split-up of the density fluctuations to the remainder of the gas. The response of the collective part of the density fluctuation to the field of the specified particle leads to two significant effects: the screening of this particle's field within a distance $\sim\lambda_D$, and the excitation of collective oscillations when the speed of the particle is greater than mean thermal speed. The particle also interacts with the individual particles component of the density fluctuations. However, the field that is the result of this component may be viewed as the sum of the screened fields of individual particles. Thus the interaction of our specified particle with the individual particles component of the density fluctuation can be described in terms of short-range collisions between pairs of particles.

We have used the random phase approximation throughout this paper. The use of this approximation is equivalent to the neglect of the damping and wavering-in-frequency of the collective oscillation resulting from the individual particles character of the electron gas. When the random phase approximation is justified, the collective component and the individual particles component of the density fluctuations will not be significantly coupled, and thus can be treated independently.

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APPENDIX I. STATISTICAL MECHANICAL TREATMENT OF DENSITY FLUCTUATIONS IN THE ELECTRON GAS

In this appendix, we shall calculate the mean amplitude of fluctuation of the Fourier components, ρ_k, of the density in an electron gas. The method used here provides an alternative approach to the screening problem, and yields certain results that we apply in the verification of the random phase approximation.

We begin with the well-known statistical mechanical expression for the probability that the electron gas is in a "microscopic" state in which the nth particle lies between $\mathbf{x}_n$ and $\mathbf{x}_n+\mathbf{dx}_n$, while its momentum lies between $\mathbf{p}_n$ and $\mathbf{p}_n+\mathbf{dp}_n$,

$$W'(\mathbf{x}_1, \mathbf{x}_2\cdots\mathbf{x}_n\cdots\mathbf{p}_1\cdots\mathbf{p}_n\cdots) \sim e^{-(E/\kappa T)}d\mathbf{x}_1d\mathbf{x}_2\cdots d\mathbf{x}_n\cdots d\mathbf{p}_1\cdots d\mathbf{p}_n\cdots, \quad \text{(A1)}$$

where E is the total energy of the system, kinetic plus potential,

$$E=\sum_n(p_n^2/2m)+\tfrac{1}{2}\sum_{\substack{mn\\ m\neq n}} V(\mathbf{x}_m-\mathbf{x}_n).$$

We can carry out the integral over momentum space, obtaining

$$W(\mathbf{x}_1, \mathbf{x}_2\cdots\mathbf{x}_n) \sim\exp\{-\tfrac{1}{2}[\sum_{\substack{mn\\ m\neq n}} V(\mathbf{x}_m-\mathbf{x}_n)]/\kappa T\}d\mathbf{x}_1\cdots d\mathbf{x}_n\cdots. \quad \text{(A2)}$$

Let us now divide the available space into cells of volume ϵ so small that no important physical property changes within them, but large enough to contain enough particles so that the number of particles in the cell can be regarded as approximately continuous. The smallest suitable value of ϵ corresponds to about 4 or 5 inter-atomic spacings. The state of the system can then be specified in terms of the number of particles $(N_1, N_2\cdots N_i\cdots)$ in each cell. It is readily shown by methods well-known in statistical mechanics that

$$W(N_1\cdots N_i\cdots) \sim\frac{[\exp\{-(1/2\kappa T)\sum_{i\neq j}N_iN_jV(\mathbf{x}_i-\mathbf{x}_j)\}]N!}{(N_1)!(N_2)!\cdots(N_i)!\cdots}, \quad \text{(A3)}$$

where $N=\sum_i N_i$. Here, $\mathbf{x}_i$ refers to the mean coordinate of the ith cell. For the Coulomb potential, we can, as is usually done in electrostatics, neglect the interaction of particles in a given cell. Using Stirling's approximation for $N_i!$ we obtain

$$W(N_1\cdots N_i\cdots)\sim\exp\{N\ln N-\textstyle\sum_i N_i\ln N_i -(1/2\kappa T)\sum_{i\neq j}N_iN_jV(\mathbf{x}_i-\mathbf{x}_j)\}. \quad \text{(A4)}$$

Let N_0 be the mean number of electrons in a cell. We shall be interested in the small fluctuations, δN_i about

this mean number. Thus we can write

$$N_i = N_0 + \delta N_i. \tag{A5}$$

We now substitute (A5) into (A4) and expand the argument of the exponential in a series of powers of $\delta N/N$, retaining only second-order terms. In doing this, we note that since the electrons are embedded in a uniform distribution of positive charge of density N_0/ϵ, the electrostatic energy is

$$e^2 \sum_{i\neq j}(N_i - N_0)(N_j - N_0)/|\mathbf{x}_i - \mathbf{x}_j| = e^2 \sum_{i\neq j} \delta N_i \delta N_j/|\mathbf{x}_i - \mathbf{x}_j|.$$

We then obtain (using $\sum_i \delta N_i = 0$ and leaving out constant factors)

$$W(N_1 \cdots N_i \cdots) \sim \exp\{-(e^2/2\kappa T)\sum_{i\neq j} \delta N_i \delta N_j/|\mathbf{x}_i - \mathbf{x}_j| - \textstyle\sum_i (\delta N_i)^2/2N_0\}. \tag{A6}$$

We now expand δN_i as a Fourier series with periodic boundary conditions in a box of unit volume.

$$\delta N_i = \sum_{k\neq 0} \rho_k \epsilon e^{i\mathbf{k}\cdot\mathbf{x}_i}, \tag{A7}$$

with $\rho_k^* = \rho_{-k}$. Since ρ_k is complex, it will be convenient to write it as

$$\rho_k = r_k e^{i\varphi_k}, \tag{A8}$$

with $r_k = r_{-k}$, $\varphi_k = -\varphi_{-k}$. Only half of the ρ_k are independent. It is readily verified that we obtain

$$(e^2/2)\sum_{i\neq j}[\delta N_i \delta N_j/|\mathbf{x}_i - \mathbf{x}_j|] = 2\pi e^2 \textstyle\sum_k (1/k^2)|\rho_k|^2 = 2\pi e^2 \sum_k (1/k^2) r_k^2, \tag{A9}$$

and

$$W \sim \exp\{-\textstyle\sum_k [(2\pi e^2/k^2\kappa T) + (1/2n)] r_k^2\}, \tag{A10}$$

where n is the average electron density.

We obtain the volume element in the space of r_k and φ_k by regarding the N_i as continuous variables, and by assuming that we have chosen $dN_i = 1$ in Eq. (A6). (A6) may therefore be multiplied by the volume element in N_i space (dN_1, $dN_2 \cdots dN_1 \cdots$), which is also unity. The volume element in the space of r_k and φ_k is then given by the Jacobian of the transformation from the N_i space. This is

$$J \sim [d(r_{k1}^2/2) d(r_{k2}^2/2) \cdots d(r_{ki}^2/2)] \times [d\varphi_{k1} \cdots d\varphi_{ki} \cdots]. \tag{A11}$$

The probability WJ is a product of separate functions, one for each k. This indicates that in the approximations used, the statistical fluctuations associated with each k are independent.

We now calculate the mean value of $|\rho_k|^2 = (r_k)^2$. In obtaining this, we note that $r_{-k}^2 = r_k^2$, so that we must replace r_k^2 in (A10) by $2r_k^2$, but sum only over half the values of k. We get

$$\langle r_k^2 \rangle_{Av} = \frac{\int\int \cdots \int W r_k^2 (\cdots d(r_k^2) \cdots)(\cdots d\varphi \cdots)}{\int\int \cdots \int W (\cdots d(r_k^2) \cdots)(\cdots d\varphi_k \cdots)} = \frac{nk^2}{(4\pi n e^2/\kappa T) + k^2}. \tag{A12}$$

For $k^2 \gg k_D^2$ we see that $\langle r_k^2 \rangle_{Av}$ reduces to n, the value which would exist in a random distribution. For $k^2 \ll k_D^2$ we obtain $\langle r_k^2 \rangle_{Av} = nk^2\lambda_D^2$. Thus, we see that for long distances, the Coulomb forces cause the fluctuations to be greatly reduced.

We can now write down the mean Coulomb potential energy associated with each k, noting from Eq. (A9), that the contributions of different k are independent. This mean potential is

$$\langle V_k \rangle_{Av} = \langle 2\pi e^2 \rho_k^2/k^2 \rangle_{Av} = \tfrac{1}{2}(\kappa T/1 + k^2\lambda_D^2). \tag{A13}$$

For $k \ll k_D$ the mean potential energy is just $\kappa T/2$. If we recall that for long wavelengths, the ρ_k act almost entirely collectively and undergo simple harmonic motion, then the previous result can easily be understood in terms of the equipartition theorem which states that the mean potential energy of a harmonic oscillator is $\kappa T/2$. For $k \gg k_D$, $\langle V_k \rangle_{Av}$ becomes $2\pi n e^2/k^2$. This is just what is obtained by assuming that the particles move with a random distribution in space. These results show in another application that for short distances, the individual particles point of view gives the right results while for long distances, the organization resulting from the Coulomb forces must be taken into account.

In thermodynamic equilibrium, the energy is actually distributed partly in the collective oscillation and partly in the random thermal motion of the individual particles. Each of the collective modes will have a mean energy of κT. This energy is usually very small in comparison to the collective energy that would be present if one of the modes were excited systematically. For even though each particle makes only a small contribution to the collective modes, the cumulative contributions of all the particles to a given mode may result in energies very much greater than κT. It also follows that the collective coordinate q_k, is not quite equal to zero in the state of thermodynamic equilibrium. But since κT is very much less than what would be present if a given q_k were systematically excited, the state, $q_K = 0$, is not far from the state of thermodynamic equilibrium.

Finally, let us note that the approximation of regarding N_i as continuous can be justified rigorously only for values of k appreciably smaller than $1/a$, where a is the interparticle spacing. Nevertheless, it turns out that for large values of k, this procedure still gives what we know to be the right value of $\langle r_k^2 \rangle_{Av}$, *viz.*, n. This

indicates that (A12) is essentially correct over all values of k.

APPENDIX II. COLLECTIVE DESCRIPTION OF PARTICLE INTERACTIONS

The approach used in Secs. II and III to study the organization brought about by the Coulomb interactions can be extended to a general repulsive particles interaction. In this appendix we indicate briefly the appropriate generalization of the results obtained in the above sections. We assume the interparticle potential is $V(|\mathbf{x}_i-\mathbf{x}_j|)$ and express it as a Fourier series in a box of unit volume with periodic boundary conditions as:

$$V(|\mathbf{x}_i-\mathbf{x}_j|)=\sum_k V_k e^{i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}. \tag{A14}$$

The equation of motion of the ith particle is thus

$$\ddot{\mathbf{x}}_i=-(i/m)\sum_{kj}\mathbf{k}V_k e^{i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}. \tag{A15}$$

The time variation of the kth density fluctuation is given by

$$d^2\rho_k/dt^2=-\sum_i(\mathbf{k}\cdot\mathbf{v}_i)^2e^{-i\mathbf{k}\cdot\mathbf{x}_i} -\sum_{k'ij}(V_{k'}/m)\mathbf{k}\cdot\mathbf{k}'\{\exp[i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}_i]\} \times\{\exp(-i\mathbf{k}'\cdot\mathbf{x}_j)\}. \tag{A16}$$

We now assume that the random phase approximation may be applied to this case. The use of this approximation cannot be justified in general, but must be considered separately for each law of force. When it is valid Eq. (A16) may be reduced to

$$d^2\rho_k/dt^2=-\sum_i(\mathbf{k}\cdot\mathbf{v}_i)^2e^{-i\mathbf{k}\cdot\mathbf{x}_i} -\sum_j(nV_k/m)k^2e^{-i\mathbf{k}\cdot\mathbf{x}_j}. \tag{A17}$$

The physical significance of the terms on the right-hand side of (A17) is the same as for the corresponding terms in Eq. (9): the first term represents the contributions from the random thermal motions of individual particles; the second term represents the effects of particles interaction and can lead to organized behavior.

Under conditions such that the individual particles term can be neglected in (A17), we have

$$d^2\rho_k/dt^2+(nk^2V_k/m)\rho_k=0, \tag{A18}$$

and the ρ_k carry out oscillations of frequency,

$$\omega=(nk^2V_k/m)^{\frac{1}{2}}. \tag{A19}$$

The criterion that the ρ_k display predominantly collective behavior, and hence that a collective description be appropriate, is

$$(nk^2V_k/m)\gg\langle(\mathbf{k}\cdot\mathbf{v}_i)^2\rangle_{\text{Av}}. \tag{A20}$$

For an isotropic velocity distribution, Eq. (A20) becomes

$$nV_k\gg(2/3)\mathcal{T}, \tag{A21}$$

where $\mathcal{T}$ denotes the mean particle kinetic energy. Thus we see that a strong interaction and a high particle density favor organized behavior, while high random thermal velocities oppose it.

Just as was done for the Coulomb interactions, we can find a collective coordinate q_k which oscillates harmonically when random thermal motions are considered, and thus obtain a more exact dispersion relation than Eq. (A19). The appropriate normalized, collective coordinate is

$$q_k=\sum_i\frac{(nk^2V_k/m)}{\omega^2-(\mathbf{k}\cdot\mathbf{v}_i)^2}e^{-i\mathbf{k}\cdot\mathbf{x}_i}, \tag{A22}$$

where ω satisfies the following dispersion relation

$$1=\sum_j(k^2V_k/m)/[\omega-(\mathbf{k}\cdot\mathbf{v}_j)]^2, \tag{A23}$$

We may obtain an approximate solution for (A23) in the limit of small $(\mathbf{k}\cdot\mathbf{v}_j/\omega)$ which is essentially the same limit as that in which the criterion (A21) is satisfied. The expansion of (A23) in powers of $(\mathbf{k}\cdot\mathbf{v}_j/\omega)$ yields

$$\omega^2=(nk^2V_k/m)+(m\langle v^2\rangle_{\text{Av}}/nV_k), \tag{A24}$$

which for sufficiently small $(\mathcal{T}/nV_k)$ reduces to the dispersion relation obtained earlier (Eq. (A19)).

It is also possible to split up the density fluctuation into its collective component q_k, and an individual particles component η_k. It can easily be verified that with the choice (A22) for q_k, one obtains:

$$\rho_k=q_k+\eta_k, \tag{A25}$$

with

$$\eta_k=\sum_i\frac{\omega^2-(\mathbf{k}\cdot\mathbf{v}_i)^2-(k^2V_kn/m)}{\omega^2-(\mathbf{k}\cdot\mathbf{v}_i)^2}e^{-i\mathbf{k}\cdot\mathbf{x}_i}. \tag{A26}$$

It may be seen that when the criterion for collective behavior (A21) is satisfied, the corresponding density fluctuation is essentially collective, and may be described by q_k.