Radiative Corrections to Compton Scattering

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Corrections of order e^6 to the differential cross section for Compton scattering of unpolarized radiation by electrons are computed. The results for corrections ascribable to virtual photons are finite, relativistically invariant, and valid at all energies, but contain a term which depends logarithmically on an assumed small photon mass λ. A cross section of the same order has also been obtained for double Compton scattering in which one of the emitted photons has an energy small compared to the rest mass of the electron (with the electron initially at rest). This contains a term depending on $\ln\lambda$ which exactly compensates the similar term arising from virtual quanta in all observable cases. Approximations for low and high energies, as well as numerical results, are given. These disagree with results obtained previously by Schafroth.

THE object of this paper is to obtain the correction to the differential cross section for Compton scattering (Klein-Nishina formula) arising from the possibility that the electron may emit and reabsorb a virtual photon in connection with the scattering process. We shall apply the methods developed by one of us[1] to obtain an explicit cross section to order e^6 for unpolarized radiation, valid (in so far as the theory is valid) at all energies.

Previous workers have shown that the high frequency divergences which enter in the straightforward application of perturbation theory to this problem can be removed by charge and mass renormalization. Schafroth[2,3] has obtained a finite e^4-order matrix element in relativistic and gauge invariant form. He also showed, following the treatment of the analogous problem for scalar particles by Corinaldesi and Jost,[4] that the infrared divergence which occurs can be removed by addition of the double Compton cross section in which the incoming photon produces two photons on interacting with the electron, and he made explicit evaluation of the cross section (but not of the double scattering) in the nonrelativistic and extreme relativistic approximations. His results, however, disagree with ours in both limits.

Since the interpretation of any experiment to measure the radiative corrections requires a knowledge of the double Compton cross section, we have computed this also, for the case that one of the emitted photons has an energy in the laboratory system which is small compared to the electron rest energy.

After a brief introduction, we shall in Sec. II write down and discuss the matrix element for the corrections. Section III will detail the evaluation of the differential cross section. Section IV will be concerned with the infrared catastrophe and the double Compton effect. Sections V and VI will discuss limiting cases and some numerical results. Mathematical details will be reserved for the appendices.

The method of calculating this effect is given by Feynman,[5] and for brevity we will not repeat the discussion here but will simply carry out the explicit evaluation of the matrix elements involved. Our notation is that of reference 1.

Some improvement has been made in the method of computing matrix elements given in reference 1(b). This is described here in detail in Appendix Y.

I. THE KLEIN-NISHINA FORMULA FOR UNPOLARIZED RADIATION

The direct Compton effect, in which a photon of momentum q_1, polarization e_1, impinges on an electron of initial momentum p_1, to be scattered as a new photon of momentum q_2, polarization e_2, is represented by a matrix element

$$W=R+S \qquad \text{(1a)}$$

with

$$R=\mathbf{e}_2(\mathbf{p}_1+\mathbf{q}_1-m)^{-1}\mathbf{e}_1,\quad S=\mathbf{e}_1(\mathbf{p}_1-\mathbf{q}_2-m)^{-1}\mathbf{e}_2. \qquad \text{(1b)}$$

The final momentum of the electron is, of course, $p_2=p_1+q_1-q_2$. The terms correspond to the diagrams of Fig. 1.

We shall call

$$p_3=p_1+q_1=p_2+q_2,\quad p_4=p_1-q_2=p_2-q_1, \qquad (2)$$

and define the important invariants κ, τ by

$$\begin{aligned} m^2\kappa&=m^2-p_3^2=-2p_1\cdot q_1=-2p_2\cdot q_2,\\ m^2\tau&=m^2-p_4^2=2p_1\cdot q_2=2p_2\cdot q_1. \end{aligned} \qquad (3)$$

In the laboratory system, with ω_1 and ω_2 the energies of the incoming and outgoing photons, κ is $-2\omega_1/m$ and τ is $2\omega_2/m$. In terms of the quantities defined in

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[1] R. P. Feynman, Phys. Rev. 76, 749 (1949); and Phys. Rev. 76, 769 (1949).

[2] M. R. Schafroth, Helv. Phys. Acta 22, 501 (1949).

[3] M. R. Schafroth, Helv. Phys. Acta 23, 542 (1950).

[4] E. Corinaldesi and R. Jost, Helv. Phys. Acta 21, 183 (1948).

[5] This problem is discussed in reference 1, Appendix D, p. 788.

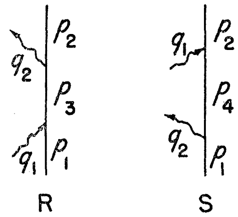

FIG. 1. Momentum diagrams for direct Compton effect.

(2) and (3) we have

$$m^2\kappa R = -e_2(p_3+m)e_1, \quad m^2\tau S = -e_1(p_4+m)e_2. \tag{4}$$

The differential cross section for the final photon to go into solid angle $d\Omega$, if the initial electron is at rest (laboratory system, $p_1 = m\gamma_t$) is

$$d\sigma = e^4 d\Omega(\omega_2^2/\omega_1^2)F \tag{5}$$

where F is the square of the matrix element of $W(1a)$,

$$F = |{}_2\langle W\rangle_1|^2. \tag{6}$$

If we are uninterested in the spin states of the electron, F may be replaced by $(2m^2)^{-1}U$ where

$$U = \tfrac{1}{4}\,\mathrm{Sp}[(p_2+m)W(p_1+m)\bar{W}]. \tag{7}$$

If, in addition, unpolarized radiation is used and the sum over polarization directions is required, e_1 can be replaced by γ_α and e_2 by γ_β in the spur and half the sum over α, β taken (reference 1(b), Sec. 8). Then the term in (7) which is second order in R is

$$\tfrac{1}{2}(2m^2\kappa)^{-2}\,\mathrm{Sp}[(p_2+m)\gamma_\beta(p_3+m)\gamma_\alpha(p_1+m)\gamma_\alpha \times(p_3+m)\gamma_\beta] = 4/\kappa^2 - \tau/\kappa - 2/\kappa. \tag{8}$$

The reduction can be accomplished by Eqs. (4a) and (36a) of reference 1(b). The term of second order in S is (8) with κ, τ interchanged, since S is obtained from R by replacing p_3 by p_4 after the average is taken on photon polarization. The cross term is

$$(2m^2\kappa)^{-1}(2m^2\tau)^{-1}\,\mathrm{Sp}[(p_2+m)\gamma_\beta(p_3+m)\gamma_\alpha \times(p_1+m)\gamma_\beta(p_4+m)\gamma_\alpha] = 8/\kappa\tau - 2/\tau - 2/\kappa. \tag{9}$$

The sum gives for $U = m^2\sum_{\mathrm{spin}}\sum_{\mathrm{pol}}|{}_2\langle W\rangle_1|^2$:

$$U = 4(\kappa^{-1}+\tau^{-1})^2 - 4(\kappa^{-1}+\tau^{-1}) - (\kappa/\tau+\tau/\kappa) \tag{10}$$

and for the Klein-Nishina formula in terms of κ, τ we have:

$$d\sigma = \frac{2\pi e_4}{m^2}\left(\frac{\tau^2}{\kappa^2}\right)\left(\frac{d\tau}{\tau^2}+\frac{d\kappa}{\kappa^2}\right)U. \tag{11}$$

In the laboratory system, in view of $\kappa = -2\omega_1/m$, $\tau = 2\omega_2/m$ and the Compton relation

$$\omega_1\omega_2(1-\cos\varphi) = m(\omega_1-\omega_2), \tag{12}$$

(11) can be written in the usual way

$$d\sigma = (e^4/2m^2)d\Omega(\omega_2^2/\omega_1^2)(\omega_1/\omega_2+\omega_2/\omega_1-\sin^2\varphi). \tag{13}$$

II. THE e^4-ORDER MATRIX ELEMENT

The diagrams of the first radiative corrections to term R of the Compton effect are given in Fig. 2. (See reference 1(b), Fig. 9.) The terms containing the analogous modifications of S can be obtained throughout by the interchange of e_1 and e_2, of q_1 and $-q_2$, and of p_3 and p_4. In the final result this means simply an interchange of κ and τ. Hence we need study only R, the S terms being obtained from the R terms immediately.

Terms N' and N'' give zero since there are no vacuum polarization effects for free photons.

Terms M' and M'' together give a factor[6] $r/2i$ times R, where

$$r = \ln(\Lambda/m) + 9/4 - 2\ln(m/\lambda) \tag{14}$$

as shown in reference 1(b), Sec. 6. The quantity Λ is a temporary high frequency cutoff, introduced so that each diagram can be separately evaluated. The final result will become independent of Λ as $\Lambda\to\infty$. The "infrared catastrophe" discussed in Sec. IV is treated, at this point, by assuming the photons to have a small rest mass λ.

The term L is

$$L = \int e_2(p_3-m)^{-1}\gamma_\mu(p_3-k-m)^{-1} \times\gamma_\mu(p_3-m)^{-1}e_1 k^{-2}d^4k\,C(k^2). \tag{15}$$

From this must be subtracted the mass correction for an electron travelling between the absorption and emission of the virtual quantum. Since (to order Δm)

$$(p-m-\Delta m)^{-1} = (p-m)^{-1} + (p-m)^{-1}\Delta m(p-m)^{-1},$$

this gives just the expression for L except that Δm replaces

$$\int \gamma_\mu(p_3-k-m)^{-1}\gamma_\mu k^{-2}d^4k\,C(k^2),$$

where Δm is the mass correction for cutoff Λ [reference 1(b), Eq. (21)]:

$$\Delta m = im[\tfrac{3}{8}+\tfrac{3}{2}\ln(\Lambda/m)]. \tag{16}$$

Since this diagram occurs for problems other than the one we consider here, we give the result in a general way. Each $(p-m)^{-1}$ propagation factor has, as a consequence of diagrams like L, a correction to the first order in e^2 given by

$$\begin{aligned}(p-m)^{-1}\int\gamma_\mu(p-k-m)^{-1}\gamma_\mu k^{-2}d^4k\,C(k^2)(p-m)^{-1} \\ -\Delta m(p-m)^{-2} \\ = (4i)^{-1}\left\{(p-m)^{-1}\left[\ln(\Lambda^2/m^2) + \frac{5}{2} - \frac{\eta}{\eta-1} + \frac{\eta(2-\eta)}{(\eta-1)^2}\ln\eta\right]\right. \\ \left. - m(p-m)^{-2}\left[\frac{\eta}{\eta-1} - \frac{\eta(3\eta-2)}{(\eta-1)^2}\ln\eta\right]\right\},\end{aligned} \tag{17}$$

where $m^2\eta = m^2 - p^2$.

[6] The factor obtained in reference 1(b) is $-(e^2/2\pi)r$, but we have reserved a factor $e^2/\pi i$ for later inclusion.

Terms K' and K'' again possess a feature common to several problems, and we will therefore first discuss it in a general way. In all problems in which an electron interacts with a potential or a free or virtual photon there will be a piece of the diagram like Fig. 3. That is, there will be a partial factor in one of the matrix elements:

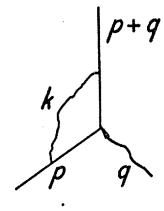

FIG. 3. Diagram for the expression T.

$$T=\int\gamma_\mu(p+q-k-m)^{-1}e(p-k-m)^{-1}\times\gamma_\mu k^{-2}d^4kC(k^2).\quad(18)$$

It would be most convenient to have this evaluated in the general case of arbitrary p and q. However, we have evaluated it only in the special case that $q^2=0$, $p^2=m^2$, with the matrix operating on a state u such that $pu=mu$. Calling $m^2\kappa=-2p\cdot q$ it is (Appendix Z):

$$\begin{aligned}8iT=&4\kappa^{-1}[m^2e+2\kappa^{-1}(e\cdot p)q]\int_{1-\kappa}^{1}\ln(1-v)dv/v\\&+2[(2m^2+pq-qp)e\\&+2\kappa^{-1}(e\cdot p)(q+m\kappa)(3\kappa-2)(\kappa-1)^{-1}](\kappa-1)^{-1}\ln\kappa\\&+[2\ln(m^2/\Lambda^2)-1]m^2e\\&-4(e\cdot p)[(q+m)(\kappa-1)^{-1}+q\kappa^{-1}].\quad(19)\end{aligned}$$

If the final, rather than the initial, state is a free electron, the matrix required is $\bar{T}$, so the result is obtained directly from (19). For term K', this T for the case $p=p_1$, $q=q_1$, $e=e_1$ is to be multiplied on the left by $e_2(p_1+q_1-m)^{-1}=e_2(p_3-m)^{-1}$. Therefore, K' and the corresponding term K'' together give

$$K=K'+K''=(8i)^{-1}[e_2(p_3-m)^{-1}T(p_1,q_1,e_1)+\bar{T}(p_2,q_2,e_2)(p_3-m)^{-1}e_1].\quad(20)$$

If we now examine the coefficients of the term $\ln(m^2/\Lambda^2)$ in K, L, and M, that is in (20), (17), and (14), we observe that K gives $(4m^2/8i)R$, L gives $(-2m^2/8i)R$, and M gives $(-2m^2/8i)R$. Therefore the terms dependent on Λ vanish. Since we shall find that the J integral is finite without cutoff, we note that the complete result is insensitive to Λ.

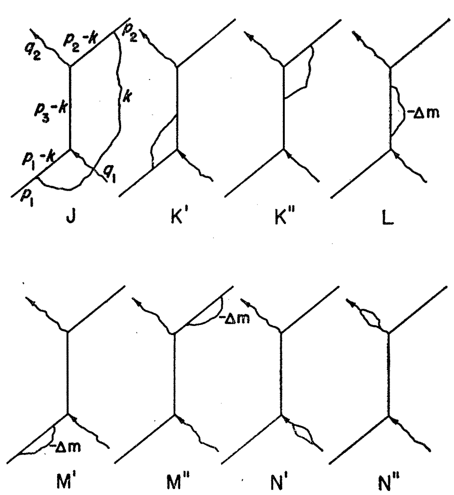

FIG. 2. Corrections to term R of Compton scattering.

The term J is given by

$$J=\int\gamma_\mu(p_2-k-m)^{-1}e_2(p_3-k-m)^{-1}\times e_1(p_1-k-m)^{-1}\gamma_\mu k^{-2}d_4k.\quad(21)$$

For large k the factors in the integrand vary as k^{-n} with $n\geqq5$ and the integration over k-space therefore converges. If we had included the convergence factor $C(k^2)$, the result would be independent of Λ as $\Lambda\rightarrow\infty$.

When the reciprocals are rationalized (e.g., $(p_2-k-m)^{-1}=(p_2-k+m)\cdot[(p_2-k)^2-m^2]^{-1}$), powers of k_μ up to the third appear in the numerator of the integrand. Therefore we shall have to evaluate integrals of the form:

$$J_{(0;\sigma;\sigma\tau;\sigma\tau\nu)}=\int(1;k_\sigma;k_\sigma k_\tau;k_\sigma k_\tau k_\nu)[(p_2-k)^2-m^2]^{-1}\times[(p_3-k)^2-m^2]^{-1}[(p_1-k)^2-m^2]^{-1}k^{-2}d_4k.\quad(22)$$

That is, for J_0 the factor $(1;k_\sigma;\cdots$ etc.) is replaced by unity, for J_σ by k_σ, for $J_{\sigma\tau}$ by $k_\sigma k_\tau$, and for $J_{\sigma\tau\nu}$ by $k_\sigma k_\tau k_\nu$. The manner in which J can be expressed in terms of these integrals is illustrated, for the case of matrix T in Appendix Z.

The J integrals can be worked out by the parametric methods described in reference 1(b) (Appendix). They involve integrals having four factors in the denominator and will lead, therefore, to integrals over three parameters. (J_0 is integrated in this manner in Appendix Y.) Generally these are very difficult to evaluate, although J_0 is particularly simple. This fact makes it possible to circumvent some of the difficulties of J_σ, $J_{\sigma\tau}$, and $J_{\sigma\tau\nu}$.

It is possible to express these other J integrals as linear combinations of the integral J_0 and of other integrals, all of which involve only three quadratic factors in the denominator. These latter, in parametric form, require only two parameters (and are much more easily evaluated than a direct attack on J_σ, say, would indicate). This technique is useful in other problems also[7] and is described in detail in Appendix Y.

III. CROSS SECTION FOR UNPOLARIZED LIGHT

If we call the sum $J+K+L+M=R^{(1)}$, then $(e^2/\pi i)R^{(1)}$ will be the correction to the matrix R of the direct effect (1). If the corresponding correction to the term S is called $(e^2/\pi i)S^{(1)}$, the corrected matrix for the Compton effect is

$$W'=W+(e^2/\pi i)W^{(1)}=R+S+(e^2/\pi i)(R^{(1)}+S^{(1)}). \quad (23)$$

The absolute square of the matrix element of W', taken between the initial and final electron states, gives the probability of transition correct to one order in e^2 higher than (6). We shall calculate in this paper only the cross section averaged over spin directions of the electron and polarization directions of the photons.

We need the spur:

$$\tfrac{1}{4}\,\mathrm{Sp}[(p_2+m)W'(p_1+m)\bar{W}'] \quad (24)$$

as in (7). Considering terms up to the first order in e^2 (which are all that are valid), (24) is

$$U-\tfrac{1}{4}\{(e^2/\pi i)\,\mathrm{Sp}[(p_2+m)W(p_1+m)\bar{W}^{(1)}] \\ -(e^2/\pi i)\,\mathrm{Sp}[(p_2+m)W^{(1)}(p_1+m)\bar{W}]\}. \quad (25)$$

In evaluating (25) for unpolarized light we have replaced $\mathbf{e}_1$ by γ_α and $\mathbf{e}_2$ by γ_β and taken one-half of the resulting sum as discussed in connection with (8). Some algebraic details are discussed in Appendix Z.[8]

The last two spurs in (25) are complex conjugates, so that the correction to U is $-e^2/\pi$ times the real part of

$$U^{(1)}=-(4i)^{-1}\,\mathrm{Sp}[(p_2+m)W^{(1)}(p_1+m)\bar{W}]. \quad (26)$$

That is, U is to be replaced in (11) by

$$U'=U-(e^2/\pi)\mathbf{R.P.}U^{(1)}. \quad (27)$$

If we let

$$P(\kappa,\tau)=-(4i)^{-1}\,\mathrm{Sp}[(p_2+m)R^{(1)}(p_1+m)\bar{W}] \quad (28)$$

then

$$U^{(1)}=P(\kappa,\tau)+P(\tau,\kappa) \quad (29)$$

since the $S^{(1)}$ diagrams are obtained from the $R^{(1)}$ diagrams (for unpolarized light) by the interchange of p_3 and p_4 and of q_1 and $-q_2$; hence the final result, simply by interchange of κ and τ.

The final result obtained in this way is:

$$\begin{aligned}P(\kappa,\tau)=&(1-2y\,\mathrm{ctnh}2y)\ln\lambda\cdot U\\ &-2y\,\mathrm{ctnh}2y[2h(y)-h(2y)]U\\ &+[-4y\sinh 2y(\kappa\tau)^{-1}(2-\cosh 2y)\\ &+2y\,\mathrm{ctnh}y]h(y)+\ln\kappa)\Bigg\{4y\,\mathrm{ctnh}2y\bigg[\frac{4}{\kappa\tau}\cosh^2 y\\ &+\frac{\kappa-6}{2\tau}\mathrm{sech}2y+\frac{4}{\kappa^2}-\frac{1}{\kappa}-\frac{\tau}{2\kappa}-\frac{\kappa}{\tau}-1\bigg]\\ &+\frac{3\tau}{2\kappa^2}+\frac{3\tau}{2\kappa}+\frac{3}{\tau}+1-\frac{7}{\kappa\tau}+\frac{8}{\kappa}-\frac{8}{\kappa^2}+\frac{2\kappa-\tau^2-\kappa^2\tau}{2\kappa^2\tau(\kappa-1)}\\ &-\frac{1}{2\tau}\frac{2\kappa^2+\tau}{(\kappa-1)^2}\Bigg\}+y^2\,\mathrm{csch}^2 y\bigg[\frac{2}{\kappa}-\frac{7}{4}\kappa-\frac{3}{4}\frac{\tau^2}{\kappa}\bigg]\\ &-4y\tanh y\left(\frac{1}{2}-\frac{1}{\kappa}\right)+4\left(\frac{1}{\kappa}+\frac{1}{\tau}\right)^2\\ &-\frac{12}{\kappa}-\frac{3}{2}\frac{\kappa}{\tau}-2\frac{\kappa}{\tau^2}+\frac{1}{\kappa-1}\left(\frac{\kappa}{\tau}+\frac{1}{2}\right)\\ &+G_0(\kappa)\bigg[\frac{\kappa^2}{\tau}+\frac{\tau}{\kappa^2}+\frac{\kappa}{\tau}+\kappa+\frac{1}{2}\tau+\frac{2}{\kappa}-\frac{3}{\tau}-1\bigg]\\ &\qquad+\text{terms antisymmetric in }\kappa,\tau, \end{aligned} \quad (30)$$

where

$$4\sinh^2 y=-(\kappa+\tau) \quad (30\mathrm{a})$$

$$h(y)=y^{-1}\int_0^y u\,du\,\mathrm{ctnh}u \quad (30\mathrm{b})$$

$$G_0(\kappa)=-2\kappa^{-1}\int_{1-\kappa}^1 \ln(1-u)du/u. \quad (30\mathrm{c})$$

This is to be added to the same expression with κ and τ interchanged (29) and the real part taken to get the correction to the Klein-Nishina formula (11). We discuss this result in the following sections.

We might note here, however, that the real part of $P(\kappa,\tau)$ is obtained by writing $\ln|\kappa|$ for $\ln\kappa$ and by writing for $G_0(\kappa)$ expression (30c) with $\ln(1-u)$ replaced by $\ln(u-1)$. Since τ is always positive, on the other hand, $P(\tau,\kappa)$ is always real. This is discussed further in Appendix W.

The imaginary part of $P(\kappa,\tau)$ is not without interest, as we shall show. This is given by π times the coefficient of $\ln\kappa$ in (30) plus $\pi\ln(1-\kappa)$ times the coefficient of $G_0(\kappa)$.

The loss of total intensity of a beam of photons is of course proportional to the total cross section for a photon to be scattered out of the beam. But this

[7] It has been applied by G. R. Lomanitz to completely evaluate the e^6 corrections to the Möller scattering cross section of electrons in his thesis *Second Order Effects in the Electron-Electron Interaction*, Cornell, 1950. Again, in the problem of scattering of light by light, the integral with unit numerator is easily done, and the other integrals can be reduced to it and simpler integrals algebraically. But here the algebraic complexity makes the problem extremely tedious.

[8] In actual evaluation it was found easier to take the spur first and perform the integrals later. Thus, in place of the expression T (Eq. 18), the expression $\mathbf{T}$ (Eq. A41) was substituted and the values of the integrals from Appendix X substituted after taking the spur. This has the advantage that some of the integrals do not appear, or appear only in simpler combinations.

decrease in forward intensity is the result of an interference between the incident photon and a photon scattered exactly in the forward direction. Therefore, as is well known, the imaginary part of the forward scattering amplitude is proportional to the total cross section (formally this is referred to as the unitary property of the S-matrix). We can use this relation to check the imaginary part of $P(\kappa, \tau)$ for the case of zero scattering angle (for which, of course, $p_1=p_2$, $q_1=q_2$, $\kappa=-\tau$).

We write $P(\kappa, \tau)$ again as a sum

$$P(\kappa, \tau)=(4m^2i)\tfrac{1}{4}\sum_{\text{spin}}\sum_{\text{pol}}\, {}_1\langle R^{(1)}\rangle_2\, {}_2\langle \bar{W}\rangle_1, \quad (31)$$

and can show easily that ${}_2\langle W\rangle_1=i/m$ if there is no spin change and no polarization change, and zero otherwise. This can be seen, aside from the phase factor i, from the fact that for small scattering angles the Klein-Nishina formula (13) is $d\sigma=r_0{}^2d\Omega$ in the laboratory system. Since ${}_1\langle R^{(1)}\rangle_2$ also vanishes when ${}_2\langle W\rangle_1$ does,

$$P(\kappa, \tau)=m\sum_{\text{spin}}\sum_{\text{pol}}\, {}_1\langle R^{(1)}\rangle_2. \quad (32)$$

But, including all factors, the complete e^4-order matrix element of $R^{(1)}$ is (according to reference 1(b)):

$$X=\frac{e^2}{\pi i}\cdot\frac{2\pi e^2}{(m\tau/2)}\cdot\frac{1}{4}\sum_{\text{spin}}\sum_{\text{pol}}\, {}_1\langle R^{(1)}+S^{(1)}\rangle_2 \quad (33)$$

and from the unitary property referred to above, it follows that the *total* cross section for Compton scattering to order e^4 is just twice the real part of X. Therefore,

$$\sigma_{\text{total}}(\text{to order } e^4)=2\mathbf{R.P.}X=(2r_0{}^2/\tau)\mathbf{I.P.}P(\kappa, \tau) \quad (34)$$

since $S^{(1)}$ has no imaginary part. That (30) satisfies this identity can be readily verified.[9]

IV. THE INFRARED CASTASTROPHE AND THE DOUBLE COMPTON EFFECT

In Sec. II we have derived the differential cross section for Compton scattering for unpolarized light, including radiative corrections, to order e^6. The cross section took the form

$$d\sigma=d\sigma_{\text{K.N.}}[1+(e^2/\pi)\delta] \quad (35)$$

with

$$\delta=-U^{(1)}/U. \quad (35a)$$

There are two reasons why this result cannot be compared directly with experiment. In the first place $U^{(1)}$ depends on the quantity λ to which no experimental significance has been attached. In the second place, it is impossible in principle to design an experiment which will guarantee that one and only one photon is emitted by the electron in the scattering process. The best one can do in an experiment is to require that if a second photon is emitted, its energy is less than some value k_{max}. This can be done, for example, by measuring the energies of the final electron and photon to some specified accuracy, the sum of the errors in the measurement being less than k_{max}. In such an experiment one would be measuring the cross section (35) plus the cross section for the double Compton effect, $d\sigma_D$, integrated over all possible directions of the second quantum and over its energy up to k_{max}.

These two difficulties, both related to quanta of low energy (if k_{max} is small), in one case virtual, in the other real, are actually related. That this should be so, can be seen physically from the fact that it is difficult to distinguish between virtual and real quanta of extremely low energy since, by the uncertainty principle, a measurement made during a finite time interval will introduce an uncertainty in the energy of the quantum, which may enable a virtual quantum to be detected as a real one. It turns out in fact that $d\sigma_D$, integrated to k_{max}, also contains an infrared divergence which just cancels the similar divergence in the radiative corrections. We are computing, of course, only to order e^6, but the multiple Compton scattering of a given higher order will also cancel all the radiative infrared catastrophes of the same order.

The problem is analogous to the perturbation theory treatment of the scattering of an electron by a potential, which has been considered by many workers, except that in our case the primary process is the Compton scattering considered in Sec. I. The cross section for emission of an additional photon $\boldsymbol{q}$ of energy ω goes for small ω as $(d^3q/\omega)(p_2/p_2\cdot q-p_1/p_1\cdot q)^2$ times the Klein-Nishina formula. Since this diverges as ω approaches zero, the probability of a single Compton process unaccompanied by such emission is zero. What is experimentally measured, however, is the probability that a Compton process occurs and that no other free photon is emitted except for a class of photons inaccessible to the experiment. This is equal, to our order of calculation, to the probability of the single process plus the probability of a double process in which one of the photons emitted is in the inaccessible class. This class is, of course, determined by the design of the experiment and a single calculation cannot suffice for all experiments. However, one feature common to all experiments will be a finite energy resolution, so that a part of the excluded class must consist of photons whose energy is less than some energy k_{max}.

We will first therefore find that part of the differential cross section for double Compton scattering which gives rise to an infrared divergence. This will be integrated over all the directions of one of the photons, and over its energy from zero to a value k_{max}, which we shall assume is small compared to the electron mass, and added to the previously obtained corrected cross section for single Compton scattering. It has already been pointed out that Schafroth[2] has demonstrated that a cancellation of the infrared divergence occurs in order e^6 when the double Compton cross section is added to

[9] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1944), p. 157, Eq. (53). Our Eq. (33) agrees with this result with τ replacing Heitler's 2γ.

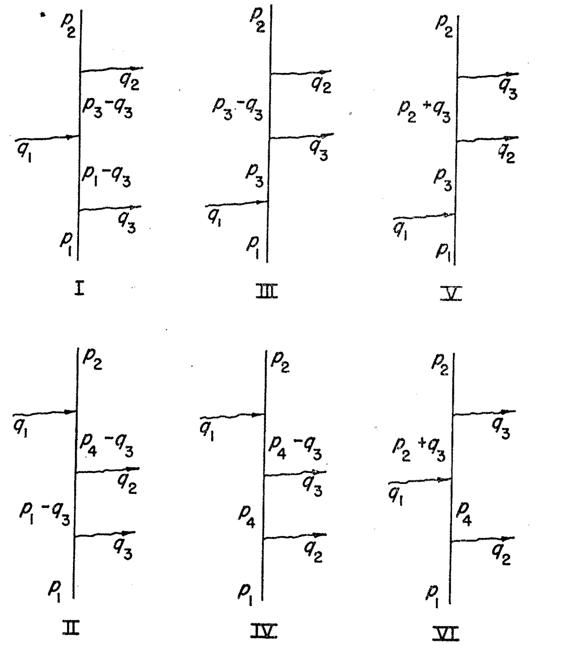

FIG. 4. Diagrams for the double Compton effect. Here $p_3=p_1+q_1$, $p_4=p_1-q_2$; the momentum condition is $p_1+q_1=p_2+q_2+q_3$.

the single scattering cross section, but we must obtain at least the zero order term in $k_{\max}$ (for $k_{\max}\ll m$) to obtain a useful result. The completely differential cross section for the double scattering has been computed by Eliezer,[10] but since we wish to make approximations and carry out an integration, it is simpler for us to obtain the desired cross section from the beginning.

Figure 4 gives the diagrams necessary for computing the cross section for double Compton scattering. The photon momentum $q_3=(\omega_3, \mathbf{q}_3)$ is assumed to be small in the following ($\omega_3\ll m$). This, of course, implies a definite coordinate system. To obtain a finite result we assume the photon has a small rest mass λ, so that $q^2=\lambda^2$. Keeping terms only to order ω_3^{-1}, we neglect q_3 occurring in the numerator of the (rationalized) matrix element terms, and terms of order ω_3 compared to $p_3^2-m^2$ and $p_4^2-m^2$ in the denominators.

We find that terms III and IV are not of the desired order. In view of the fact that we are to make matrix elements between the free electron states u_1, and u_2, a factor $(\boldsymbol{p}_1+m)\boldsymbol{e}_3$ (with e_3 the polarization vector of q_3) operating on the left of u_1, is equivalent to $2p_1\cdot e_3$ and a factor $\boldsymbol{e}_3(\boldsymbol{p}_2+m)$ operating on the right of u_2 is equivalent to $2p_2\cdot e_3$. Thus, with q_3 small, we get

$$\begin{aligned}\langle\mathrm{I}\rangle&=-Rp_1\cdot e_3/p_1\cdot q_3, & \langle\mathrm{II}\rangle&=-Sp_1\cdot e_3/p_1\cdot q_3,\\ \langle\mathrm{V}\rangle&=Rp_2\cdot e_3/p_2\cdot q_3, & \langle\mathrm{VI}\rangle&=Sp_2\cdot e_3/p_2\cdot q_3.\end{aligned} \quad (36)$$

Adding these we find the matrix for the double Compton process:

$$(R+S)\left(\frac{p_2\cdot e_3}{p_2\cdot q_3}-\frac{p_1\cdot e_3}{p_1\cdot q_3}\right), \quad |\mathbf{q}_3|\ll m. \quad (37)$$

Taking the absolute square of (37) and averaging over polarizations and spins in the usual manner, it is clear that we obtain the Klein-Nishina cross section $d\sigma_{\mathrm{K.N.}}$ (Eq. (11)) multiplied by the following factors: (a) $d^3\mathbf{q}_3/(2\pi)^3$, the density of states for $\boldsymbol{q}_3$ (neglecting its effect on the momentum balance, and therefore assuming it is emitted independently of $\boldsymbol{q}_2$), (b) e^2, from the additional interaction vertex, (c) $2\pi/\omega_3$, the normalization factor for the photon $\boldsymbol{q}_3$, (d)

$$\sum\nolimits_{\mathrm{pol}}\left(\frac{p_2\cdot e_3}{p_2\cdot q_3}-\frac{p_1\cdot e_3}{p_1\cdot q_3}\right)^2=-\left(\frac{p_2}{p_2\cdot q_3}-\frac{p_1}{p_1\cdot q_3}\right)^2.$$

We collect these factors and integrate the photon momentum over all angles and from $\mathbf{q}_3=0$ to the sphere $|\mathbf{q}_3|=k_{\max}$, where $k_{\max}\ll m$. Thus,[11]

$$d\sigma_D=-\frac{e^2}{(2\pi)^2}d\sigma_{\mathrm{K.N.}}\times\int_{|\mathbf{q}_3|=0}^{|\mathbf{q}_3|=k_{\max}}\left(\frac{p_2}{p_2\cdot q_3}-\frac{p_1}{p_1\cdot q_3}\right)^2\frac{d^3\mathbf{q}_3}{(\mathbf{q}_3^2+\lambda^2)^{\frac{1}{2}}}. \quad (38)$$

Observe that if we replace $d\sigma_{\mathrm{K.N.}}$ by the cross section $d\sigma_0$ for an arbitrary process of one electron, the result (37) is valid for that process with one additional photon of small energy in the final state. For if the "small" photon $\boldsymbol{q}_3$ is emitted from an electron line of momentum $\boldsymbol{p}$ where $p^2\neq m^2$, its effect will be negligible. The only diagrams contributing to the process with $\boldsymbol{q}$ emitted will be those of the original process, modified by the emission of $\boldsymbol{q}$ either before or after the original process. Thus the factor of $(R+S)$ in (37) will always be the factor modifying the original matrix element (for small $\boldsymbol{q}$) and the result (38) is the general result for an arbitrary process of one electron, $d\sigma_0$ replacing $d\sigma_{\mathrm{K.N.}}$.

It is most convenient to impose our restriction $k_{\max}\ll m$ in the laboratory system.[12] In this case, the result of integrating (38) (expressed in terms of invariants) is

$$d\sigma_D=-(e^2/\pi)d\sigma_{\mathrm{K.N.}}\{2(1-2y\,\mathrm{ctnh}2y)[\ln(2k_{\max}/\lambda)-\tfrac{1}{2}]+4y\,\mathrm{ctnh}2y[h(2y)-1]\}, \quad (39)$$

with y and $h(y)$ defined in (30).

[10] C. J. Eliezer, Proc. Roy. Soc. (London) **A187**, 210 (1946). When the class of photons inaccessible to the experiment does not consist simply of those below a given very small energy $k_{\max}$ (but consists, for example, of those in a given solid angle, or with a limited momentum component, or having energies too large to permit the approximations we have made) the contribution which these events make to the measured cross section can be obtained from Eliezer's formula. Explicitly, one must add to our result (39) the cross section for the double process given by Eliezer, integrated over all the photons in the class inaccessible to the experiment but which also exceed some arbitrary very small energy $k_{\max}$. The sum, of course, will not depend on $k_{\max}$.

[11] This expression (with $\lambda=0$) has been obtained previously. See, for example, R. Jost, Phys. Rev. **72**, 815 (1947), and F. Bloch and A. Nordsieck, Phys. Rev. **52**, 54 (1937).

[12] J. Schwinger, Phys. Rev. **76**, 790 (1949) has integrated (38) for the case of the scattering of an electron in a potential.

When (39) is added to (35) the effect is to replace the quantity

$$\{2(1-2y\,\mathrm{ctnh}2y)\ln\lambda-4y\,\mathrm{ctnh}2y[2h(y)-h(2y)]\}U$$

in $U^{(1)}$ by

$$\{2(1-2y\,\mathrm{ctnh}2y)[\ln(2k_{\max})-\tfrac{1}{2}] +8y\,\mathrm{ctnh}2y[h(2y)-h(y)-\tfrac{1}{2}]\}U. \quad (40)$$

We have now arrived at a physically understandable result as the quantity $k_{\max}$ which replaces λ in $U^{(1)}$ is the sum of the experimental uncertainties in the measurement of the final energies of the Compton scattered photon and electron. Our result can be compared with an experiment providing the energy resolution[10] is known and $k_{\max}$ is sufficiently small.

Under the limits of validity of our formula, the term in (40) containing $k_{\max}$ is positive, and thus makes a negative contribution to the cross section. As the energy resolution of an experiment improves, it is thus found that the measured Compton cross section gets smaller. This is reasonable since we are eliminating from our observations more double Compton events.

The expression (35a) for δ with $U^{(1)}$ given by (29) and U given by (10) is valid also for the correction to the two-quantum pair annihilation and the two-quantum pair production processes, provided that for the former problem we replace κ by $-\kappa$ and for the latter τ by $-\tau$. This occurs because in writing down the matrix element we represent the emission of a photon by $-\boldsymbol{q}$ and its absorption by $+\boldsymbol{q}$, and because a matrix $\boldsymbol{p}$ representing an electron also represents a positron of four-momentum $-p$. However, the infrared divergences in these problems are not compensated by the corresponding three-quantum processes[13] (which are not divergent) but by the effect of Coulomb interaction. This will not be discussed further in this paper.

V. EXTREME RELATIVISTIC LIMIT

The Compton formula (12) can be written in the laboratory system, with scattering angle φ, as

$$(\kappa+\tau)/\tau=\tfrac{1}{2}\kappa(1-\cos\varphi). \quad (41)$$

We assume $|\kappa|\gg1$ and consider the three cases listed in Table I. This table also lists the approximations made in obtaining the formulas for U and $U^{(1)}$ for the three cases and the corresponding conditions on the laboratory and center-of-mass scattering angles. The energies (in units of mc^2) of the incoming and outgoing quanta are represented in the laboratory system by ω_1 and ω_2, respectively, and in the c.m. system by ν. The results are as follows:

Case I.

$$U=2$$

$$U^{(1)}=4(1-2y\,\mathrm{ctnh}2y)\ln\lambda-8y\,\mathrm{ctnh}2y[2h(y)-h(2y)] +4yh(y)\,\mathrm{ctnh}y+\ln|\kappa|(4y\tanh y-1) -2y^2-4y\tanh y+3-(\ln|\kappa|)^2-\pi^2/6. \quad (42)$$

[13] Note that the two quantum pair processes are symmetric with respect to interchange of p_1 and p_2 so that (37) vanishes.

Case II.

$$U=-[(\kappa/\tau)+(\tau/\kappa)]$$

$$U^{(1)}=U\{(1-2y)(\tfrac{3}{2}+2\ln\lambda)+2y^2-\pi^2/6\} +\left(1+\frac{\tau}{2\kappa}+\frac{\kappa}{\tau}\right)\left\{\left[\ln\left(1+\frac{\tau}{\kappa}\right)\right]^2-\ln\left(1+\frac{\tau}{\kappa}\right) +2\ln\frac{\tau}{|\kappa|}\right\}+\left(1+\frac{\tau}{\kappa}+\frac{\kappa}{2\tau}\right)\left\{\left(\ln\left|1+\frac{\kappa}{\tau}\right|\right)^2 -\ln\left(1+\frac{\tau}{\kappa}\right)-\ln\left|\frac{\tau}{\kappa}\right|+\pi^2\right\}. \quad (43)$$

Case III.

$$U=-\kappa/\tau$$

$$U^{(1)}=U\left\{2(1-2y)\ln\lambda+\ln\tau\left[2y-\frac{3}{2}\frac{\tau+1}{\tau}+\frac{1}{2\tau(\tau-1)}\right] -\frac{\pi^2}{3}-G_0(\tau)\left(\frac{1}{\tau}+\frac{\tau}{2}\right)+\frac{3}{2}+\frac{2}{\tau}\right\}. \quad (44)$$

The corrected cross sections, in the relativistic limit are given in the laboratory system by

$$d\sigma=(r_0^2/2)d\Omega(\tau^2/\kappa^2)[U-(e^2/\pi)U^{(1)}] \quad (45)$$

and in the c.m. system by

$$d\sigma=(r_0^2/8\nu^2)d\Omega[U-(e^2/\pi)U^{(1)}] \quad (46)$$

with $r_0=e^2/mc^2$.

As we have explained in the previous section, for actual comparison with experiment one must add to (45) the cross section for double Compton scattering (35) which is valid, of course, only in the laboratory system with $k_{\max}\ll m$. If we write (39) as

$$d\sigma_D=(-e^2/\pi)(r_0^2/2)d\Omega(\tau^2/\kappa^2)U_D \quad (47)$$

then we must replace $U^{(1)}$ in (45) and (46) by $U^{(1)}+U_D$. For our three cases, we get:

TABLE I. Approximation made in obtaining the extreme relativistic limit of (30), expressed in invariants, laboratory system quantities, and c.m. system quantities.

Case	Defined by	Leads to	Lab system conditions	c.m. conditions
I	$\lvert(\kappa+\tau)/\tau\rvert\ll1$	$\lvert\kappa\rvert\approx\tau\gg1$	$\omega_2\approx\omega_1$, $1-\cos\varphi\ll1/\omega_1$ (φ near 0)	$\tan^2(\theta/2)\ll1$
II	$\lvert(\kappa+\tau)/\tau\rvert\sim1$	$\tau\gg1$, $\lvert\kappa+\tau\rvert\gg1$	ω_2 near $\omega_1/2$, $1-\cos\varphi\sim1/\omega_1$ (φ near $(2/\omega_1)^{\frac{1}{2}}$)	$\nu^2\cos^2(\theta/2)\gg1$, $\nu^2\sin^2(\theta/2)\gg1$
III	$\lvert(\kappa+\tau)/\tau\rvert\gg1$	$\lvert\kappa+\tau\rvert\gg1$, $\lvert\kappa\rvert\gg\tau$	ω_2 near 1, $1-\cos\varphi\gg1/\omega_1$ ($\varphi\gg(2/\omega_1)^{\frac{1}{2}}$)	$\tan^2(\theta/2)\gg1$

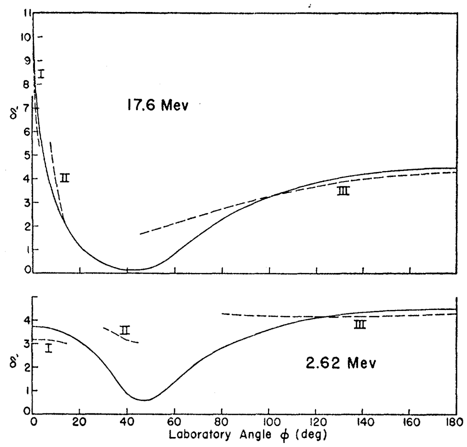

FIG. 5. Plot of $\delta' = -U^{(1)}/U - 2(1-2y \text{ ctnh} 2y) \ln\lambda$ for 2.62 Mev and 17.6 Mev as calculated from the exact expressions (29) and (30) (solid curve). The dotted curves are calculated from the extreme relativistic formulas (42), (43), and (44), and are numbered accordingly.

Case I.

Same as (39).

Cases II and III.

$$U_D = U\{2(1-2y)[\ln(2k_{\max}/\lambda) - \tfrac{1}{2}] + 4y[y + (\pi^2/24y) - 1]\}. \quad (48)$$

In Fig. 5 we have plotted a comparison of the exact expression for $-U^{(1)}/U$ (leaving out the term proportional to $\ln\lambda$) with the limiting cases expressed by (42), (43), and (44). That is, if we write $U^{(1)} = a \ln\lambda + b$, we have plotted $-b/U$. Especially simple formulas result for b in the extreme relativistic limit for the cases $\varphi = 0$ and $\varphi = 180°$. At zero angle (where incidentally, the double Compton effect and the $\ln\lambda$ term in $U^{(1)}$ vanish):

$$b(0°) = -\ln\tau(1+\ln\tau) + 1.355. \quad (49)$$

At 180 degrees,

$$b(180°) = -4.225U. \quad (50)$$

Another simple case results from the condition $\kappa = -2\tau$, in the extreme relativistic limit. This corresponds to 90° scattering in the c.m. system. Here we get $U = 5/2$ and

$$b(90°, \text{c.m.}) = (2y^2 - 3y - 2.29)U. \quad (51)$$

TABLE II. Percent correction to the Compton cross section for unpolarized light arising from Eq. (30), excluding the term proportional to $\ln\lambda$, at zero degrees and at ninety degrees in the c.m. system; computed as a function of the laboratory energy of the incident photon from the special equations (49) and (51).

Laboratory energy Mev	$-(e^2/\pi)b(0°)/U$ percent	$-(e^2/\pi)b(90°, \text{c.m.})/U$ percent
50	3.80	−0.32
150	5.26	−0.87
300	6.41	−2.13
1000	8.80	−4.35

A few results for high energies computed from (49) and (51) are given in Table II. At 180°, the quantity $-(e^2/\pi)b/U$ is +0.98 percent.

VI. NONRELATIVISTIC LIMIT

In the nonrelativistic or Thompson limit our results will be equally valid in the laboratory and c.m. systems, since we will keep only the first nonvanishing terms. In the c.m. system, we let the scattering angle be φ and $|\mathbf{q}_1| = |\mathbf{q}_2| = \omega$. Then

$$\kappa = -2p_1 \cdot q_1 = -2[(1+\omega^2)^{\frac{1}{2}}\omega + \omega^2]$$
$$\tau = 2p_1 \cdot q_2 = 2[(1+\omega^2)^{\frac{1}{2}}\omega + \omega^2 \cos\varphi], \quad (52)$$

so that for $\omega \ll 1$:

$$\kappa = -2\omega(1+\omega+\omega^2/2+\cdots)$$
$$\tau = 2\omega(1+\omega\cos\varphi+\omega^2/2+\cdots). \quad (53)$$

Since κ depends only on ω and $d\tau = \omega^2 d\Omega/\pi$, Eq. (11) becomes

$$d\sigma = (e^4/2m^2)d\Omega(1-2\omega+2\omega^2+\cdots)[U - (e^2/\pi)U^{(1)}]. \quad (54)$$

In Table III we give the nonrelativistic limits of functions occurring in $U^{(1)}$ (Eq. 30). A straightforward calculation then yields:

$$U = 1 + \cos^2\varphi + 0(\omega) \quad (55)$$

$$U^{(1)} = -(4/3)\omega^2(1-\cos\varphi)U\ln\lambda + (1+\cos\varphi+\cos^2\varphi - \tfrac{1}{3}\cos^3\varphi)4\omega^2\ln\omega + 0(\omega^2). \quad (56)$$

The angular dependence of the $4\omega^2 \ln\omega$ term is plotted as $f(\varphi)$ in Fig. 6. Expression (56) disagrees with the result of Schafroth.[2]

Since $(1/\kappa)+(1/\tau) = \frac{1}{2}(1-\cos\varphi) - (\omega/2)\sin^2\varphi$ in the c.m. system and the same invariant is $\frac{1}{2}(1-\cos\theta)$ in the laboratory system, with θ the laboratory scattering angle, it is clear that $\cos\theta = \cos\varphi + 0(\omega)$ for given κ, τ. Also, since $\kappa + \tau = -2\omega_1\omega_2(1-\cos\theta) = -2\omega^2(1-\cos\varphi)$ and $\omega_1 = \omega_2 + 0(\omega_1^2)$, we get that $\omega_1 = \omega + 0(\omega^2) \approx \omega_2$ and therefore (56) is valid with φ interpreted as the laboratory scattering angle and ω the incident photon energy. Equation (55) is valid, with this interpretation, to order ω^2.

The double Compton effect (39) gives in this limit (with U_D defined as in (47))

$$U_D = -(4/3)\omega^2(1-\cos\varphi)U\ln(2k_{\max}/\lambda) + 0(\omega^2). \quad (57)$$

It will be observed that all the corrections vanish in the zero energy limit.

APPENDIX W. ON THE TRANSCENDENTAL FUNCTIONS $G_0(\kappa)$ AND $h(y)$

The complete expression for the radiative correction (29) is expressed in terms of the relatively unfamiliar transcendental integrals $G_0(\kappa)$ and $h(y)$. These can both be expressed, however, in terms of one of the so-called Spence functions,[14] namely

$$L(x)=\int_0^x \ln(1-u)du/u \qquad \text{(A1)}$$

which we shall consider briefly.

It is well known that $L(-1)=\pi^2/12$, $L(1)=-\pi^2/6$. If $x<1$,

$$L(x)=-\int_0^x (u+\tfrac{1}{2}u^2+\cdots)du/u$$
$$=-\left(x+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\cdots\right). \qquad \text{(A2)}$$

If $x>1$,

$$L(x)=L(1)+\int_1^x \ln(1-u)du/u$$
$$=L(1)+\int_1^x [\ln|1-u|\pm i\pi]du/u$$
$$=\bar{L}(x)\pm i\pi \ln x \qquad \text{(A3)}$$

where

$$\bar{L}(x)=\int_0^x \ln|1-u|du/u.$$

For computational convenience, we can also write for $x>1$:

$$\bar{L}(x)=L(1)+\int_{1/x}^1 \ln\left(\frac{1-v}{v}\right)\frac{dv}{v}$$
$$=-\tfrac{1}{3}\pi^2-L(1/x)+\tfrac{1}{2}(\ln x)^2, \qquad \text{(A4)}$$

and in a similar manner

$$L(-x)=(\pi^2/6)-L(-1/x)+\tfrac{1}{2}(\ln x)^2. \qquad \text{(A5)}$$

Since $\kappa=(m^2-p_3^2)$ is always negative, $G_0(\kappa)=2\kappa^{-1}[L(1-\kappa)-L(1)]$ has an imaginary part whose sign is not determined by (A3). To fix the sign we must recall that according to the scheme of reference 1(b), all photons and electrons are considered to have a small additional negative imaginary mass. Thus κ has a small positive imaginary part, i.e., $\kappa=-|\kappa|+i\delta$ with δ vanishingly small. Therefore

$$G_0(\kappa)=2\kappa^{-1}[\bar{L}(1-\kappa)-L(1)]+i\pi 2\kappa^{-1}\ln(1-\kappa). \qquad \text{(A6)}$$

(Similarly, the term $\ln\kappa$ in (30) is equal to $\ln|\kappa|+i\pi$.) We might also note that since τ is always positive, $G_0(\tau)$ has no imaginary part.

Thus if $-\kappa\gg1$, $\tau\gg1$ (from A4, A5):

$$G_0(\kappa)\simeq 2\kappa^{-1}[\tfrac{1}{2}(\ln|\kappa|)^2-(\pi^2/6)+i\pi\ln|\kappa|] \qquad \text{(A7)}$$

$$G_0(\tau)\simeq 2\tau^{-1}[\tfrac{1}{2}(\ln\tau)^2+(\pi^2/3)]. \qquad \text{(A8)}$$

[14] For references see Fletcher, Miller, and Rosenhead, *An Index of Mathematical Tables* (Scientific Computing Service, Ltd., London, 1946), p. 343.

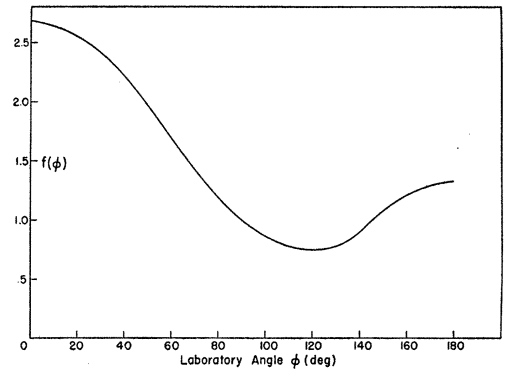

FIG. 6. Angular dependence of nonrelativistic limit of $U^{(1)}$ (without $\ln\lambda$ term).

Our other transcendental function

$$h(y)=y^{-1}\int_0^y udu\,\mathrm{ctnh}u$$

can also be expressed in terms of $L(x)$. Integration by parts gives

$$h(y)=\ln(\sinh y)-y^{-1}\int_0^y \ln(\sinh u)du. \qquad \text{(A9)}$$

Letting $t=e^{-2u}$ yields $\ln(\sinh u)=\ln(1-t)-\ln 2t$ and the integral is obtained directly:

$$h(y)=\ln(2\sinh y)-y/2+(2y)^{-1}[\pi^2/6+L(e^{-2y})]. \qquad \text{(A10)}$$

If y is sufficiently large that we can neglect e^{-2y} compared to unity, we get

$$h(y)\simeq(y/2)+(\pi^2/12y). \qquad \text{(A11)}$$

APPENDIX X. TABLE OF INTEGRALS

In this appendix we will simply list the integrals which enter this problem, reserving for Appendix Y a discussion of the methods used. To simplify the presentation of the integrals (which occur also in other problems) we introduce the following definitions:

For factors of the denominator we write $k^2-2p_1\cdot k=(1)$, $k^2-2p_2\cdot k=(2)$, $k^2-2p_3\cdot k-\kappa=(\kappa)$, $k^2=(0)$. We write for frequently occurring vectors

$$p_3=p_1+q_1, \qquad p_4=p_1-q_2,$$
$$2p_0=p_1+p_2, \qquad 2q_0=q_1+q_2,$$
$$2Q=p_1-p_2=q_2-q_1.$$

TABLE III. Nonrelativistic limits of functions occurring in Eq. (30). These expressions are valid in either the c.m. or laboratory system with φ the scattering angle and ω the incident photon energy.

$$y^2=-\tfrac{1}{4}(\kappa+\tau)-(1/48)(\kappa+\tau)^2+0(\omega^6)$$
$$h(y)=1+y^2/9+\cdots$$
$$G_0(\kappa)=2(1+\kappa/2+\kappa^2/9+\cdots)-2\ln|\kappa|(1+\kappa/2+\kappa^2/3+\cdots)$$
$$\ln|\kappa|=\ln 2\omega+\omega+0(\omega^3)$$
$$\ln\tau=\ln 2\omega+\omega\cos\varphi+\tfrac{1}{2}\omega^2(1-\cos^2\varphi)+0(\omega^3)$$

The integrals are defined by

$$A=8i\int d^4k/(1)(2),\qquad B^{(1)}=8i\int d^4k/(1)(\kappa)$$

$$C^{(1)}=8i\int d^4k/(1)(0),\qquad D=8i\int d^4k/(0)(\kappa)$$

$$F=8i\int d^4k/(1)(2)(\kappa),\quad G^{(1)}=8i\int d^4k/(1)(\kappa)(0)$$

$$H=8i\int d^4k/(1)(2)(0),\qquad J=8i\int d^4k/(1)(2)(\kappa)(0).$$

Integrals $B^{(2)}$, $C^{(2)}$, $G^{(2)}$ are defined as $B^{(1)}$, $C^{(1)}$, $G^{(1)}$ but with (2) replacing (1). Their values are obtained from those of $B^{(1)}$, $C^{(1)}$, $G^{(1)}$ by replacing p_1 by p_2, q_1 by q_2, and Q by $-Q$. We use the notation, as in reference 1(b), that

$$F_{(0;\sigma;\sigma\tau)}=8i\int(1;k_\sigma;k_\sigma k_\tau)d^4k/(1)(2)(\kappa),\text{ etc.}$$

Let:

$$\mu=Q^2=-\sinh^2 y=\tfrac{1}{4}(\kappa+\tau),\quad (y\text{ is real as }Q^2<0)$$

$$a=-\ln\Lambda^2$$

$$b=1-y\coth y$$

$$c=(\kappa/\kappa-1)\ln\kappa$$

$$d=2y\,\mathrm{csch}\,2y=(1-b)/(1-\mu)$$

$$L(x)=\int_0^x\frac{du}{u}\ln(1-u)$$

$$h(y)=(1/y)\int_0^y u\coth u\,du$$

$$v=\kappa^2+4\mu(1-\kappa).$$

Integrals

$$A_0=2a-4b+2$$

$$B_0^{(1)}=2a+2=B_0$$

$$C_0^{(1)}=2a-2=C_0$$

$$D_0=2a-2+2c$$

$$F_0=y^2\,\mathrm{csch}^2 y$$

$$G_0^{(1)}=(2/\kappa)[L(1-\kappa)-L(1)]=G_0$$

$$H_0=2d[-\ln\lambda+h(2y)-h(y)]$$

$$J_0=(2d/\kappa)[2h(y)-h(2y)-\ln(\kappa/\lambda)]$$

$$A_\sigma=(2a-4b+3)p_{0\sigma}$$

$$B_\sigma^{(1)}=(a+\tfrac{3}{2})(p_{1\sigma}+p_{3\sigma})$$

$$C_\sigma^{(1)}=(a+\tfrac{1}{2})p_{1\sigma}$$

$$D_\sigma=[(\kappa/\kappa-1)(c-1)+a+\tfrac{1}{2}]p_{3\sigma}$$

$$F_\sigma=F_0p_{0\sigma}+[F_0-(2b/\mu)]q_{0\sigma}$$

$$G_\sigma^{(1)}=[G_0-(2c/\kappa)]p_{1\sigma}+(2/\kappa)[G_0-(2-\kappa/\kappa)c-2]q_{1\sigma}$$

$$H_\sigma=2dp_{0\sigma}$$

$$vJ_\sigma=[2\mu F_0-(2\mu-\kappa)Z-\kappa G_0]p_{0\sigma}+[(2\mu-\kappa)F_0+2(1-\mu)Z-(2-\kappa)G_0]q_{0\sigma}$$

with

$$Z=H_0+\kappa J_0=2d[-\ln\kappa+h(y)]$$

$$F_{\sigma\tau}=F_\sigma p_{0\tau}+[(F_0-2b/\mu)p_{0\sigma}+(F_0+Y_0-2b/\mu)q_{0\sigma}]q_{0\tau}-Y_0Q_\sigma Q_\tau+[\mu Y_0+\tfrac{1}{2}a+\tfrac{3}{4}]\delta_{\sigma\tau}$$

with

$$Y_0=(F_0-2b-1)/2\mu$$

$$G_{\sigma\tau}^{(1)}=p_{1\sigma}p_{1\tau}\left[G_0-\frac{3\kappa-2}{\kappa-1}\frac{c}{\kappa}+\frac{1}{\kappa-1}\right]+(p_{1\sigma}q_{1\tau}+q_{1\sigma}p_{1\tau})\left[\frac{3}{\kappa}G_0+\frac{2\kappa^2-9\kappa+6}{\kappa(\kappa-1)}\frac{c}{\kappa}+\frac{6-5\kappa}{\kappa-1}\frac{1}{\kappa}\right]$$

$$+q_{1\sigma}q_{1\tau}\left[\frac{6}{\kappa^2}G_0+\frac{\kappa^3+4\kappa^2-18\kappa+12}{\kappa^2(\kappa-1)}\frac{c}{\kappa}-\frac{2\kappa^2+9\kappa-12}{\kappa(\kappa-1)}\frac{1}{\kappa}\right]+\frac{1}{4}\delta_{\sigma\tau}\left[2G_0-3+2a+2\frac{\kappa-2}{\kappa}c\right]$$

$$H_{\sigma\tau}=dp_{0\sigma}p_{0\tau}+(b/\mu)Q_\sigma Q_\tau+\tfrac{1}{2}\delta_{\sigma\tau}(a-2b+\tfrac{1}{2})$$

$$J_{\sigma\tau}=\alpha_\sigma Q_\tau+\beta_\sigma p_{0\tau}+\gamma_\sigma p_{3\tau}+\epsilon s_{\sigma\tau}$$

where:

$$\epsilon=F_0-\alpha_\sigma Q_\sigma-\beta_\sigma p_{0\sigma}-\gamma_\sigma P_{3\sigma}$$

$$4\mu\alpha_\sigma=G_\sigma^{(1)}-G_\sigma^{(2)}$$

$$v\beta_\sigma=\kappa F_\sigma-(2-\kappa)(H_\sigma+\kappa J_\sigma)+(1-\kappa)(G_\sigma^{(1)}+G_\sigma^{(2)})$$

$$v\gamma_\sigma=(2\mu-\kappa)F_\sigma+2(1-\mu)(H_\sigma+\kappa J_\sigma)-\tfrac{1}{2}(2-\kappa)(G_\sigma^{(1)}+G_\sigma^{(2)})$$

$$s_{\sigma\tau}=\delta_{\sigma\tau}-(Q_\sigma Q_\tau/\mu)+4(1-\kappa)(p_{0\sigma}p_{0\tau}/v)-2(2-\kappa)[(p_{0\sigma}p_{3\tau}+p_{3\sigma}p_{0\tau})/v]+4(1-\mu)(p_{3\sigma}p_{3\tau}/v)$$

$$[s_{\sigma\sigma}=1]$$

$$J_{\sigma\tau\rho}=\alpha_{\sigma\tau}Q_\rho+\beta_{\sigma\tau}p_{0\rho}+\gamma_{\sigma\tau}p_{3\rho}+\epsilon_\sigma s_{\tau\rho}+\epsilon_\tau s_{\sigma\rho}$$

where

$$\epsilon_\sigma=F_\sigma-\alpha_{\sigma\tau}Q_\tau-\beta_{\sigma\tau}p_{0\tau}-\gamma_{\sigma\tau}p_{3\tau}$$

$$4\mu\alpha_{\sigma\tau}=G_{\sigma\tau}^{(1)}-G_{\sigma\tau}^{(2)}$$

$$v\beta_{\sigma\tau}=\kappa F_{\sigma\tau}-(2-\kappa)(H_{\sigma\tau}+\kappa J_{\sigma\tau})+(1-\kappa)(G_{\sigma\tau}^{(1)}+G_{\sigma\tau}^{(2)})$$

$$v\gamma_{\sigma\tau}=(2\mu-\kappa)F_{\sigma\tau}-2(1-\mu)(H_{\sigma\tau}+\kappa J_{\sigma\tau})-\tfrac{1}{2}(2-\kappa)(G_{\sigma\tau}^{(1)}+G_{\sigma\tau}^{(2)}).$$

APPENDIX Y. ON THE INTEGRALS IN THE CORRECTED CROSS SECTION

In this section we will describe the methods by which the integrations occurring in the e^4-order matrix element have been performed and will give some examples of the calculations.

Those integrals which are scalars (those with no k in the numerator) have been done by the parametric method discussed in the appendix to reference 1(b). Those which are tensors (having one or more k's in the numerator) can be done either by the parametric method, or can be derived by an algebraic procedure described below from those of lower tensor order.

We now do several examples of the integrals to indicate the methods employed.

(a) The Two-Denominator Integrals

These all have the form $\chi(\Lambda^2)$ (where we need only the case of Λ much greater than any of the momenta involved in the problem), with

$$\chi=8i\int(1;k_\sigma)(k^2-2p_1\cdot k-\Delta_1)^{-1}\times(k^2-2p_2\cdot k-\Delta_2)^{-1}(-\Lambda^2)(k^2-\Lambda^2)^{-1}d^4k. \quad \text{(A12)}$$

To reduce this integral and those considered below we use the methods of reference 1(b).

In the first place we combine the denominators by making use of the relation

$$1/ab=\int_0^1 dy[ay+b(1-y)]^{-2} \quad \text{(A13)}$$

and similar expressions for $1/ab^2$, $1/ab^3$, etc., obtainable from (A13) by differentiation. Thus χ becomes

$$\chi=8i\int_0^1\int_0^1\int(1;k_\sigma)dy2zdz(-\Lambda^2)d^4k\times[k^2-2zp_y\cdot k-z\Delta_y-(1-z)\Lambda^2]^{-3} \quad \text{(A14)}$$

with $p_y=p_1y+p_2(1-y)$; $\Delta_y=\Delta_1y+\Delta_2(1-y)$. Using (12a) of reference 1(b), we get

$$\chi=\int_0^1\int_0^1 2zdzdy(-\Lambda^2)(1;zp_{y\sigma})\times[z^2p_y^2+z\Delta_y+(1-z)\Lambda^2]^{-1}. \quad \text{(A15)}$$

To facilitate the work we observe that in the limit of very large Λ^2 (with $\delta\ll1$)

$$\int_0^1\frac{(1-z)^ndz(-\Lambda^2)}{P(z)+\Lambda^2(1-z)}\cong-\int_0^{1-\delta}(1-z)^{n-1}dz+\int_{1-\delta}^1\frac{(1-z)^ndz(-\Lambda^2)}{P(1)+\Lambda^2(1-z)}$$
$$\cong-1/n,\quad n>0$$
$$\cong\ln[P(1)/\Lambda^2],\quad n=0. \quad \text{(A16)}$$

Writing

$$(1;zp_{y\sigma})=(1-(1-z);[(1-z)^2-2(1-z)+1]p_{y\sigma}),$$

(A15) becomes

$$\chi=\int_0^1 2dy\left\{(1;p_{y\sigma})\ln\frac{p_y^2+\Delta_y}{\Lambda^2}+(1;\tfrac{3}{2}p_{y\sigma})\right\}. \quad \text{(A17)}$$

The second term gives $(2;3(p_{1\sigma}+p_{2\sigma}))$. In the first term

$$\chi_a=\int_0^1 2dy(1;p_{y\sigma})\ln[(p_y^2+\Delta_y)/\Lambda^2], \quad \text{(A18)}$$

notice that with $2Q=(p_1-p_2)$,

$$p_y^2=p_2^2+Q^2y^2+2p_2\cdot(p_1-p_2)y,$$
$$\Delta_1=m^2-p_1^2,\quad \Delta_2=m^2-p_2^2$$

so that

$$p_y^2+\Delta_y=m^2+4Q^2(y^2-y)=m^2+Q^2[(2y-1)^2-1].$$

Now let $m=1$, $Q^2=\sin^2\theta$, and $2y-1=\tan\alpha/\tan\theta$, so that $dy(\sec^2\alpha/2\tan\theta)d\alpha$. We get

$$p_y^2+\Delta_y=\cos^2\theta\sec^2\alpha$$

so that

$$\chi_a=\int_{-\theta}^{\theta}(\sec^2\alpha/\tan\theta)d\alpha(1;p_{y\sigma}(\alpha))\times\ln(\cos^2\theta\sec^2\alpha/\Lambda^2). \quad \text{(A19)}$$

This integral can be done easily. For example,

$$\int_{-\theta}^{\theta}\sec^2\alpha d\alpha\ln(\sec^2\alpha)=\int_{-\tan\theta}^{\tan\theta}dy\ln(1+y^2)$$
$$=4\tan\theta[\ln(\sec\theta)-1]+4\theta,$$

the last integral being performed by parts. In this manner we obtain integrals A, B, C, D.

(b) The Intregal $G_0^{(1)}$

As an example of the three denominator integrals we integrate $G_0^{(1)}$. The parameterization method gives

$$G_0^{(1)}=8i\int d^4k(k^2-2p_1\cdot k)^{-1}(k^2-2p_3\cdot k-\kappa)^{-1}(k^2)^{-1}$$
$$=8i\int_0^1\int_0^1\int d^4k2dyxdx(k^2-2p_x\cdot k-\Delta_x)^{-3} \quad \text{(A20)}$$

with

$$p_y=(1-y)p_1+yp_3=p_1+yq_1,\quad p_x=xp_y,\quad \Delta_x=xy\kappa.$$

Using (12a) of reference 1(b) again,

$$G_0^{(1)}=\int_0^1\int_0^1 2dxdy(xp_y^2+y\kappa)^{-1}. \quad \text{(A21)}$$

Since

$$\int_0^1 dx(ax+b)^{-1}=a^{-1}\ln(b+a)/b,$$

and since $p_y^2=1-\kappa y$

$$G_0^{(1)}=-\int_0^1 2dy(1-\kappa y)^{-2}\ln\kappa y$$

$$=-\frac{2}{\kappa}\int_{1-\kappa}^1 \frac{dv}{v}\ln(1-v). \quad \text{(A22)}$$

In the last step we have let $v=1-\kappa y$. We obtain finally:

$$G_0^{(1)}=G_0^{(2)}=G_0=(2/\kappa)[L(1-\kappa)-L(1)]. \quad \text{(A23)}$$

(c) The Integrals H

These are the same as those done in the radiationless scattering problem. They are given in reference 1(b), appendix. Equations (23a), (24a), and (25a) should have the signs of their left-hand sides changed.

(d) The Integral J_0

$$J_0=8i\int d^4k(k^2-2p_1\cdot k)^{-1}(k^2-2p_3\cdot k-\kappa)^{-1}$$

$$\times(k^2-2p_2\cdot k)^{-1}(k^2-\lambda^2)^{-1}. \quad \text{(A24)}$$

We have given the photon k a small mass λ as an infrared cutoff. When this integral is parameterized it becomes

$$J_0=8i\int_0^1\int_0^1\int_0^1\int 6dxdydz(1-x)z^2d_4k$$

$$\times[k^2-2zp_x\cdot k-xz\kappa-\lambda^2(1-z)]^{-4}. \quad \text{(A25)}$$

By (13a) of reference 1(b),

$$8i\int d^4k(k^2-2p\cdot k-\Delta)^{-4}=-\tfrac{1}{3}(p^2+\Delta)^{-2}$$

so that

$$J_0=-\int_0^1\int_0^1\int_0^1 2dxdydz(1-x)z^2$$

$$\times[z^2p_x^2+xz\kappa+\lambda^2(1-z)]^{-2} \quad \text{(A26)}$$

with $p_x=(1-x)p_y+xp_3$, $p_y=yp_1+(1-y)p_2$.

We break the x integration into two regions ($\epsilon\ll1$):

$$\int_0^1 dx\int_0^1 dy\int_0^1 dz=\int_0^1 dy\int_\epsilon^1 dx\int_0^1 dz+\int_0^1 dy\int_0^\epsilon dx\int_0^1 dz.$$

$$\text{(I)} \qquad\qquad \text{(II)}$$

In (I) we let $\lambda\rightarrow0$, getting

$$\text{(I)}=\int_0^1 dy\int_\epsilon^1 dx\int_0^1 2dz(1-x)(zp_x^2+x\kappa)^{-2}$$

$$=2\int_0^1\int_\epsilon^1\frac{(1-x)dxdy}{\kappa x(p_x^2+\kappa x)}. \quad \text{(A27)}$$

In region (II), since x is small, we neglect x^2 compared to x:

$$\text{(II)}=\int_0^1 dy\int_0^1 dz\int_0^\epsilon 2z^2dx[z^2p_y^2+2xz^2a$$

$$+z\kappa x+\lambda^2(1-z)]^{-2}$$

$$=2\int_0^1 dy\int_0^1 dz\epsilon z^2[z^2p_y^2+\lambda^2(1-z)]^{-1}$$

$$\times[z^2p_y^2+2\epsilon z^2a+z\kappa\epsilon+\lambda^2(1-z)]^{-1} \quad \text{(A28)}$$

with $a=p_y\cdot(p_3-p_y)$.

In (II) we now break the z integration into two regions; $0\leq z\leq z_c$ and $z_c\leq z\leq 1$ such that $\lambda^2\ll z_c^2p_y^2\ll z_c\kappa\epsilon$. Thus for $z<z_c$ we neglect z relative to unity and $z^2p_y^2$ relative to $z\kappa\epsilon$. For $z>z_c$ we neglect λ. There results:

$$\int_{z_c}^1\frac{2\epsilon dzdy}{p_y^2z[z(p_y^2+2\epsilon a)+\kappa\epsilon]}=\frac{2}{\kappa p_y^2}\ln\frac{\kappa\epsilon}{z_cp_y^2},$$

$$\int_0^{z_c}\frac{2\epsilon z^2dzdy}{(z^2p_y^2+\lambda^2)(z\kappa\epsilon+\lambda^2)}$$

$$=-\int_0^{z_c}\frac{2\epsilon dz}{(\lambda^2p_y^2+\kappa^2\epsilon^2)}\left(\frac{\lambda^2-\kappa\epsilon z}{z^2p_y^2+\lambda^2}-\frac{\lambda^2}{\kappa\epsilon z+\lambda^2}\right)$$

$$=\frac{1}{\kappa p_y^2}\ln\frac{z_c^2p_y^2}{\lambda^2}\text{(neglecting terms of order }\lambda\text{)}.$$

Adding these together we get

$$\text{(II)}=\int_0^1\frac{dy}{\kappa p_y^2}\ln\frac{\kappa^2\epsilon^2}{\lambda^2p_y^2}. \quad \text{(A29)}$$

We can now use the same substitution for y as that leading to (A8). Notice that in this case $\Delta_y=0$. We get $dy/\kappa p_y^2=d\alpha/\sin2\theta$, $p_y^2=\cos^2\theta/\cos^2\alpha$ so that (II) becomes:

$$\text{(II)}=\int_{-\theta}^{\theta}\frac{2d\alpha}{\sin2\theta}\left[\ln\frac{\kappa\epsilon}{\lambda\cos\theta}+\ln(\cos\alpha)\right]$$

$$=\frac{4\theta}{\sin2\theta}\ln\frac{\kappa\epsilon}{\lambda\cos\theta}+\int_0^{\theta}\frac{4d\alpha}{\sin2\theta}\ln(\cos\alpha). \quad \text{(A30)}$$

We have still to finish evaluating (I), Eq. (A27). First investigate the denominator $p_x^2+\kappa x$. Since $p_3^2=1-\kappa$ and $2p_3\cdot p_y=2-\kappa$,

$$p_x^2=(1-x)^2p_y^2+x^2(1-\kappa)+x(1-x)(2-\kappa)$$
$$p_x^2+\kappa x=(1-x)^2(p_y^2-1)+1.$$

Letting $1-x=\sin\phi/\sin\theta$, $2y-1=\tan\alpha/\tan\phi$ with $Q^2=\sin^2\theta$, we get

$$p_y^2=\sin^2\theta[(\tan^2\alpha/\tan^2\phi)-1],$$
$$p_x^2+\kappa x=\sin^2\phi\ \mathrm{ctn}^2\phi(\sec^2\alpha-\sec^2\phi)+1=\cos^2\phi\sec^2\phi,$$
$$-dx=\cos\phi d\phi/\sin\phi,\quad dy=\sec^2\alpha d\alpha/2\tan\phi.$$

The integrand of (I) becomes

$$\frac{1}{\kappa}\left(\frac{\sin\theta}{\sin\theta-\sin\phi}\right)\left(\frac{\sin\phi}{\sin\theta}\right)\left(\frac{\cos^2\alpha}{\cos^2\phi}\right)\left(\frac{\cos\phi d\phi}{\sin\theta}\right)\left(\frac{\sec^2\alpha d\alpha}{\tan\phi}\right)=\frac{1}{\kappa\sin\theta}\frac{d\phi d\alpha}{\sin\theta-\sin\phi}$$

and

$$(\mathrm{I})=\int_0^{\theta-\epsilon\tan\theta}\frac{d\phi}{\kappa}\int_{-\phi}^{\phi}\frac{d\alpha}{\sin\theta(\sin\theta-\sin\phi)}=\int_0^{\theta-\epsilon\tan\theta}\frac{2\phi d\phi}{\kappa\sin\theta(\sin\theta-\sin\phi)}.\quad (\mathrm{A31})$$

Integrate (A31) by parts, using No. 436, Dwight's *Tables of Integrals*, to get

$$(\mathrm{I})=-\frac{4\theta}{\kappa\sin2\theta}\ln\left(\frac{\epsilon\tan\theta}{2\cos\theta}\right)+\int_0^\theta\frac{4d\phi}{\sin2\theta}\ln\left(\frac{\sin\frac{1}{2}(\theta-\phi)}{\cos\frac{1}{2}(\theta+\phi)}\right)\quad (\mathrm{A32})$$

which can be put in the form

$$(\mathrm{I})=-\frac{4\theta}{\kappa\sin2\theta}\ln\frac{\epsilon\sin\theta}{\cos^2\theta}-\int_0^\theta\frac{4d\alpha}{\sin2\theta}\ln(\cos\alpha)+\int_0^\theta\frac{4d\alpha}{\sin2\theta}\ln(\tan\alpha)$$

and can now be added to (A19).

This gives the result,

$$J_0=\frac{-4\theta}{\kappa\sin2\theta}\left[\ln\frac{\kappa}{\lambda\tan\theta}+\theta^{-1}\int_0^\theta dv\ln(\tan v)\right].\quad (\mathrm{A33})$$

Another integration by parts and the substitution $\theta=iy$ gives the result in the table:

$$J_0=(2d/\kappa)[2h(y)-h(2y)-\ln(\kappa/\lambda)].\quad (\mathrm{A34})$$

(e) The Integrals J_σ, $J_{\sigma\tau}$, $J_{\sigma\tau\nu}$

From the preceding work on J_0 it may be supposed that to attempt these more complicated integrals by the parametric method would involve great labor. Fortunately there is a way to reduce these integrals to a combination of integrals of a lower tensor order, and those of a smaller number of denominators. We will do J_σ as an example but it should be clear from this how $J_{\sigma\tau}$ and $J_{\sigma\tau\nu}$ are done. This method can of course, be applied also to G_σ, $G_{\sigma\tau}$, etc. We shall be able to express J_σ in terms of J_0 and the integrals F, G having only three denominators.

Using the notation indicated in Appendix X for the denominators, $(1)=k^2-2p_1\cdot k$, etc., we write

$$J_\sigma=8i\int\frac{k_\sigma d^4k}{(1)(2)(\kappa)(0)}=\alpha p_{1\sigma}+\beta p_{2\sigma}+\gamma p_{3\sigma},\quad (\mathrm{A35})$$

α, β, γ being scalar functions of p_1, p_2, and p_3. The vectors p_1, p_2, p_3 will in general define a three-space. It is clear that the vector J_σ cannot have a component in the direction P which is perpendicular to this three-space, since for k_σ in the P direction, the integrand is an odd function and therefore J_σ must vanish.

If we now take the scalar products of J_σ with p_1, p_2, p_3, since $2p_1\cdot k=(0)-(1)$, etc., we get

$$2p_1\cdot J=8i\int\frac{2p_1\cdot kd^4k}{(1)(2)(\kappa)(0)}=8i\int\frac{d^4k}{(1)(2)(\kappa)}-8i\int\frac{d^4k}{(\kappa)(2)(0)}=F_0-G_0$$
$$2p_2\cdot J=8i\int\frac{2p_2\cdot kd^4k}{(1)(2)(\kappa)(0)}=8i\int\frac{d^4k}{(1)(2)(\kappa)}-8i\int\frac{d^4k}{(\kappa)(2)(0)}=F_0-G_0\quad (\mathrm{A36})$$
$$2p_3\cdot J=8i\int\frac{2p_3\cdot kd^4k}{(1)(2)(\kappa)(0)}=8i\int\frac{d^4k}{(1)(2)(\kappa)}-8i\int\frac{d^4k}{(1)(2)(0)}-8i\int\frac{\kappa d^4k}{(1)(2)(\kappa)(0)}=F_0-H_0-\kappa J_0.$$

Taking also the scalar products with the right-hand side of (A35), we get the set of linear equations:

$$F_0-G_0=2\alpha+\beta(1-\tfrac{1}{2}\kappa-\tfrac{1}{2}\tau)+\gamma(2-\kappa)$$
$$F_0-G_0=\alpha(1-\tfrac{1}{2}\kappa-\tfrac{1}{2}\tau)+2\beta+\gamma(2-\kappa)\quad (\mathrm{A37})$$
$$F_0-H_0-\kappa J_0=(\alpha+\beta)(2-\kappa)+2\gamma(1-\kappa).$$

J_σ can now be readily obtained by solving these equations for α, β, and γ.

To obtain $J_{\sigma\tau}$ we write

$$J_{\sigma\tau}=8i\int\frac{k_\sigma k_\tau d^4k}{(1)(2)(\kappa)(0)}$$
$$=\alpha_\sigma p_{1\tau}+\beta_\sigma p_{2\tau}+\gamma_\sigma p_{3\tau}+\epsilon\delta_{\sigma\tau},\quad \text{(A38)}$$

α_σ, β_σ, γ_σ being vector functions of p_1, p_2, p_3, and ϵ a scalar function of the same variables. The tensor $\epsilon\delta_{\sigma\tau}$ now occurs on the right-hand side as it is possible for $J_{\sigma\tau}$ to have nonzero components depending on P. (If $k_\sigma=k_\tau=k_P$, $J_{\sigma\tau}$ need not vanish.) If we take inner products with p_1, p_2, and p_3 now we get

$$F_\sigma-G_\sigma^{(1)}=2\alpha_\sigma+2\beta_\sigma(1-\tfrac{1}{2}\kappa-\tfrac{1}{2}\tau)+\gamma_\sigma(2-\kappa)+2\epsilon p_{1\sigma}$$

for $2p_{1\tau}J_{\sigma\tau}$ and similar equations for p_2 and p_3. This gives us three equations for the four quantities α_σ, β_σ, γ_σ, ϵ. However, there is the additional independent result obtained by summing $J_{\sigma\sigma}$ over σ:

$$J_{\sigma\sigma}=8i\int\frac{k_\sigma k_\sigma d^4k}{(1)(2)(\kappa)(0)}$$
$$=F_0=\alpha_\sigma p_{1\sigma}+\beta_\sigma p_{2\sigma}+\gamma_\sigma p_{3\sigma}+4\epsilon.\quad \text{(A39)}$$

Solving these four equations we obtain $J_{\sigma\tau}$ algebraically in terms of simpler integrals. In a similar manner $J_{\sigma\tau\nu}$ can be expressed in the form given in Appendix X.

APPENDIX Z. EXAMPLES OF THE CALCULATIONS

We shall here illustrate by two examples the method of evaluation of the transition amplitude.

The matrix T (18) can be written, if we rationalize the denominator, as ($p^2=m^2$, $q^2=0$)

$$T=\int(k^2-2p\cdot k-2q\cdot k-\kappa)^{-1}(k^2-2p\cdot k)^{-1}$$
$$\times d^4kk^{-2}C(k^2)\mathbf{T}\quad \text{(A40)}$$

with

$$\mathbf{T}=\gamma_\mu(\mathbf{p}+\mathbf{q}-\mathbf{k}+m)\mathbf{e}(\mathbf{p}-\mathbf{k}+m)\gamma_\mu.\quad \text{(A41)}$$

$\mathbf{T}$ can now be split into terms involving no $\mathbf{k}$, one $\mathbf{k}$, and two $\mathbf{k}$'s, and the results of the integrations over $\mathbf{k}$-space inserted from Appendix Y. Thus,

$$\mathbf{T}=\gamma_\mu(\mathbf{p}+\mathbf{q}+m)\mathbf{e}(\mathbf{p}+m)\gamma_\mu-\gamma_\mu\mathbf{k}\mathbf{e}(\mathbf{p}+m)\gamma_\mu$$
$$-\gamma_\mu(\mathbf{p}+\mathbf{q}+m)\mathbf{e}\mathbf{k}\gamma_\mu+\gamma_\mu\mathbf{k}\mathbf{e}\mathbf{k}\gamma_\mu$$
$$=2\mathbf{p}(\mathbf{p}+\mathbf{q}+m)\mathbf{e}-2\mathbf{p}\mathbf{k}\mathbf{e}+2\mathbf{k}\mathbf{e}(\mathbf{p}+\mathbf{q})$$
$$-4m(e\cdot k)-2\mathbf{k}\mathbf{e}\mathbf{k},$$

where we have used Eq. (4a) of reference 1(b) and the fact that the matrix T operates from the left on a state u such that $\mathbf{p}u=mu$. Inserting the integrals and grouping terms we now have,

$$8iT=2\mathbf{p}(\mathbf{p}+\mathbf{q}+m)\mathbf{e}G_0-2(\mathbf{p}\gamma_\sigma\mathbf{e}-\gamma_\sigma\mathbf{e}(\mathbf{p}+\mathbf{q})+2me_\sigma)G_\sigma$$
$$-2\gamma_\sigma\mathbf{e}\gamma_\tau G_{\sigma\tau}.\quad \text{(A42)}$$

This expression can now be further simplified. For example, the term in G_σ is (using Appendix X)

$$-2[G_0-(2c/\kappa)][\mathbf{p}^2\mathbf{e}-\mathbf{p}\mathbf{e}(\mathbf{p}+\mathbf{q})+2m(e\cdot p)]$$
$$-(4/\kappa)[G_0-(2-\kappa/\kappa)c-2][\mathbf{p}\mathbf{q}\mathbf{e}-\mathbf{q}\mathbf{e}(\mathbf{p}+\mathbf{q})]$$

with $\kappa=-2p\cdot q$. Since $\mathbf{p}^2=m^2$, $\mathbf{q}^2=0$, $e\cdot q=0$, and quite generally $\mathbf{a}\mathbf{b}+\mathbf{b}\mathbf{a}=2a\cdot b$, this term becomes finally

$$-2[G_0-(2c/\kappa)](2m^2\mathbf{e}-\mathbf{p}\mathbf{e}\mathbf{q})$$
$$+(4/\kappa)[G_0-(2-\kappa/\kappa)c-2]\cdot[2(e\cdot p)\mathbf{q}+\kappa\mathbf{e}].$$

Combining this with the terms in G_0 and $G_{\sigma\tau}$ in (A42) (expanded in the same manner) we obtain the expression for T given in the text (19).

To illustrate the simplification that occurs upon taking the spur, for unpolarized light, consider the term J (21). This may be decomposed, as was T above, into a sum of terms involving various numbers of $\mathbf{k}$'s in the numerator. For example, the term involving three $\mathbf{k}$'s is

$$-\int\gamma_\mu\mathbf{k}\mathbf{e}_2\mathbf{k}\mathbf{e}_1\mathbf{k}\gamma_\mu d^4k/(1)(2)(\kappa)(0)\quad \text{(A43)}$$

using the notation of Appendix X. If we replace $\mathbf{e}_1$, by γ_α and $\mathbf{e}_2$ by γ_β, this gives a contribution to the spur $P(\kappa,\tau)$, Eq. (28), of a numerical factor times

$$\int g(k)d^4k/(1)(2)(\kappa)(0)\quad \text{(A44)}$$

with

$$g(k)=\mathrm{Sp}[(\mathbf{p}_2+m)\gamma_\mu\mathbf{k}\gamma_\beta\mathbf{k}\gamma_\alpha\mathbf{k}\gamma_\mu(\mathbf{p}_1+m)\overline{W}]$$
$$=\kappa^{-1}\{4(p_1\cdot k)(p_2\cdot k)(p_3\cdot k)$$
$$+k^2[\tfrac{1}{2}m^2\kappa(p_1+p_2)\cdot k-(p_1\cdot p_2+2m^2)(p_3\cdot k)]\}$$
$$+\tau^{-1}\{4(p_1\cdot k)(p_2\cdot k)(p_4\cdot k)$$
$$-m^2k^2(p_1+p_2+p_3)\cdot k\}.\quad \text{(A45)}$$

It will now be seen that the factor k^2 will cancel the factor (0) in the denominator of (A44) leading to the integral F_σ (Appendix X). Also, we can write

$$2p_1\cdot k=-(k^2-2p_1\cdot k)+k^2=-(1)+(0)$$

leading to integrals $G_{\sigma\tau}^{(2)}$ and $F_{\sigma\tau}$. Thus it is not necessary, for unpolarized light, to use integral $J_{\sigma\tau\rho}$. (A44) can now be written

$$2(\kappa^{-1}p_{2\sigma}p_{3\tau}+\tau^{-1}p_{2\sigma}p_{4\tau})(F_{\sigma\tau}-G_{\sigma\tau}^{(2)})$$
$$+[\tfrac{1}{2}m^2(p_{1\sigma}+p_{2\sigma})-\kappa^{-1}(p_1\cdot p_2+2m^2)p_{3\sigma}$$
$$-m^2\tau^{-1}(p_{1\sigma}+p_{2\sigma}+p_{3\sigma})]F_\sigma.\quad \text{(A46)}$$