The Force between Particles in a Nonlinear Field Theory

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It is shown for a general class of scalar nonlinear classical field theories, in which singularities are excluded and particles are represented by small regions in which the field is intense, that the interaction between two particles is described by the Yukawa potential at large distances.

1. INTRODUCTION

In recent years a number of attempts have been made to set up a nonlinear field theory of elementary particles in which singularities are excluded.[1–4] In such a theory a particle is represented by a small portion of space in which some function representing matter density has a large value. Such a theory can be expected to be free from at least some of the divergences present in theories in which particles are represented by points, i.e., by singularities. It also has the advantage that the equations of motion of a particle are a consequence of the field equations.

It is of interest to know something concerning the nature of the force which one particle exerts on another according to a nonlinear field theory. For this purpose a very simple classical nonlinear theory has been investigated. The field in this case is taken to be described by a complex scalar ψ, the behavior of which is determined by the Lagrangian density function

$$\mathfrak{L}=-\frac{\partial\psi}{\partial x_\lambda}\frac{\partial\psi^*}{\partial x_\lambda}-\sigma^2\psi\psi^*+\tfrac{1}{2}g\psi^2\psi^{*2}, \tag{1}$$

where σ^2 and g are positive constants and

$$(x_1, x_2, x_3, x_4)=(x, y, z, ict).$$

This field was recently discussed to some extent by Finkelstein, LeLevier, and Ruderman[4] incidentally to the treatment of a more complicated case.

If one uses the Lagrangian density (1) in a four-dimensional variational principle, one gets for the field equation

$$(\Box^2-\sigma^2)\psi+g\psi^2\psi^*=0. \tag{2}$$

By the usual methods one also finds that the charge-current density vector is given by

$$s_\mu=-i\epsilon(\psi\partial\psi^*/\partial x_\mu-\psi^*\partial\psi/\partial x_\mu), \tag{3}$$

where ϵ is a constant, and the energy-momentum density tensor is given by

$$T_{\mu\nu}=\frac{\partial\psi}{\partial x_\mu}\frac{\partial\psi^*}{\partial x_\nu}+\frac{\partial\psi}{\partial x_\nu}\frac{\partial\psi^*}{\partial x_\mu}+\mathfrak{L}\delta_{\mu\nu}. \tag{4}$$

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[1] N. Rosen, Phys. Rev. **55**, 94 (1939).
[2] R. J. Finkelstein, Phys. Rev. **75**, 1079 (1949).
[3] S. D. Drell, Phys. Rev. **79**, 220 (1950).
[4] Finkelstein, LeLevier, and Ruderman, Phys. Rev. **83**, 326 (1951); henceforth referred to as FLR.

2. SINGLE PARTICLE

In the stationary spherically symmetric case one sets, in polar coordinates,

$$\psi=\theta(r)e^{i\omega t}, \tag{5}$$

where θ and ω are real, and ω is constant. The field equation becomes

$$d^2\theta/dr^2+(2/r)d\theta/dr-\alpha^2\theta=-g\theta^3, \tag{6}$$

where $\alpha^2=\sigma^2-\omega^2/c^2$ will be assumed to be positive.

It follows that in this case there exist solutions that are everywhere analytic and go to zero exponentially at infinity.[5] The two simplest solutions, one without nodes, the other with one node, obtained by numerical integration, are shown in Fig. 1.[6] We shall restrict ourselves hereafter to the nodeless solution, as representing the ground state of the particle.

From the solution describing a particle, on the basis of Eqs. (3) and (4), one obtains for the charge of the particle,

$$Q\equiv-ic^{-1}\int s_4dV=2\epsilon\omega c^{-2}\int\theta^2dV, \tag{7}$$

and for the energy of the particle, after making use of the field equation and carrying out an integration by

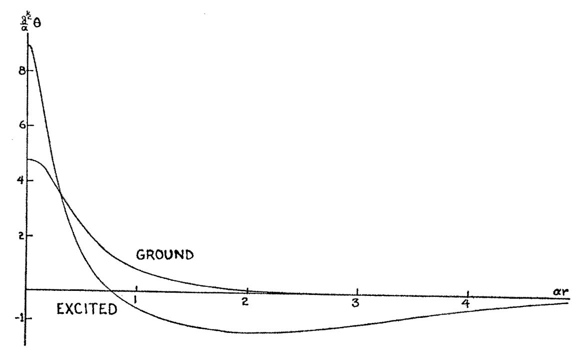

FIG. 1. The two simplest particle-like solutions of Eq. (6).

[5] This is shown by FLR by an analysis of the solutions in the phase plane. It follows from their arguments that there exists at least one particle-like solution with any given number of nodes. This result may perhaps be more directly seen by transforming (6) into $\eta''/\eta=1-(\eta/x)^2$ by $x=\alpha r$, $\eta=g^{\frac{1}{2}}r\theta$, and considering solutions with $\eta(0)=0$ as functions of increasing $\eta'(0)$.
[6] The solution exhibited in Fig. 2 of FLR is, as nearly as can be determined in the absence of a scale, identical with our nodeless solution.

parts,

$$W \equiv -\int T_{44} dV = \int (2\omega^2 c^{-2}\theta^2 + \tfrac{1}{2} g\theta^4) dV. \quad (8)$$

Thus the energy of the particle is positive definite, although the energy density $-T_{44}$ is not.

From Eq. (6) it is seen that, for large values of r for which θ is small so that the right-hand side is negligible, the particle-like solution will have a behavior given by

$$\theta = A g^{-\frac{1}{2}} e^{-\alpha r}/r, \quad (9)$$

where A is a constant. A rough numerical integration gives $A \sim 2.5$.

It is seen that the particle-like solutions of (6) depend on the frequency ω. In the case of a neutral particle Eq. (7) shows that $\omega = 0$, so that ψ is real. For a charged particle some criterion would have to be adopted to fix the value of ω, the sign of which depends on the sign of the charge.

3. INTERACTION BETWEEN PARTICLES

We now consider the question of the interaction between two identical particles which are nearly at rest. The exact way of dealing with two particles would be to find a solution of the field equation (2) for which there are two small regions in which $|\psi|$ has an appreciable value. However, this is not feasible. Instead, we limit ourselves to the case in which the two particles are far apart and take as the approximate solution of (2),

$$\psi = e^{i\omega t}[\theta(r_A) + \theta(r_B)], \quad (10)$$

where the points A and B are the centers of the two particles, r_A and r_B are the distances from A and B to a point in the field, and $\theta(r)$ is the ground-state solution of (6) for a single particle. The assumed form of the solution (10) is justified to some extent by the fact that for two particles far apart the nonlinear term in (2) is small in the region where $\theta(r_A)$ and $\theta(r_B)$ overlap, and hence the sum of the two solutions is approximately a solution. In order for (10) to be a good approximation in spite of the nonlinearity of the equation, the points A and B must move in a suitable way in the course of time.

From this expression for ψ one can calculate the force exerted by one particle on the other. One way of doing this is to make use of the divergence equation satisfied by $T_{\mu\nu}$,

$$\partial T_{\mu\lambda}/\partial x_\lambda = 0. \quad (11)$$

In particular, if for $\mu = k = 1, 2, 3$ we integrate this equation over a three-dimensional volume V bounded by a closed surface S and make use of Gauss's theorem we get

$$\frac{d}{dx_4}\int_V T_{k4} dV = -\int_S T_{kn} dS, \quad (12)$$

where the subscript n indicates the component in the direction of the outward normal. Since the left-hand side of (12) represents the time-rate of change of momentum contained in V, the right-hand side represents the force acting on this region, which also follows from the fact that the space components of $T_{\mu\nu}$ represent stress components. Thus we can write the force components

$$F_k = -\int_S T_{kn} dS. \quad (13)$$

If we choose the surface S so that it encloses one particle (this will be practically the case if the smallest distance from the particle center to the surface is large compared to $1/\alpha$), then (13) will give the force on this particle.

If one chooses the surface S sufficiently large so that everywhere on it Eq. (9) is valid for both $\theta(r_A)$ and $\theta(r_B)$, the calculation becomes rather simple. Thus, suppose that at a certain moment A and B are located on the z-axis each a distance $R/2$ from the origin, R being the inter-particle distance, and we take for S the plane of symmetry midway between the particles (closed with an infinite hemisphere, which gives no contribution). It is enough to calculate

$$F_z = -\int_S T_{zz} dS,$$

and in the expression for T_{zz} it is enough to consider only the terms involving products of $\theta(r_A)$ and $\theta(r_B)$ or their derivatives to get the particle interaction. For a sufficiently large value of R one can neglect the fourth-degree terms in T_{zz}, since on S these will be small compared to the remaining terms. Under these conditions, making use of the symmetry of the problem, one finds that

$$T_{zz} = -4\alpha^2\theta^2(r_A), \quad (14)$$

and one gets for the force of attraction

$$F = 8\pi A^2\alpha^2 g^{-1}\int_0^\infty \exp[-2\alpha(\rho^2 + R^2/4)^{\frac{1}{2}}] \times (\rho^2 + R^2/4)^{-1}\rho d\rho \quad (15)$$
$$= -8\pi A^2\alpha^2 g^{-1}\,\mathrm{Ei}(-\alpha R),$$

or, assuming αR to be large, the force of attraction is given by

$$F(R) = 8\pi A^2 \alpha g^{-1} e^{-\alpha R}/R, \quad (16)$$

and the interaction potential energy therefore by

$$U(R) = -8\pi A^2 g^{-1} e^{-\alpha R}/R. \quad (17)$$

For a neutral particle, $\omega = 0$, $\alpha = \sigma$, this gives

$$U(R) = -8\pi A^2 g^{-1} e^{-\sigma R}/R. \quad (18)$$

We see then that the interaction between the two particles at large distances is described by the Yukawa potential.[7]

An inspection of the method used to obtain this result shows that it would also be obtained under more general conditions than those assumed here. If, in the Lagrangian density (1), the last term were replaced by some other function which also led to particle-like solutions, and which was negligible compared to the other terms at large distances from the particle center, then the same result would be obtained for the interaction, except for the numerical coefficient. For example, this would be the case if the last term were taken proportional to $(\psi\psi^*)^n$ for any $n>1$.

[7] H. Yukawa, Proc. Phys.-Math. Soc. Japan (3) 17, 48 (1935).

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Fine Structure of the Hydrogen Atom.* III

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The third paper of this series provides a theoretical basis for analysis of precision measurements of the fine structure of hydrogen and deuterium. It supplements the Bechert-Meixner treatment of a hydrogen atom by allowing for the presence of a magnetic field, as well as radiative corrections. The theory of hyperfine structure is somewhat extended. Stark effects due to motional and other electric fields are calculated. Possible radiative and nonradiative corrections to the shape and location of resonance peaks are discussed. Effects due to the finite size of the deuteron are also considered.

A theory of the sharp resonances $2^2S_{\frac{1}{2}}(m_s=\frac{1}{2})$ to $2^2S_{\frac{1}{2}}(m_s=-\frac{1}{2})$ is given which leads to an understanding of the peculiar shapes of resonance curves shown in Part II. In this connection, a violation of the "no-crossing" theorem of von Neumann and Wigner is exhibited for the case of decaying states.

THE earlier Parts[67] I and II of this paper have described some qualitative studies of the fine structure of hydrogen and deuterium made by a microwave method. In order to prepare the ground for analysis of much more highly precise measurements in Part IV, it is necessary to make available a more refined theory of the hydrogen atom than was used previously. The object of Part III is to supply this need, as well as to treat a number of other theoretical problems which arise in the work. Frequent references to Parts I and II are made. Chapters, sections, figures, tables, equations, and footnotes of Part III are numbered consecutively after those of Parts I and II.

J. ENERGY LEVELS OF A HYDROGEN-LIKE ATOM

48. General Program

The results of theory for the energy levels of an ideal hydrogen atom were given in Part I assuming an infinitely heavy nucleus, thereby neglecting reduced mass effects as well as magnetic and retarded interaction between electron and nucleus. In addition, a number of other approximations were made. The calculation of hyperfine structure was oversimplified by assumption of Back-Goudsmit and Russell-Saunders coupling. In the theories of Zeeman effect and doublet separation $P_{\frac{3}{2}}-P_{\frac{1}{2}}$ the anomalous magnetic moment of the electron was neglected. Shifts of levels due to Stark effect and relativistic and higher order corrections to Zeeman splitting were ignored.

There is no one place in the literature where a treatment of all these effects may be found. One may only form a patchwork Hamiltonian by collecting separate terms from papers by various authors who have been concerned with limited aspects of the problem. It would probably not be justified here to give a detailed systematic theory, but it does seem worthwhile to indicate the basis of the rather provisional treatment which is now possible. The object is to write down all terms known at present having a potential magnitude of 0.1 Mc/sec or larger in the discussion of the precision experiments of Part IV.

The electron and proton should be allowed to interact with one another through their intermediate coupling with the quantized electromagnetic field and the vacuum of occupied negative energy states for electrons and protons. By eliminating these effects from the theory, one hopes to find an equivalent two-body problem in which the two particles have a velocity and spin dependent interaction with one another, and the particles themselves have somewhat changed properties (renormalization of charge and mass, anomalous magnetic moments, etc.).

At present, this program has not been fully carried out. Those terms of low orders in the fine structure constant which have been found will be incorporated into the following discussion. It should be relatively easy to make the small corrections necessary when any missing terms have been calculated.

The starting point is here taken to be a two-body Dirac equation for electron and nucleus. Even when the nucleus is a proton and not a deuteron there might be grave doubt that it would obey an equation of the Dirac type in view of its anomalous magnetic moment.

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[67] W. E. Lamb, Jr., and R. C. Retherford, Part I, Phys. Rev. 79, 549 (1950), and Part II, 81, 222 (1951).