Mean-Value Calculations for Projected Multiple Scattering*

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The multiple-scattering theories of Molière and Snyder-Scott are compared and the equivalence of the mathematical development stated. Using the preferable single-scattering probability of the Molière theory, results are quoted of interest to experimenters. Several mean-value quantities are given: mean arithmetic angle, median angle, half-width, $1/e$ width, angle $1/P_0\pi^{\frac{1}{2}}$ related to the zero-angle amplitude, and mean arithmetic angle with a cutoff at 4 times the mean. These quantities are given for both the projected tangent angle and projected chord angle distributions, in the form of linear relationships between the square of the angle divided by Ω and the logarithm of Ω, where Ω is the mean number of scatterings undergone by the particle in question. The linear relationships are good to 1 percent for Ω from 10^2 to 10^5. Information is also given on smoothed-out distributions, and on an estimate of the error for the cut-off arithmetic mean angle. The scattering constant K is given for several methods of measurement, for Ilford G-5 emulsions.

THE multiple-scattering theories of Molière[1] and Snyder and Scott[2,3] are based on the same mathematical development. Molière has an extensive discussion of the single-scattering law used in this development, and his results are superior to those of Snyder and Scott in this regard. It is the purpose of this article to state explicitly the relations between the two theories and to quote some formulas for various types of mean value that have been derived from the numerical tables of references 2 and 3.

We shall use the same notation as given in the previous papers. Molière's single-scattering formula may be obtained from that of Snyder and Scott by (1) replacing the angular unit η_0 by $\chi_\gamma = 1.13\eta_0\sqrt{(1.13+3.76\gamma^2)}$ and (2) replacing the mean number of scatterings z by[4] $\Omega = z\eta_0^2/\chi_\gamma^2 = z(1.13)^{-2}(1.13+3.76\gamma^2)^{-1}$. Here $\gamma = ZZ'/137\beta$; Z, Z' are, respectively, the atomic numbers of the scattered particle and the scattering nucleus, βc is the velocity of the scattered particle, the factor 1.13 in front of the radical is the ratio of the Bohr radius to the Fermi-Thomas radius for hydrogen, and the numbers 1.13 and 3.76 under the radical sign arise from an empirical fit[1] to a Fermi-Thomas single-scattering calculation. We can thus use our numerically calculated functions[2] $W(\eta|z)$ for the projected angular deviation η after a pathlength corresponding to z, with Molière's corrections, by writing for the distribution in actual angle Φ:

$$P(\Phi)d\Phi = W(\Phi/\chi_\gamma|\Omega)d\Phi/\chi_\gamma. \tag{1}$$

The small-angle approximation is used, and the functions are so normalized that

$$\int_{-\infty}^{\infty} Pd\Phi = 1.$$

Molière's calculations depend on the introduction of a quantity B, which is a function of Ω, and a subsequent expansion in inverse powers of B, using as an angular variable the quantity $\varphi = \Phi/\chi_\gamma(B\Omega)^{\frac{1}{2}}$.

For completeness, we give here the formulas for Ω and χ_γ:

$$\Omega = \frac{4\pi NtZ^{4/3}Z'^2\hbar^2}{1.13^2m^2c^2\beta^2(1.13+3.76\gamma^2)}, \tag{2a}$$

$$= 7830\sigma Z^{4/3}Z'^2/A\beta^2(1+3.33\gamma^2); \tag{2b}$$

$$\chi_\gamma = 1.13me^2Z^{\frac{1}{3}}(1.13+3.76\gamma^2)^{\frac{1}{2}}/\hbar p \text{ radians}, \tag{3a}$$

$$= 0.257Z^{\frac{1}{3}}(1+3.33\gamma^2)^{\frac{1}{2}}/pc \text{ degrees}. \tag{3b}$$

In case the scattering occurs in a mixture of materials, we must find χ_γ and Ω from the separate nuclear species (denoted by index i) by the following:[5]

$$\log\chi_\gamma = \Sigma_i(N_iZ_i^2 \log\chi_{\gamma i})/\Sigma_i(N_iZ_i^2) \tag{4}$$

and

$$\chi_\gamma^2\Omega = 4\pi te^4Z'^2(pc\beta)^{-2}\Sigma_iN_iZ_i^2 \text{ radians}^2. \tag{5a}$$

$$= 515\sigma Z'^2\langle Z^2\rangle_{\text{Av}}/Ap^2c^2\beta^2 \text{ deg}^2. \tag{5b}$$

In these formulas, N is the number of scattering nuclei per unit volume, t is the length of scattered track, σ is the track length measured in grams/cm^2 of scatterer, A is the atomic weight of the scatterer, m and e refer to the electron mass and charge, p is the momentum of the scattered particle, and pc in formulas (3b) and (5b) is measured in Mev/c. The case of energy loss in the scattering material may be included by the method of Molière.[1] The effect of electron-electron scattering can be included for those cases in which it is pertinent (small angles, small Z) by writing $Z_i^2+Z_i$ for Z_i^2 in Eqs. (4) and (5a). This correction has been discussed by Hanson, Lanzl, Lyman, and Scott.[6]

We have compared our tables with the results of the first three terms of Molière's expansion and find excellent numerical agreement for six values of Ω from 100 to 84000—in most cases better than 1 percent and always within about 2 percent. Molière's functions $f^{(1)}$

* Work performed under the auspices of the AEC.
[1] G. Molière, Z. Naturforsch. 2a, 133 (1947), and 3a, 78 (1948).
[2] H. S. Snyder and W. T. Scott, Phys. Rev. 76, 220 (1949).
[3] W. T. Scott and H. S. Snyder, Phys. Rev. 78, 223 (1950).
[4] Ω is the same as Ω_b in reference 1.
[5] Reference 1, second paper, Eq. (6.10′).
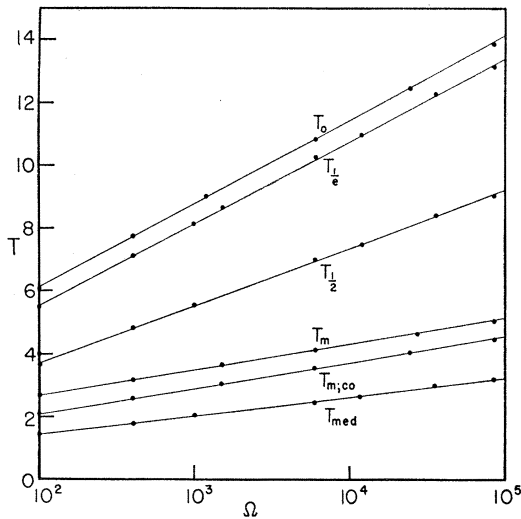[6] Hanson, Lanzl, Lyman, and Scott, Phys. Rev. 84, 634 (1951).

FIG. 1.

and $f^{(2)}$ are not tabulated to much greater accuracy. We have also extended his tables of these functions for φ from 4 to 13.5 in order to compare the tails of our curves, and again have found excellent agreement. The asymptotic formula in reference 2 can be readily shown to agree with that of reference 1; however, the arrangement of terms is different. The expansion given in reference 2 is equivalent to the terms in the same powers of φ in the $f^{(1)}, f^{(2)},$ and $f^{(3)}$ terms of reference 1. Terms in higher inverse powers of φ are given by Molière for $f^{(1)}$ and $f^{(2)}$, and it is with these terms that our numerical tables agree. Butler[7] has found an asymptotic development by another method which is almost identical with Molière's $f^{(1)}$ term alone. The $f^{(2)}$ term for $\Omega=100$ contributes 3 percent at $\varphi=4$ and 1 percent at $\varphi=13$. These relative contributions are reduced to one-half at $\Omega=24000$.

The use of numerical tables or graphs, however, is tedious for purposes of comparison with experiment. We have therefore calculated several mean-value expressions that may be of use in various experiments. It turns out that if we plot, for any of these values ψ, the quantity $T_\psi=\psi^2/\chi_\gamma^2\Omega$ as a function of $\log\Omega$, an almost straight line results. Consequently we give our results in the form of such linear relationships, which are sufficiently accurate for most experimental purposes.

The quantities chosen are the following: mean absolute angle, median absolute angle, angle for which $P=P_0/2$ ("half-width"), angle for which $P=P_0/e$ ("$1/e$ width"), the angle $\Phi_0=1/P_0\pi^{\frac{1}{2}}$ derived from P_0 (the value of P at $\Phi=0$), and a cut-off mean absolute angle calculated with P cutoff at 4 times the cut-off mean. The corresponding values of ψ and T_ψ will be denoted by Φ_m, T_m; $\Phi_{\rm med}$, $T_{\rm med}$; $\Phi_{\frac{1}{2}}$, $T_{\frac{1}{2}}$; $\Phi_{1/e}$, $T_{1/e}$; Φ_0, T_0; Φ_{mco}, T_{mco}.

We have in addition calculated the corresponding quantities for the distribution in the angle α between

[7] S. T. Butler, Proc. Phys. Soc. (London) **63A**, 599 (1950).

two adjacent chords, as a function of the mean number of scatterings in the track containing the two chords.[3] The quantities will be labeled α_m, C_m; $\alpha_{\rm med}$, $C_{\rm med}$; etc. (T for tangents and C for chords).

Another way of presenting multiple-scattering results is in terms of the "scattering constant." If some type of mean angle ψ is measured, the corresponding scattering constant $K_\psi=\sqrt{2}\psi pv/l^{\frac{1}{2}}$, where p and v refer to the scattered particle and t is the length of track. (The $\sqrt{2}$ is needed because, in the case of chord angles, our t includes *two* chords.) It is well known that K_ψ is a function only of $\frac{1}{2}t\beta^{-2}$. In terms of T_ψ (or C_ψ) we can write, using Eq. (4),

$$K_\psi^2=2p^2v^2\chi_\gamma^2\Omega T_\psi/t=(8\pi e^4 Z'^2\Sigma_i N_i Z_i^2)T_\psi. \quad (6)$$

We find for the T's and C's:

$$T_m=\langle|\Phi|\rangle_{\rm Av}^2/\chi_\gamma^2\Omega=1.044+0.809\ \log_{10}\Omega; \quad (7a)$$

$$T_{\rm med}=|\Phi|_{\rm med}^2/\chi_\gamma^2\Omega=0.222+0.596\ \log_{10}\Omega; \quad (7b)$$

$$T_{\frac{1}{2}}=(\Phi_{\frac{1}{2}})^2/\chi_\gamma^2\Omega=0.035+1.831\ \log_{10}\Omega; \quad (7c)$$

$$T_{1/e}=(\Phi_{1/e})^2/\chi_\gamma^2\Omega=0.253+2.636\ \log_{10}\Omega; \quad (7d)$$

$$T_0=(\pi P_0^2\chi_\gamma^2\Omega)^{-1}=0.806+2.656\ \log_{10}\Omega; \quad (7e)$$

$$T_{m;co}=\langle|\Phi|\rangle_{{\rm Av};co}^2/\chi_\gamma^2\Omega=0.418+0.818\ \log_{10}\Omega; \quad (7f)$$

$$C_m=\langle|\alpha|\rangle_{\rm Av}^2/\chi_\gamma^2\Omega=0.299+0.269\ \log_{10}\Omega; \quad (8a)$$

$$C_{\rm med}=|\alpha|_{\rm med}^2/\chi_\gamma^2\Omega=0.033+0.200\ \log_{10}\Omega; \quad (8b)$$

$$C_{\frac{1}{2}}=(\alpha_{\frac{1}{2}})^2/\chi_\gamma^2\Omega=-0.123+0.613\ \log_{10}\Omega; \quad (8c)$$

$$C_{1/e}=(\alpha_{1/e})^2/\chi_\gamma^2\Omega=-0.105+0.879\ \log_{10}\Omega; \quad (8d)$$

$$C_0=(\pi P_0^2\chi_\gamma^2\Omega)^{-1}=0.088+0.885\ \log_{10}\Omega; \quad (8e)$$

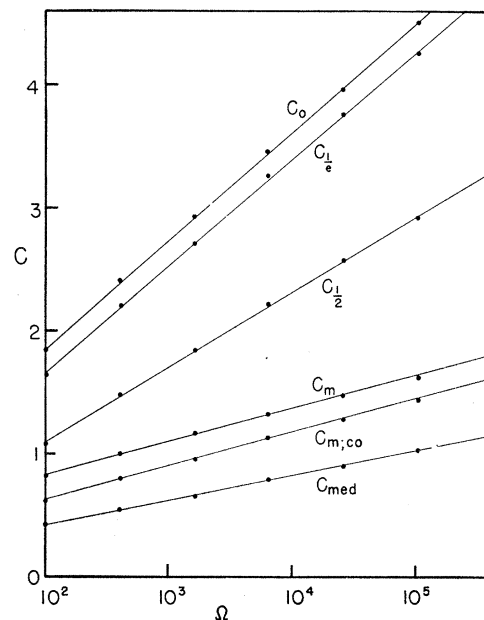$$C_{m;co}=\langle|\alpha|\rangle_{{\rm Av};co}^2/\chi_\gamma^2\Omega=0.090+0.272\ \log_{10}\Omega. \quad (8f)$$

FIG. 2.

Figs. 1 and 2 show the calculated results and fitted lines for the T's and C's.

With minor exceptions, the formulas agree with the calculations within 1 percent over the range of our tables, from $\Omega=100$ to $\Omega=84,000$ for the tangent distribution and 100 to 100,000 for the chord distribution. They were each fitted for six or seven values of Ω, by a least-square-relative-error method. The formula for T_{med} is off by 1.2 percent at one point, that for C_m by 1.3 percent at one point, and that for C_{med} by 1.6 percent at one point, with no other deviations as much as 1 percent.

The values for the mean absolute angle were compared with the analytic formula of Molière, and it was found that the linear relation is good to 1 percent out to $\Omega=10^7$, and to 3 percent to $\Omega=10^9$. The values for $T_{1/e}$, $T_{\frac{1}{2}}$, and T_0 are closely related to Molière's B, which obeys the linear relationship $B=1.153+2.583\log_{10}\Omega$ good to 3 percent out to $\Omega=10^9$. Hence we believe that all our formulas may be safely extrapolated for large Ω. Their accuracy is, however, not good for Ω much below 100.

Figure 3 shows a plot of B against $\log\Omega$, with Molière's values and some further calculations of ours, along with fitted line.

In the Gaussian approximation and in the limit of large Ω for the exact small-angle case, there is a relation between the C's and the T's. The factor as usually stated is $(3/2)^{\frac{1}{2}}$, but since Ω here for the chord case refers to two chords and the angles are squared, we find a factor 3. The coefficients of the logarithms in Eqs. (7) are all within 0.4 percent of 3 times the corresponding coefficients in (8). However, the constant terms show no such simple relationship.

We have not made calculations for the distribution of angles between "best-fit" or "smoothed-out" chords, as used by various investigators using photographic emulsions. However, Molière has dealt with this case in considerable detail.[8] The smoothing-factors for the Gaussian part of the calculation are given by Molière as $(3/2)^{\frac{1}{2}}$ for the chord case, $(35/26)^{\frac{1}{2}}$ for the angles between successive best-fitting (least-squares) lines, and $(70/61)^{\frac{1}{2}}$ for the angles between alternate best-fitted chords. We can thus approximate the effect of using successive best-fitted lines by multiplying our chord functions (8a)–(8f) by the factor $3/(35/13)=39/35$, and the effect of using alternate chords in half-cells by multiplying the tangent functions (7a)–(7f) (which give the nearest results) by $61/70$. For this last calculation the length t in Eq. (5a) refers to the center-to-center distance of two alternate chords. Goldschmidt-Clermont[9] uses a smoothing factor of 0.96, for which $(61/70)^{\frac{1}{2}}=0.934$ is a closer approximation.
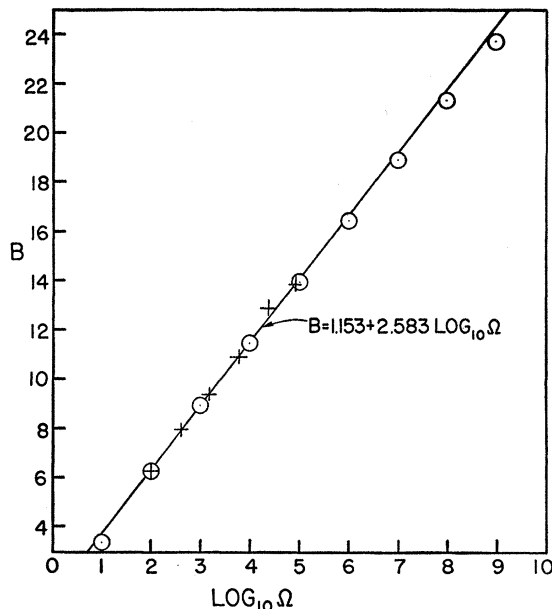
The extent to which the Gaussian approximation is

[8] G. Molière, unpublished manuscript. I am indebted to Dr. Molière for communicating his results to me in prepublication form.
[9] Y. Goldschmidt-Clermont, Nuovo Cimento 7, 331 (1950).

FIG. 3.

valid can be shown by correspondences among the various T's. If we characterize a Gaussian distribution by its $1/e$ width $\Phi_{1/e}=w$, the various other quantities are given by $\langle|\Phi|\rangle_{Av}=w/\pi^{\frac{1}{2}}$; $|\Phi|_{med}=0.4769w$; $\Phi_{\frac{1}{2}}=w(\log_e 2)^{\frac{1}{2}}$; $(P_0\pi^{\frac{1}{2}})^{-1}=w$; and $\langle|\Phi|\rangle_{Av;\,co}=0.5614w$. Calculating w from the various T's by these formulas gives for the coefficient of $\log_{10}\Omega$, in order, (a) 2.593, (b) 2.637, (c) 2.642, (d) 2.636, (e) 2.656, (f) 2.595. For large Ω, then, the criteria (7b), (7c), (7d), and (7e) agree within 1 percent showing that the central part of the curve has indeed a Gaussian shape. The effect of the plural scattering tail is still felt to an extent of about 2 percent for large Ω.

It will be noted that we do not refer to a mean square angle. One of the simplifications used in the theory results in the fact that the single-scattering law does not possess a mean square unless a somewhat arbitrary cutoff is introduced. This fact has a negligible effect on the calculations unless one should wish a multiple-scattering mean square angle. However, the statistical errors introduced by using the mean square as an experimental criterion are considerable and make this calculation less useful than the simpler mean absolute angle.

On the other hand, a mean-square angle can be of use in calculating the errors involved in a series of repeated measurements. Specifically, if a series of chord angles are measured using $n+1$ cells on the same track, as in an emulsion, and a mean absolute angle calculated, the rms deviation of this result from the expected mean can be found by the well-known theorem on the variance of a sample mean for an arbitrary distribution—namely, that the rms deviation is $n^{-\frac{1}{2}}$ times the rms deviation expected for a single measurement. This result requires

the successive angles to be independent, which is true for the tangent case but not for the chord case.

However, in an earlier calculation concerned with a measurement of mean-square angle of scattering,[10] a correlated, Gaussian-approximation formula was used for 8 chords and yielded an rms spread about 5 percent greater than the uncorrelated calculation. Thus, the uncorrelated error calculation for the mean absolute chord angle method should be good to 5 percent or better if 10 or more cells are used.

We have calculated for both the tangent and chord angle cases the rms deviation of the cut-off mean absolute angle, and quote here the results

$$\langle \Phi^2 \rangle_{Av;\,co}/\chi_\gamma^2\Omega = 0.998 + 1.221\,\log_{10}\Omega; \tag{7g}$$

$$\langle \alpha^2 \rangle_{Av;\,co}/\chi_\gamma^2\Omega = 0.202 + 0.422\,\log_{10}\Omega. \tag{8g}$$

These formulas are also good to 1 percent. Using (7f), (7g) and (8f), (8g) we find specifically

$$[\langle \Phi^2 \rangle_{Av;\,co} - \langle |\Phi| \rangle_{Av;\,co}{}^2]^{\frac{1}{2}}/\langle |\Phi| \rangle_{Av;\,co} = 0.82 \quad \text{for} \quad \Omega = 10^2$$
$$= 0.77 \quad \text{for} \quad \Omega = 10^5;$$
$$[\langle \alpha^2 \rangle_{Av;\,co} - \langle |\alpha| \rangle_{Av;\,co}{}^2]^{\frac{1}{2}}/\langle |\alpha| \rangle_{Av;\,co} = 0.80 \quad \text{for} \quad \Omega = 10^2$$
$$= 0.77 \quad \text{for} \quad \Omega = 10^5.$$

Thus, no serious misestimation of error will be made if we use for n angular measurements of any sort on a track, a standard deviation for the arithmetic mean of $0.85n^{-\frac{1}{2}}$ times the mean.

For the case of Ilford G-5 emulsions, we have calculated the scattering constant K. In agreement with Snyder,[11] we find that $\Sigma_i N_i Z_i^2 = 3.70 \times 10^{25}$, or $\langle Z^2 \rangle_{Av}^{\frac{1}{2}} = 21.34$. However, using Eq. (4) instead of an average of $Z^{4/3}$ we find, combining with Eq. (5) and $\beta = 1$, that $\Omega = 631 s_0$ where s_0 is the cell-length $t/2$ in units of 100 microns. This result is equivalent to using $Z = 40.6$ in Eq. (3b), and differs from the value $z = 1248 s_0$ of reference 11, which was calculated from Eq. (2a) without the factors 1.13^2 and $1.13 + 3.76\gamma^2$, and involved $\langle Z^{4/3} \rangle_{Av} = (16.22)^{4/3}$. For $s_0 = 1$, $\Omega = 631$, we find instead of the value[11] 26.8: (a) complete mean value, $K_m = 25.8$ from (8a); (b) cut-off mean value, $K_{mco} = 23.2$ from (8f). If we wish to compare the results of using Eqs. (7) and a factor $\frac{1}{3}$, we find: (c) complete mean value from $T_m/3$, $K_m' = 26.4$; (d) cut-off mean value from $T_{mco}/3$, $K_{mco}' = 23.9$.

With somewhat less reliability, we can calculate scattering constants using Molière's smoothing factors. For best-fitted lines (assumed to be the same as a least-squares fitting would yield), we have: (e) complete

mean value from $(39/35)C_m$, $K_{m-\text{smooth}} = 27.2$; (f) cut-off mean value from $(39/35)C_{mco}$, $K_{mco-\text{smooth}} = 24.5$.

Finally, for angles between alternate smoothed chords at separations $2s$, we have, dividing by 2 to reduce to cell-length s: (g) complete mean value from $(61/140)T_m$, $K_{m-\text{alt}} = 30.2$; (h) cut-off mean value from $(61/140)T_{mco}$, $K_{mco-\text{alt}} = 27.4$. The use of $Z_i^2 + Z_i$ instead of Z_i in these results would increase $\langle Z^2 \rangle_{Av}$ by 0.29 percent and would have a negligible effect on Ω. Hence all the results above can be increased by about 0.1 because of this effect. The variation of K with cell-length is to be found, of course, from the appropriate one of Eqs. (7) or (8).

The variation of Ω with β^2 is given to sufficient accuracy by Fig. 2 of the paper by Gottstein *et al.*[12] The values given by this figure must be multiplied by 2 for use in our formulas.

We have compared our results with those of Williams' scattering theory[13] as discussed by Voyvodic and Pickup.[14] The results of this theory can be written in similar form to Eqs. (7). In particular, the mean and cut-off mean values of Williams' theory fit the following formulas:

$$T_m{}^W = 1.63 + 0.79\,\log_{10}\Omega, \quad T_{mco}{}^W = 1.18 + 0.79\,\log_{10}\Omega.$$

The differences from the present results will evidently be greatest for small Ω.

Work is progressing on formulas similar to Eqs. (7) for the spatial (unprojected) tangent angle distribution, and will be reported later.

We should like to take the opportunity here to point out that the graphs of the two Snyder-Scott distributions printed in reference 3, Figs. 5 and 6, have had their legends interchanged.

Note added in proof: Recent, unpublished words by S. Olbert on the inclusion in Molière's theory of the effect of the finite size of nuclei indicates that the tails of the multiple scattering distribution are considerably reduced by this effect. Thus our results may need modification, especially for the mean absolute angles; to a lesser extent for the cut-off mean median and $(P_0\pi)^{-\frac{1}{2}}$ values; and probably very little for the $\frac{1}{2}$ and $1/e$ widths. I am indebted to Dr. Bruno Rossi for communicating these results to me.

[10] W. T. Scott, Phys. Rev. **75**, 1763 (1949).
[11] H. S. Snyder, Phys. Rev. **83**, 1068 (1951).
[12] Gottstein, Menon, Mulvey, O'Ceallaigh, and Rochat, Phil. Mag. **42**, 716 (1951).
[13] E. J. Williams, Phys. Rev. **58**, 292 (1940).
[14] E. Pickup and L. Voyvodic, Phys. Rev. **84**, 1190 (1951).