π-Meson Production by Protons on Nuclei*

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The meson production cross section for protons incident on nuclei is formulated in terms of two-particle transition rates and the struck nucleon momentum distribution in the nucleus. Three different momentum densities are assumed. They are a modified Chew-Goldberger, a Gaussian, and a Fermi distribution. With these, it is attempted to fit the experimental π^+ and π^- meson spectra obtained by bombarding C^{12} with 345-Mev protons. The effect of the exclusion principle is estimated on the basis of a single particle model for the nucleus. Meson reabsorption and scattering, after production, are also taken into account, but only roughly. The calculations show that the gaussian distribution approximates nuclear conditions best, and that the fundamental proton-neutron and proton-proton production transitions matrices are most probably not equal.

I. INTRODUCTION

EXPERIMENTAL studies of the production spectrum and angular distribution for π-mesons have been made for protons incident on various elements, notably hydrogen,[1-3] carbon,[4,5] and lead[6] at energies of 345 and 381 Mev. In this paper an attempt is made to explain these spectra in terms of an analysis of the production of mesons in free nucleon-nucleon collisions as described by Watson and Brueckner,[7] (hereafter referred to as W.B.). The treatment is similar to that of Lax and Feshbach for photomeson production in nuclei.[8]

The analysis need not depend on any particular meson theory, but assumes that the matrix element for meson production in proton-proton ($P-P$), and proton-neutron ($P-N$) collisions are known in detail. Unfortunately, little is presently known about the latter, nor is the excitation curve known for either types of meson production. On the other hand, a great deal of evidence (see reference 7 for example) seems to indicate that both the charged and neutral mesons are pseudoscalar. The spin dependence of the matrix for the production from either proton-proton or proton-neutron collisions is based on this assumption, but the calculation can easily be extended to apply to any other type of meson.

The evaluation of the transition matrix of the problem is carried out by performing closure over all but the interacting nucleons. Except near threshold, the production spectrum at a given angle of meson emission is then shown to be proportional to the momentum distribution of the struck particle in the nucleus, folded into the free particle production rate. The consequence of the exclusion principle is to reduce the momentum space available, and is calculated on the basis of a single particle model for the nucleus.

The absorption and scattering of the meson by the nucleus, after production, are treated as separate processes.

At sufficiently high energies above threshold, the cross section is expected to be equal to the number of protons and the number of neutrons in the nucleus multiplied by their respective free-particle production cross sections. As the energy of the incident proton is decreased, not all the nucleons can contribute to the process (i.e., the whole momentum distribution is not available), and the "production efficiency" may be said to decrease, approaching zero at threshold. This efficiency is a consequence of the dynamics of the problem and is independent of the exclusion effects, which will tend to decrease the cross section still further near threshold. At a given proton energy, the efficiency is a function of the angle of emission of the meson. It is largest for mesons emitted forward. For mesons emitted at 90° to an incident proton beam of 341 Mev, it is only somewhat over 0.5.

The theoretical results derived in the first part are applied to carbon bombarded by 341-Mev protons. It is shown that the spectrum obtained for meson production at 90° to the incident beam depends mainly on the nuclear momentum density. At 0°, however, for meson energies up to 80 Mev, the spectrum depends largely on the free particle meson production matrices. For the sake of simplicity these are assumed to differ only by a numerical constant for $P-N$ and $P-P$ collisions, even though this hypothesis will be shown to be inadequate. It is then attempted to deduce some information both about the matrix elements and about the nuclear momentum distribution involved in the cross section.

II. FORMULATION

A. Free Particle Meson Production

Before considering the actual problem of meson production in nuclei, it is worthwhile to review some of the features of creation in free particle interactions. The cross section for meson formation in a nucleon-nucleon collision can be expressed in terms of an

* This work was performed under the auspices of the AEC.
[1] Cartwright, Richman, Whitehead, and Wilcox, Phys. Rev. **78**, 823 (1950).
[2] Peterson, Iloff, and Sherman, Phys. Rev. **81**, 647(A) (1951).
[3] M. N. Whitehead and C. Richman, Phys. Rev. **83**, 855 (1951).
[4] C. Richman and H. A. Wilcox, Phys. Rev. **78**, 496 (1950)—345 Mev.
[5] Block, Passman, and Havens, Phys. Rev. **83**, 167 (1951)—381 Mev.
[6] M. Weissbluth, Phys. Rev. **78**, 86(A) (1950).
[7] K. M. Watson and K. A. Brueckner, Phys. Rev. **83**, 1 (1951).
[8] M. Lax and H. Feshbach, Phys. Rev. **81**, 189 (1951).

R-matrix.[7,9] W.B. show that, due to the short range of interaction required for meson production, a zero range approximation can be introduced for the nucleons in the final state. If the meson is assumed to come off as a plane wave, and momentum conservation is factored out (in units of $\hbar = c = 1$)

$$d\sigma = [(2\pi)^4/v_R] |(2\pi)^{\frac{3}{2}}\chi_{p'}(0)|^2 |(\mathbf{q}', \mathbf{p}'|r|\mathbf{p})|^2 \times \delta(\mathbf{q}+\mathbf{n}_1'+\mathbf{n}_2'-\mathbf{n}_1-\mathbf{n}_2)d\mathbf{J}, \quad (1)$$

where $\mathbf{q}$ is the meson momentum, $\mathbf{p}'$ and $\mathbf{p}$ the final and initial relative momenta, respectively, and v_R the relative velocity of the interacting nucleons. If μ, and M are the meson and nucleon masses, $\mathbf{n}_1$ and $\mathbf{n}_2$ the individual particle momenta,

$$\mathbf{q}' = \mathbf{q} - (\mu/2M)(\mathbf{n}_1+\mathbf{n}_2),$$
$$\mathbf{p}' = \tfrac{1}{2}(\mathbf{n}_1' - \mathbf{n}_2'),$$
$$\mathbf{p} = \tfrac{1}{2}(\mathbf{n}_1 - \mathbf{n}_2).$$

The $|(2\pi)^{\frac{3}{2}}\chi_{p}'(0)|^2$ factor in Eq. (1) ($\mathbf{p}'$ refers to the eigenvector and 0 is used to indicate the zero-range approximation) comes from the final state nucleon interaction, on the hypothesis that only their relative motion S-state need be considered.[10] If q_0 is the meson energy, T_F and T_I the final and initial nucleon kinetic energies, respectively, then the phase space volume available,

$$d\mathbf{J} = d\mathbf{q}d\mathbf{n}_1'd\mathbf{n}_2'\delta(q_0+T_F-T_I).$$

A sum over the final spin states and an average over the initial ones is implied in Eq. (1).

A partial wave analysis of the final nucleon states has been used. If a similar consideration is applied to the meson angular momentum, and only P- and S-states are included, the matrix $|r|^2$ in the cross-section formula can be written as:

$$|r^P|^2 = a(q^2/\mu^2)\cos^2\theta + b(q^2/\mu^2) + e, \quad (2a)$$

for meson production in $P-P$ collisions, and as

$$|r^N|^2 = a'(q^2/\mu^2)\cos^2\theta + b'(q^2/\mu^2) + c'(q^2/\mu^2)\cos\theta + d'(q/\mu)\cos\theta + e', \quad (2b)$$

for mesons produced in $P-N$ collisions. The coefficients, a, b, e, etc., are numerical factors which depend on the initial momentum, $\mathbf{p}$, and on the spin and isotopic spin states of the reaction. θ is the meson angle in the center-of-mass system of the interacting particles. No $\cos\theta$ terms appear in Eq. (2a) because of the exclusion principle.

W.B. investigated the consequences of charge independence for nuclear forces, taken together with angular momentum and parity conservation, as well as the Pauli exclusion principle, for the process considered. For mesons created by incident protons their results are summarized in Table I. The case of π^0 production

TABLE I. Transitions permitted by angular momentum and parity conservation, as well as by the Pauli exclusion principle for pseudoscalar mesons, when the relative final nucleon angular momentum is an S-state. M_1, M_2, and M_3 are the isotopic spin representation of the R-matrices. The notation used is that of W.B.

	Meson in S-state		Meson in D-state	
$P+P,\ \pi^+$	${}^3P_0 \to {}^1S_0$ ${}^3P_1 \to {}^3S_1$	$M_1 = (1^+\ t^0\vert r^P\vert t^-)$ $M_3 = (1^+\ s\vert r^P\vert t^-)$	$\left.\begin{matrix}{}^1S_0\\{}^1D_2\end{matrix}\right\} \to {}^3S_1$	$M_3 = (1^+\ s\vert r^P\vert t^-)$
$P+N,\ \pi^+$	${}^3P_0 \to {}^1S_0$	$M_1 = (1^+\ t^+\vert r^N\vert t^0)$	$\left.\begin{matrix}{}^3S_1\\{}^3D_1\end{matrix}\right\} \to {}^1S_0$	$M_2 = (1^+\ t^+\vert r^N\vert s)$
$P+N,\ \pi^-$	${}^3P_0 \to {}^1S_0$	$M_1 = -(1^-\ t^-\vert r^N\vert t^0)$	$\left.\begin{matrix}{}^3S_1\\{}^3D_1\end{matrix}\right\} \to {}^1S_0$	$M_2 = (1^-\ t^-\vert r^N\vert s)$
$P+P,\ \pi^{0\text{a}}$	${}^3P_0 \to {}^1S_0$	$M_1 = -(1^0\ t^-\vert r^N\vert t^-)$	forbidden	$M_1 = -(1^0\ t^-\vert r^P\vert t^-)$
$P+N,\ \pi^0$	${}^3P_0 \to {}^1S_0$	0	$\left.\begin{matrix}{}^3S_1\\{}^3P_{1,2}\end{matrix}\right\} \to {}^3S_1$	0
	${}^3P_1 \to {}^3S_1$	$M_3 = (1^0\ s\vert r^N\vert t^0)$	$\left.\begin{matrix}{}^3S_1\\{}^3D_1\end{matrix}\right\} {}^1S_0$	$M_2 = (1^0\ t^0\vert r^N\vert s)$
	${}^1P_1 \to {}^3S_1$	0	$\left.\begin{matrix}{}^1S_0\\{}^1D_2\end{matrix}\right\} \to {}^3S_1$	$M_3 = (1^0\ s\vert r^N\vert t^0)$

[a] Bjorkland, Crandall, Moyer, and York, Phys. Rev. 77, 213 (1950) show experimental evidence of the suppression of this reaction.

will not be studied in any further detail here, though the treatment can easily be extended to include it as well.

W.B. also obtained the general spin dependence of the R-matrices on the assumption of pseudoscalar mesons. If the index "1" refers to one of the interacting particles and "2" to the other, this dependence is given by a combination of

$$A(1) = \mathbf{T}(\mathbf{p},\mathbf{q})\cdot\boldsymbol{\sigma}^{(1)} + \mathbf{U}(\mathbf{p},\mathbf{q})\cdot\boldsymbol{\sigma}^{(1)}\mathbf{V}(\mathbf{p},\mathbf{q})\cdot\boldsymbol{\sigma}^{(2)} + \mathbf{W}(\mathbf{p},\mathbf{q})\cdot\boldsymbol{\sigma}^{(1)}\times\boldsymbol{\sigma}^{(2)}, \quad (3)$$

and similar quantities $A(2)$ obtained by interchanging "1" and "2" and replacing $\mathbf{p}$ by $-\mathbf{p}$. The R-matrix must be symmetric with respect to nucleon exchange. Thus, for a π^+ meson P-state, if T is an even function of $\mathbf{p}$, we would take $\mathbf{T}\cdot(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)})$, since the isotopic spin dependence (see Table I) of R is odd.

In this manner, then, except for numerical factors, an almost complete specification of the free particle production matrix is possible.

B. Production in Nuclei

For meson production in a fairly heavy nucleus, at energies of the order of 350 Mev, the interaction can be assumed to take place between the incoming proton and a single nucleon. The wavelength of the incoming proton, as seen by the nucleus (if $A \gg 1$, the center-of-mass system and the laboratory frame are approximately equivalent) is $\lambda\!\!\!^{-} = 2.3\times10^{-14}$ cm. Since this is quite small compared to internuclear distances ($\sim 2.8 \times 10^{-13}$ cm) the above approximation is justified. On the basis of an impulse approximation[11] then, the probability amplitude for meson production in a nucleus, A, is proportional to

$$\left(G_F, \sum_{j=1}^{A} R_j G_I\right),$$

[9] C. Møller, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. 23, 1 (1945).

[10] See Appendix, K. Bruckner, Phys. Rev. 82, 598 (1951).

[11] G. F. Chew, Phys. Rev. 80, 196 (1950).

where R_j is the appropriate two-particle matrix of the form

$$(\mathbf{q}, \mathbf{n}_1', \mathbf{n}_2'|R|\mathbf{n}_1, \mathbf{n}_2).$$

Momentum conservation can be factored out, giving[12]

$$(\mathbf{q}', \mathbf{p}'|r|\mathbf{p})\delta(\mathbf{q}+\mathbf{n}_1'+\mathbf{n}_2'-\mathbf{n}_1-\mathbf{n}_2).$$

G_F and G_I are the final and initial antisymmetrized nuclear wave functions in momentum space. Thus, if the momentum, $\mathbf{n}''$, spin and isotopic spin, ξ, of a particle, i, is indicated by η_i'',

$$G_I=\frac{1}{(A+1)^{\frac{1}{2}}}\sum_{i=1}^{A+1}(-1)^P P_{i,A+1} \times\psi_I(\eta_1''\cdots\eta_i''\cdots\eta_A'')\epsilon(\eta_{A+1}''), \quad (4)$$

where ψ_I is antisymmetric and $(-1)^P P_{i,A+1}$ is the particle permutation operator which insures that G_I is also antisymmetric. ϵ is the wave function of the incoming particle. This is a plane wave, and may be written as

$$\epsilon(\eta_{A+1}'')=\delta_{\mathbf{P}}(n_{A+1}'')\epsilon'(\xi_{A+1}),$$

where ϵ' represents the spin, isotopic spin wave function.

The cross section for meson creation is then

$$d\sigma_A=\frac{(2\pi)^4}{v_0}\sum_F\left|\left(G_F, \sum_{j=1}^{A} R_jG_I\right)\right|^2\delta(E_F-E_I)d\mathbf{q}d\mathbf{P}', \quad (5)$$

where v_0 is the relative velocity of the incident proton and the struck nucleus, $\mathbf{P}'$ is the total momentum of the interacting nucleons in the final state, and $\sum_F$ indicates a sum over all possible final nuclear states, consistent with over-all energy conservation.

Since the meson-producing interacting is a strong, short-range one, it is expected that the excitation energy of the final nucleus, excluding the two colliding particles will be small. If the effect of this slight excitation on the energy conservation is neglected, we can sum over all final states of $(A-1)$ particles, and thus obtain a partial closure approximation to the cross section. The exact energy conservation condition is then replaced by:

$$\delta(T_F+q_0+\bar{B}_F-T_I-B_I), \quad (6)$$

where T_F is the final kinetic energy of the interacting nucleons, and T_1 the initial kinetic energy of the incident proton, since the initial nucleus is at rest. B_I and $\bar{B}_F$ are the initial and average final nuclear binding energies, respectively.

Due to the large momentum transfer involved in the interaction, we can take the final state wave function to be separable. Then, if ψ_F and χ are antisymmetric wave functions

$$G_F(\eta_1\cdots\eta_{A+1})=\frac{1}{A}\sum_{r\neq s}\sum_{s\neq r}(-1)^P P_{A,r \atop A+1,s} \times\psi_F(\eta_1\cdots\eta_r\cdots\eta_s\cdots\eta_{A-1})\chi(\eta_A, \eta_{A+1}). \quad (7)$$

[12] In order not to obtain the square of the δ-function, the artifice employed is that of using Q instead of q in the second δ-function, and integrating over this momentum.

With this separation, the evaluation of the matrix involved in the cross section Eq. (5) is carried out as follows:

$$M=\sum_F|(G_F, \sum_j R_jG_I)|^2\equiv\sum_F|(G_F, \phi_I)|^2. \quad (8)$$

This defines $\phi_I\equiv\sum_j R_jG_I$, which must also be antisymmetric. If closure is performed over all final states of ψ_F, and a sum over the final spins of the struck nucleons, M is given by

$$M=\frac{1}{A^2}\sum_{r\neq s}\sum_{s\neq r}{}'\langle\eta_1\cdots\eta_r'\cdots\eta_s'\cdots\eta_{A+1} \times|\chi^*(\mathbf{n}_r, \mathbf{n}_s)\chi(\mathbf{n}_r', \mathbf{n}_s')\delta(\xi_r'-\xi_r)\delta(\xi_s'-\xi_s)| \times\eta_1\cdots\eta_r\cdots\eta_s\cdots\eta_{A+1}\rangle', \quad (9)$$

where $|\eta_1\cdots\eta_{A+1}\rangle'$ represents ϕ_I, and an integration over all variables is implied. All terms in the sum are similar. The matrix may thus be replaced by a typical element

$$M={}'\langle\eta_1'\eta_2'\eta_3\cdots\eta_{A+1}|\chi^*(\mathbf{n}_1, \mathbf{n}_2)\chi(\mathbf{n}_1', \mathbf{n}_2') \times\delta(\xi_1'-\xi_1)\delta(\xi_2'-\xi_2)|\eta_1\eta_2\eta_3\cdots\eta_{A+1}\rangle'.$$

If ψ_I is represented by $|\eta_1\cdots\eta_A\rangle$, and if the free particle R-matrices are given an added index to indicate both interacting particles, the meson production matrix becomes

$$M=\frac{1}{A+1}\sum_{j\neq i}\sum_i\sum_{l\neq k}\sum_k\langle\eta_1'\eta_2'\eta_3\cdots\eta_l'''\cdots\eta_A| \times\epsilon^*(\eta_k''')R_{kl}^\dagger\chi^*(\mathbf{n}_1, \mathbf{n}_2)\chi(\mathbf{n}_1', \mathbf{n}_2')\delta(\xi_1'-\xi_1) \times\delta(\xi_2'-\xi_2)R_{ij}\epsilon(\eta_i'')|\eta_1\eta_2\cdots\eta_j''\cdots\eta_A\rangle.$$

C. Diagonal Elements

The matrix M has both diagonal and off-diagonal elements. The diagonal terms represent the main contribution to the cross section, except near threshold, since the off-diagonal elements would not occur at all if correlation effects did not exist for the location of nucleons and if wave interference is neglected. The diagonal terms, M^0 are considered first. If the center-of-mass motion of $\chi(\mathbf{n}_i, \mathbf{n}_j)$ is factored out

$$M^0=\frac{1}{A+1}\sum_{j\neq i}\sum_i\langle\psi_I|\epsilon'^*(\xi_i')R_{ij}\delta_{\mathbf{P}'}(\mathbf{n}_i+\mathbf{n}_j) \times\chi^*(\mathbf{n}_i-\mathbf{n}_j)\delta_{\mathbf{P}}(\mathbf{n}_i'+\mathbf{n}_j')\chi(\mathbf{n}_i'-\mathbf{n}_j')R_{ij}\epsilon'(\xi_i)|\psi_I\rangle. \quad (10)$$

Though the colliding nucleons may have final momenta considerably larger than in the free particle case, it can be argued, that due to their intimate interaction, the zero range hypothesis of W.B. can still be used. The final state of these nucleons is again approximated by considering relative angular momentum S-waves only. $\chi(\mathbf{n}_i-\mathbf{n}_j)$ is thus replaced by $\chi_{p'}(0)$ for S-states, and is taken as zero otherwise. With this simplification, and if momentum conservation is factored out,

$$M^0=\frac{1}{A+1}\sum_{j\neq i}\sum_i\langle\psi_I|\epsilon^{*\prime}(\xi_i')|r_{ij}|^2|(2\pi)^{\frac{3}{2}}\chi_{p'}(0)|^2 \times\delta(\mathbf{q}+\mathbf{P}'-\mathbf{P}-\mathbf{k})\epsilon'(\xi_i)|\psi_I\rangle, \quad (11)$$

where $\mathbf{k}$ is the struck nucleon momentum eigenvector.

The spin and isotopic spin sums in Eq. (11) can be performed by making use of the representation of the R-matrices shown in Sec. A. Thus, by means of Table I, the isotopic spin dependence of M^0 is completely specified. For mesons emitted in a P-state, (omitting the δ-function and the factor $|(2\pi)^{\frac{3}{2}}\chi_{p'}(0)|^2$ for the moment),

$$M^0(\pi^+)=\langle\psi_I|\sum_{j\neq i}\sum_i\{(1^+s|r_{ij}^P|t^-)^2+(1^+t^+|r_{ij}|s)^2\}|\psi_I\rangle, \quad (12)$$

$$M^0(\pi^-)=\langle\psi_I|\sum_{j\neq i}\sum_i(1^-t^-|r_{ij}^N|s)^2|\psi_I\rangle. \quad (13)$$

Similar expressions are obtained if the meson is emitted in an S-state. The actual spin dependence, as shown in Eq. (3) is not required. It can be shown that, for the transitions involved (see Table I) we need only consider

$$r_{ij}^P=\mathbf{K}\cdot\boldsymbol{\sigma}_j \quad\text{and}\quad r_{ij}^N=\mathbf{L}\cdot\boldsymbol{\sigma}_j,$$

operating on the initial nucleus wave function, where $\mathbf{K}$ and $\mathbf{L}$ are functions of the meson and colliding nucleon momenta. All other forms cancel for pseudoscalar mesons.

Averaging over all nuclear spins, and performing the isotopic spin sum, we obtain for both meson P- and S-states

$$M^0(\pi^+)=\left\langle\psi_I\left|ZK^2+\frac{A-Z}{2}L^2\right|\psi_I\right\rangle, \quad (14)$$

$$M^0(\pi^-)=\left\langle\psi_I\left|\frac{A-Z}{2}L^2\right|\psi_I\right\rangle, \quad (15)$$

where Z is the number of protons, and A the total number of nucleons in the nucleus.

These matrix elements can be expressed in terms of the normalized momentum distribution of the struck nucleon, $\rho(\mathbf{k})$ by integrating over all nuclear coordinates on which the R-matrix does not depend.

The contribution of the diagonal elements to the cross section is thus given by (if the omitted factors are again taken into account):

$$d\sigma_A{}^0(\pi^+)=\frac{(2\pi)^4}{v_0}\int\left(ZK^2+\frac{A-Z}{2}L^2\right)\times|(2\pi)^{\frac{3}{2}}\chi_{p'}(0)|^2\rho(\mathbf{k})d\mathbf{k}\delta(\mathbf{q}+\mathbf{P}'-\mathbf{k}-\mathbf{P})\times\delta(E_F-E_I)d\mathbf{P}'d\mathbf{p}'d\mathbf{q}, \quad (16)$$

since $d\mathbf{q}=q_0qdTd\Omega_q$, where T is the meson kinetic energy and $d\Omega_q$ is an element of solid angle about the direction of emission of the meson, we have, in terms of the free particle cross sections

$$\frac{d\sigma_A{}^0(\pi^+)}{dTd\Omega_q}=\frac{1}{v_0}\int\left[Z\frac{d\sigma_{P-P}}{dTd\Omega_q}v_R+(A-Z)\frac{d\sigma_{P-N}}{dTd\Omega_q}v_R\right]\rho(\mathbf{k})d\mathbf{k}, \quad (17a)$$

$$\frac{d\sigma_A{}^0(\pi^-)}{dTd\Omega_q}=\frac{1}{v_0}\int(A-Z)\frac{d\sigma_{P-N}}{dTd\Omega_q}v_R\rho(\mathbf{k})d\mathbf{k}. \quad (17b)$$

The correct energy conservation condition, as given by Eq. (6), is implied in these equations, and it is assumed that the excitation function of the free particle cross section is known.

On the basis of the diagonal terms alone, if $K^2=L^2$, the ratio of

$$d\sigma_A{}^0(\pi^+)/d\sigma_A{}^0(\pi^-)=(A+Z)/(A-Z).$$

For a nucleus such as carbon, where $Z=A/2$, this ratio is 3:1.

D. Correlation Effects

It remains to consider the off-diagonal elements, $\bar{M}$, of the matrix M. These occur because of wave interference effects and due to the antisymmetrization of the wave functions. Since the production process involves large momentum transfers, the former type is expected to be small. It is neglected in the actual calculations, where a single-particle model is used to evaluate the correlation terms. For the present, however, these restrictions will not be imposed.

The magnitude of the matrix $\bar{M}$ decreases as the energy of the incident proton increases above threshold. Since the latter is at approximately 165 Mev,[13] $\bar{M}$ is not expected to alter the cross section appreciably at 341 Mev.

If an average is performed over the spins of the incoming particle, the general off-diagonal matrix element, referring to Eq. (9), can be written as

$$\bar{M}=\frac{1}{A+1}\sum_{l\neq j\neq i}\sum_{j\neq i}\sum_i\langle\eta_1'\eta_2'\eta_3\cdots\eta_l'''\cdots\eta_A|\epsilon'^*(\xi_i)\times R_{il}^\dagger\chi^*(\mathbf{n}_1,\mathbf{n}_2)\chi(\mathbf{n}_1',\mathbf{n}_2')\delta(\xi_1'-\xi_1)\times\delta(\xi_2'-\xi_2)R_{ij}\epsilon'(\xi_i)|\eta_1\cdots\eta_j''\cdots\eta_A\rangle. \quad (18)$$

$\bar{M}$ contains several types of terms, corresponding physically to an exclusion of either or of both of the interacting particles in the final state. With reference to Eq. (18) the three distinct off-diagonal matrices which occur may be represented by:

(1) M'; this occurs if $i=1$; $l\neq j\neq 1$ or 2;
(2) M''; this occurs if $j=1$; $l\neq i\neq 1$ or 2;
(3) M'''; this occurs if $l\neq j\neq i\neq 1$ or 2.

[13] W. H. Barkas, Phys. Rev. 75, 1109 (1949). He obtains 155 Mev for π^+ production and 178 Mev for π^- production.

M' and M'' involve an overlap integral over a single particle variable, whereas M''' involves one over two variables. If these overlaps are large, the contribution of the off-diagonal matrices will be important, and in a direction to cancel the diagonal ones.

The spin and isotopic spin sums in M' and M'' can be evaluated as by Lax and Feshbach[2] by methods due to Wigner and Feenberg.[14] M', for example, involves pair correlation functions $\rho(\mathbf{n}_j, \mathbf{n}_l)$. To a first approximation only momentum space symmetric and antisymmetric correlations, $\rho(|\mathbf{n}_j-\mathbf{n}_l|)$ are differentiated. The actual dependence of M' on $\mathbf{n}_l$ and $\mathbf{n}_j$ can only be obtained if χ is known, and if the overlap is performed. If $\chi(\mathbf{n}_i, \mathbf{n}_2)$ is replaced by individual plane waves:

$$\chi(\mathbf{n}_i, \mathbf{n}_2)=\delta_{p2}(\mathbf{n}_2)\delta_{p_i}(\mathbf{n}_i),$$

we obtain for M'

$$M'=\frac{1}{A+1}\sum_{l\neq j\neq i}\sum_{j\neq i}\sum_{i}\langle\psi_I|\epsilon'(\xi_i)r_{il}^\dagger r_{ij}\delta(\mathbf{n}_j''-\mathbf{n}_l''')$$
$$\times\delta(\mathbf{P}-\mathbf{q}-\mathbf{p}_i)\epsilon(\xi_i)|\psi_I\rangle. \quad (19)$$

The exclusion effect is now implicit in the wave function ψ_I, as will be shown in more detail, later. A similar result is obtained for M'', except that the final momentum involved in the δ-function is now that of the other nucleon in the final state, $\mathbf{p}_2$.

If V_{jl} is defined as the momentum space orientation mean value of $\delta(\mathbf{n}_j''-\mathbf{n}_l'')\delta(\mathbf{P}-\mathbf{q}-\mathbf{p}_i)$ and an average over nuclear spins is performed, we obtain for mesons emitted in P-states:

$$M'(\pi^-)=\frac{1}{A+1}\sum_{l\neq j\neq i}\sum_{j\neq i}\sum_{i}\langle\psi_I|\epsilon'^*(\xi_i)(L^2\boldsymbol{\sigma}_j\cdot\boldsymbol{\sigma}_l)$$
$$\times(s_{il}|0|1^-t^-)(1^-t^-|0|s_{ij})\epsilon(\xi_i)|\psi_I\rangle, \quad (20)$$

where 0 is the isotopic spin representation of the R-matrices, and the notation is that of Table I. A similar, but more complicated result is obtained for $M'(\pi^+)$, since contributions occur from both $P-P$ and $P-N$ collisions, though there is no interference between these two since their 0-operators are orthogonal (M_2, M_3) in isotopic spin space. For mesons emitted in S-states, such interference is possible as can be seen by referring to Table I.

The state $|\psi_I\rangle$ is now broken up into its symmetric and antisymmetric parts $|\psi_I\rangle=|\psi_I^{(s)}+|\psi_I^{(a)}\rangle$ and the spin and isotopic spin sums are carried out. If

$$V_s\equiv\langle\psi_I^{(s)}|V_{jl}|\psi_I^{(s)}\rangle \quad\text{and}\quad V_a=\langle\psi_I^{(a)}|V_{jl}|\psi_I^{(a)}\rangle,$$

the leading terms, of order A and A^2 become for mesons in P-states

$$M_{P-P}'(\pi^+)=-\tfrac{1}{2}K^2A\{\tfrac{1}{2}(V_s+V_a)+\tfrac{1}{16}A(V_s-V_a)\}, \quad (21a)$$
$$M_{P-N}'(\pi^\pm)=-\tfrac{1}{4}L^2A\{\tfrac{1}{2}(V_s+V_a)+\tfrac{1}{16}A(V_s-V_a)\}, \quad (21b)$$

where $M_{P-P}'(\pi^+)$ refers to π^+ mesons and $M_{P-N}'(\pi^\pm)$ to π^+ or π^- mesons created in $P-P$ and $P-N$ collisions, respectively. The results obtained for S-state mesons is exactly the same, to this order, and so is that for M''. M''', on the other hand, is really a three-particle problem, and will not be evaluated, since it is expected to be small at 341 Mev. It may be important, however, in explaining the experimentally observed increase of the π^+ to π^- ratio in carbon from 5.1±0.8:1 at 341 Mev[15] to approximately 14:1 at 278 Mev.[16] (The inverse of this ratio was actually measured, by bombarding carbon with neutrons.) A general argument has been advanced by Chew and Steinberger[17] to explain this considerable increase near threshold. The cause is ascribed to the exclusion principle, which should be less important for π^+ production than for π^- production in $P-N$ collisions (i.e., only one particle is excluded on an alpha-particle model in the former case, whereas two are in the latter). It is therefore to be expected that M''' will be larger for π^- than for π^+ production by $P-N$ collisions in nuclei. The energy dependence of the π^+ to π^- ratio, which is due to an effective raising of the threshold for π^- production on the argument of Chew and Steinberger can also be deduced. Near threshold we may expect M''' to be important. At 341 Mev, however, the energy remaining to the struck nucleon in the final state is larger than 80 Mev, as long as mesons of kinetic energy less than 100 Mev are studied. It is thus quite probable that at least one particle will hardly "feel" the exclusion effect and will be ejected from the nucleus. Part of the increase of the π^+ to π^- ratio with decreasing proton energy however may also be accounted for by the difference of the free-particle excitation function and matrix elements involved in the $P-P$ and $P-N$ production processes. Some experimental evidence of this is seen in the fact that Blockman, Passman, and Havens, at Columbia, obtain a ratio of 11±3:1 for carbon bombarded by 381-Mev protons.[5] This increase can hardly be ascribed to an exclusion effect.

At 341 Mev, then, it is expected that either M' or M'' will be of physical interest. If one of these matrices is large, the other one will be small, in general. In order to evaluate either of these matrices explicitly, and hence V_s and V_a, a particular nuclear model must be chosen. We take a rather simple one here, namely a single-particle model, and neglect wave interference.

If the nuclear wave functions are taken as plane waves enclosed in a box in space, of volume $V=(4/3)\pi R^3$ where R is the nuclear radius (Fermi gas model), the matrix M' becomes

$$M'(\pi^+)=-\int[ZK^2+\tfrac{1}{2}(A-Z)L^2]V_K\rho(\mathbf{k})\rho(\mathbf{p}_2)$$
$$\times\delta(\mathbf{P}+\mathbf{k}-\mathbf{q}-\mathbf{p}_i-\mathbf{p}_2)d\mathbf{k}d\mathbf{p}_2, \quad (22a)$$

[14] E. P. Wigner and E. Feenberg, Reports on Progress in Physics, Phys. Soc., London 8, 308 (1941).

[15] Richman, Weissbluth, and Wilcox (to be published).
[16] Bradner, O'Connel, and Rankin, Phys. Rev. 79, 720 (1950).
[17] G. F. Chew and J. L. Steinberger, Phys. Rev. 78, 497 (1950).

$$M'(\pi^-) = -\int \tfrac{1}{2}(A-Z)L^2 V_K \rho(\mathbf{k})\rho(\mathbf{p}_2) \times \delta(\mathbf{P}+\mathbf{k}-\mathbf{q}-\mathbf{p}_i-\mathbf{p}_2)d\mathbf{k}d\mathbf{p}_2, \quad (22b)$$

where $V_K=(4/3)\pi K_{\max}^3$ is the volume of a sphere of maximum momentum, corresponding to the space volume V. The spin sums were evaluated as before, in obtaining Eq. (21), and are correct to the same order. A similar result is obtained for M'', with the roles of $\mathbf{p}_i$ and $\mathbf{p}_2$ interchanged.

$$\rho(k)=\begin{cases}1/V_K & \text{if } 0<k<K_{\max} \\ 0 & \text{if } K_{\max}<k.\end{cases} \quad (23)$$

In the case of a nucleus represented by an excited Fermi gas (i.e., temperature larger than 0°K) we assume that the nuclear wave functions can still be represented by plane waves inside a nuclear volume. In that case (k) is to be interpreted as the momentum of a nucleon on this new model. In this manner the correlation effects can be computed for a general momentum distribution. Equation (22) still holds, but the momentum distribution is now unspecified.

The explicit effect of the exclusion principle is now apparent. If the average value of $\rho(\mathbf{p}_2)$ for the transition process is large, then $M'\sim M$ and the cross section becomes very small. Appropriately, $\langle\rho(\mathbf{p}_2)\rangle_{\mathrm{Av}}$ is large when $\mathbf{p}_2\sim\mathbf{k}$, and both are small in magnitude.

The use of the single particle model in the calculation of the correlation terms, rather than a more general one, neglects the final state interaction of the two colliding particles. Thus, on the Fermi model the cross section does not go to zero, as it should, if the final particles are excluded. In a strictly consistent formulation, both the diagonal and the off-diagonal terms should be treated in the same manner. Hence, on a more general nuclear model than the one used for evaluation of the off-diagonal matrix, a final state interaction term should occur. Since the exclusion terms are expected to be small in any case, we arbitrarily insert a final state interaction here, so that the cross section will at least go to zero correctly on the Fermi model.

If M'' and M''' can be neglected at 341 Mev, the cross section for π^+ and π^- mesons in a nucleus, A, can be written as:

$$\frac{d\sigma_A(\pi^+)}{dTd\Omega_q}=\frac{1}{v_0}\int\left[Z\frac{d\sigma_{P-P}}{dTd\Omega_q}+(A-Z)\frac{d\sigma_{P-N}}{dTd\Omega_q}\right] \times v_R\rho(\mathbf{k})[1-V_K\rho(\mathbf{k}+\boldsymbol{\Delta})]d\mathbf{k}, \quad (24a)$$

$$\frac{d\sigma_A(\pi^-)}{dTd\Omega_q}=\frac{1}{v_0}\int(A-Z)\frac{d\sigma_{P-N}}{dTd\Omega_q} \times v_R\rho(\mathbf{k})[1-V_K\rho(\mathbf{k}+\boldsymbol{\Delta})]d\mathbf{k}, \quad (24b)$$

where $\boldsymbol{\Delta}=\mathbf{P}-\mathbf{q}-\mathbf{p}_i$.

With the inclusion of the correlation effects as treated, it is seen that if L^2 is equal to K^2 the ratio of π^+ to π^- production is still $(Z+A)/(Z-A)$.

E. Absorption and Scattering of Mesons

So far the outgoing meson has been represented by an undamped plane wave. Evidence from photomeson production,[18,19] however, indicates that the meson may interact with other nucleons before it is emitted from the nucleus in which it was produced. The problem of the interaction of mesons with nuclear matter has been treated by Brueckner, Serber, and Watson,[20] who calculated the effect of meson absorption on the production cross section. They quote evidence that this effect is similar for π^+ and π^- mesons, and conclude, furthermore, that the scattering effects are, in general, small compared to the reabsorption ones. They find that the mean free path, λ_a, for meson absorption is of the order of 2–3 a_0, where $a_0=1.4\times10^{-13}$ cm is the effective nucleon radius.

The time taken for a 40-Mev meson to travel a distance of 3 a_0 is of the order of 10^{-23} sec. Since this is small compared to the characteristic nuclear time (taken here as a measure of the time taken for a nuclear disturbance to cross the nucleus, approximately 10^{-22} sec), the probability of meson reabsorption before the nucleus feels the effect of its production is large. Rather than using damped plane waves for the emitted mesons, we therefore treat the meson absorption problem as a separate process.

The information available on absorption, to date, is still scant. For this reason, and due to the other uncertainties involved in the production problem, we neglect the meson energy dependence of the reabsorption process. The analysis of Brueckner, Serber, and Watson shows that the absorption cross section decreases somewhat with meson energy, but is substantially independent of this energy for larger than 40-Mev mesons. On this assumption, and using the model of Fernbach, Serber, and Taylor,[21] they find that the production cross section is reduced by a factor, f_a

$$f_a=3\{(1/2x)-(1/x^3)+(1/x^3)(1+x)e^{-x}\}, \quad (25)$$

where $x\equiv2R/\lambda_a$, and R is the nuclear radius. For $\lambda_a=3a_0$, and $R=a_0A^{\frac{1}{3}}$, this factor is equal to 0.6.

Experiments performed by Bernardini, Booth, and Lederman at Columbia[22] indicate that, whereas the elastic scattering of mesons by nuclei is indeed small, and approximately energy independent, the inelastic scattering cross section increases rather rapidly with meson energy beyond 30–50 Mev. Thus, the cross section for inelastic scattering at 100–110 Mev is about four times that in the range of 30–50 Mev. The net

[18] R. F. Mozley, Phys. Rev. **80**, 493 (1950).
[19] R. M. Littauer and D. Walker, Phys. Rev. **83**, 206 (A) (1951).
[20] Brueckner, Serber, and Watson, Phys. Rev. **84**, 258 (1951).
[21] Fernbach, Serber, and Taylor, Phys. Rev. **75**, 1352 (1949).
[22] Bernardini, Booth, and Lederman, Phys. Rev. **83**, 1075 (1951).

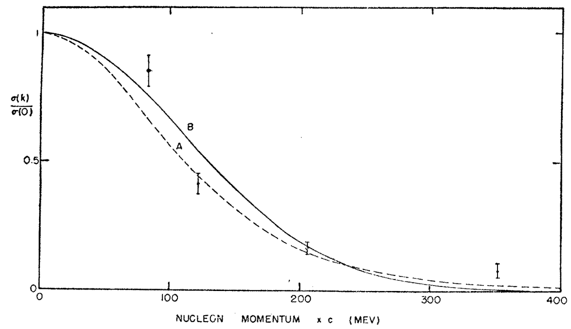

FIG. 1. Modified Chew-Goldberger (A) and Gaussian (B) momentum distributions for a nucleon in C^{12}.

effect of this inelastic scattering is to increase the low energy meson spectrum at the expense of the high energy part. The order of magnitude of this effect can be computed, simply by taking λ_a in x of Eq. (25) to represent the total mean free path for a meson interaction to occur after creation. λ_a decreases then somewhat with meson energy. The effect on f_a is slight, however, for meson energies smaller than 80 Mev. If $f_a=0.6$ at 20 Mev it is only reduced to ~ 0.55 at 80 Mev. The consequences of scattering are therefore neglected in the subsequent calculations.

III. COMPARISON WITH EXPERIMENT

A. Method of Calculation

There is a fair amount of experimental information available at present for meson production in nuclei. The fundamental nucleon-nucleon production data is still rather scarce, however. Whereas the production of positive mesons by $P-P$ collisions has been, and is still being fairly extensively studied both experimentally and theoretically,[1–3, 7, 23] little is known as yet about the production of either positive or negative mesons in $P-N$ collisions. For this reason, only the experiments of meson production in carbon,[4, 15] at 341 Mev are analyzed in terms of the theory developed in the previous section. Both the element and the energy are chosen because of the availability of experimental information.

The equations derived for the meson cross section (both $d\sigma_A{}^0$ and $d\sigma_A$) involve not only the free transition matrix elements, but also the momentum distribution of the bound, struck nucleon. Theoretically, it is possible to deduce this momentum density from the experimental data. It is simpler, however, to start with a given normalized momentum distribution, such as that of Chew and Goldberger,[24] and to fit it to the experimental cross section. From an analysis of the deuteron pick-up data of York,[25] Chew and Goldberger arrive at the distribution

$$\rho(\mathbf{k})d\mathbf{k}=\frac{\alpha_p}{\pi^2(\alpha_P{}^2+k^2)^2}d\mathbf{k}, \qquad (26a)$$

where α_P corresponds to an energy of 18 Mev. As pointed out in their paper, this momentum density is in doubt at the high end. This distribution, furthermore, gives an infinite average kinetic energy to a nucleon in the nucleus. A preliminary analysis,[26] showed that, due to its long tail, Eq. (26a) did not seem to fit the 90° meson production spectrum obtained by Richman and Wilcox.[4] For this reason, in the calculation made here, the distribution was modified so as to give a finite average kinetic energy of 48.1 Mev, and still fit the experimental points obtained from York's data fairly well. Then

$$\rho_A(\mathbf{k})d\mathbf{k}=C_A\frac{d\mathbf{k}}{(\alpha_P{}^2+k^2)^2(\beta^2+k^2)^2}, \qquad (26b)$$

where C_A is the normalization constant, α_P and β are constants with $\beta=2.5\alpha_P$. This distribution represents the square of the Fourier transform of the wave function $\phi(r)\sim(e^{-\alpha_P r}-e^{-\beta r})/r$. It is shown as curve A in Fig. 1, together with the experimental points of York.

Two other momentum distributions were also investigated. One was a Gaussian type, with an average kinetic energy of 19.3 Mev, chosen so as to still fit the low momentum points obtained by Chew and Goldberger

$$\rho_B(\mathbf{k})d\mathbf{k}=C_B\exp(-\alpha k^2)d\mathbf{k}, \qquad (27)$$

where C_B is the normalization constant. ρ_B is shown as curve B in Fig. 1. The third distribution chosen was that for a 0°K Fermi degenerate gas model of the nucleus, with a maximum momentum of 200 Mev/c. A further momentum density has been suggested by Heidmann,[27] namely that of an excited Fermi gas at a temperature corresponding to 9 Mev. This distribution resembles the Gaussian one somewhat, but was not treated, chiefly because of mathematical complications.

Using these momentum densities, assumed to be the same for a proton and a neutron in the nucleus, it is then possible to calculate the energy spectrum at a given angle of emission of the meson. Since the free nucleon production excitation function is unknown, the dependence of K^2, L^2, or of the factors a, b, e, etc., in Eq. (2) on the initial particle momentum is replaced by an average, constant value, determined by fitting the calculated free particle meson spectra to the measured ones at various meson angles. Furthermore, since L^2 is not known we set it equal to ΛK^2 and note whether it is at all possible to fit the experimental carbon data with this choice. It will appear later that this last assumption is not a very good one. This is really not surprising, since

[23] K. M. Watson and C. Richman, Phys. Rev. **83**, 1256 (1951).
[24] G. F. Chew and M. L. Goldberger, Phys. Rev. **77**, 470 (1950).
[25] H. York, Phys. Rev. **75**, 1467 (A) (1949).
[26] E. M. Henley and R. H. Huddlestone, Phys. Rev. **82**, 754 (1951).
[27] J. Heidmann, Phys. Rev. **80**, 171 (1950).

no odd angular dependence can occur in the case of $P-P$ collisions, but may be present in $P-N$ interactions. Thus $c'd'$ in Eq. (2b) need not be zero. Taking $L^2=\Lambda K^2$ implies, in fact, that $\Lambda=a'/a=b'/b=e'/e$ and $c'=d'=0$. Nevertheless, the calculations are made with this choice of L^2, due to lack of more information. The numerical value of Λ is fixed by the π^+ to π^- ratio. In order to fit the observed value of [15] 5.1±0.8:1, $\Lambda=\frac{1}{2}$ is chosen so that the calculated ratio, using either Eq. (17) or Eq. (24) becomes 5:1 (since $d\sigma_{P-N}=\frac{1}{2}\Lambda d\sigma_{P-P}$).

Deuteron formation is neglected in calculating the meson production spectrum in nuclei, since it is expected that the probability of this process is considerably reduced by the nuclear structure. Experimentally, no deuterons have been observed in meson production from carbon.

It follows from the theoretical analysis of W.B. and from the experimental information of Whitehead and Richman[3] that the predominant term in the partial wave analysis of π^+ production in free $P-P$ collisions is one with the meson emitted in a P-state with a $\cos^2\theta$ distribution in the center-of-mass system. In fact, Whitehead and Richman show that the differential cross section in the center-of-mass system of the process as a function of θ, the meson angle is

$$d\sigma_{P-P}/d\Omega=(3.20\pm0.78)(0.071\pm0.068+\cos^2\theta)\times10^{-29}\text{ cm}^2\text{ sterad}^{-1}. \quad (28)$$

If we consider meson S- and P-states only, the observed symmetrical component may be due to either $b(q^2/\mu^2)$ or e (or both) in Eq. (2a). It was shown by Brueckner, Serber, and Watson[28] that some S-state is necessary to explain the experimental results of Panofsky, Aamodt, and Hadley[29] for the capture of π^- mesons in deuterium. They show, in fact, that the ratio of meson S- to P-state [e/a in Eq. (2a)] is approximately $\frac{1}{8}$.

The calculation of typical spectrum—that for π^+ mesons emitted into P-states at 90° to the incident proton beam—is shown in Appendix A. Exclusion effects are not considered in this computation.

The evaluation of the correlation terms can be simplified considerably by choosing an average value of $\mathbf{p}_2$ in Eq. (22). Thus, Passman, Block, and Havens, Jr.[30] assumed that the meson is always emitted with its maximum possible momentum. The justification of this hypothesis rests on the large final state nucleon interaction which occurs when their relative momentum is zero. In this case the magnitude of $\bar{M}$, in general, is very small for mesons emitted both at 90° and at 0° with respect to the incident 341-Mev proton beam. In fact, on the Fermi model, the off-diagonal terms do not contribute at all.

An approximate upper limit to the exclusion effect, on the other hand, is obtained if one of the two final nucleons takes most of the momentum so that little remains to the other one. In this case M'' is certainly small if M' is large and vice versa. The evaluation of M' is carried out in Appendix B, where the approximations employed are also shown. A more thorough treatment of the correlations is not carried out for the present, since the sparseness of fundamental information is not thought to justify it.

[28] Brueckner, Serber, and Watson, Phys. Rev. 81, 515 (1951).
[29] Panofsky, Aamodt, and Hadley, Phys. Rev. 81, 565 (1951).
[30] Passman, Block, and Havens, Jr., Phys. Rev. 83, 167 (1951).

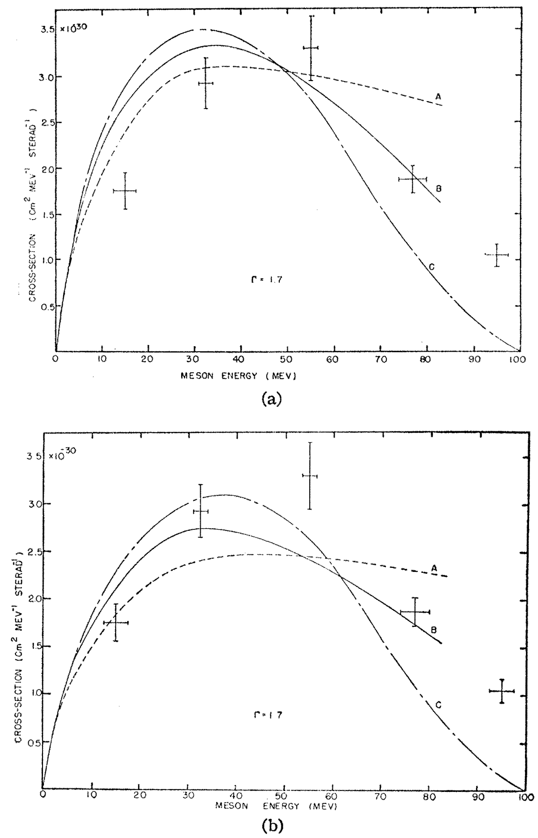

FIG. 2. 90° π^+ spectrum for a meson $P-\cos^2\theta$ state. Exclusion results are neglected in (a) and their approximate maximum effect is shown in (b). Curves A, B, C are for a modified Chew-Goldberger, a Gaussian, and a Fermi distribution, respectively, in this and all subsequent figures.

B. 90° Meson Spectra

On the assumptions discussed in the previous section, the spectra for mesons emitted into P- and S-states at 90° to the incident beam are shown in Figs. 2–4, both without, and with maximum exclusion. In each of these plots, curves A, B, C are the spectra obtained with a modified Chew-Goldberger, a Gaussian, and a Fermi momentum distribution, respectively. The curves are normalized to the free $P-P$ production spectrum, and the absorption factor, $f_a=0.6$ is taken into account. The graphs are multiplied by an arbitrary renormalization constant Γ, chosen so as to obtain an approximate fit to the experimental points, somewhere in between the no-exclusion and the approximate maximum exclusion

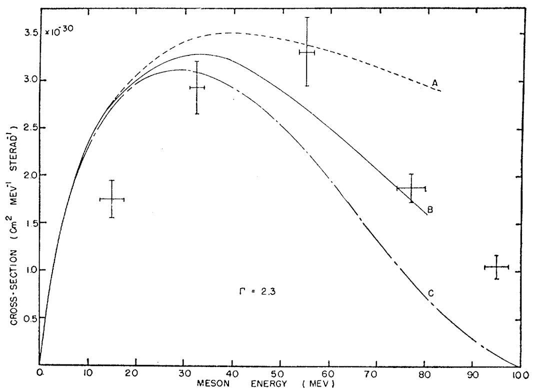

FIG. 3. 90° π^+ spectrum for meson S-state. Exclusion effects are not shown, but are similar to those in Fig. 2. The cross section is normalized to $e/a=1$ in Eq. (2a).

curves. Theoretically, Γ should be equal to unity. The experimental points have been corrected for nuclear absorption, taken as nuclear area.

Figures 2(a) and (b) show the spectra obtained for mesons emitted into P-states with a $\cos^2\theta$ distribution in the center of mass of the interacting nucleons. If $L^2=\Lambda K^2$ this term should represent the main contribution to the cross section. It is noted, however, that $\Gamma=1.7$, so that the calculated spectrum falls considerably too low. It might be remarked at this point that the continuum part of the free $P-P$ cross section, as calculated by W.B. was also too small by 20–30 percent. Not too much emphasis will thus be placed on the values of Γ, though they will be indicated in all instances. Whereas the over-all best fit is obtained with a Gaussian type momentum distribution, it is seen that the calculated spectrum maximum occurs at meson energies which are too small. The calculated spectra also rise too sharply at small meson energies, though the exclusion effect corrects this to a certain degree. At the high energy end of the spectrum, the Fermi momentum distribution cuts the cross section off too rapidly, while the tail of the modified Chew-Goldberger momentum distribution still contributes too much. The Gaussian distribution, on the other hand, follows the spectrum decrease fairly well. At energies beyond those considered, the exclusion effects again lower the calculated spectrum considerably.

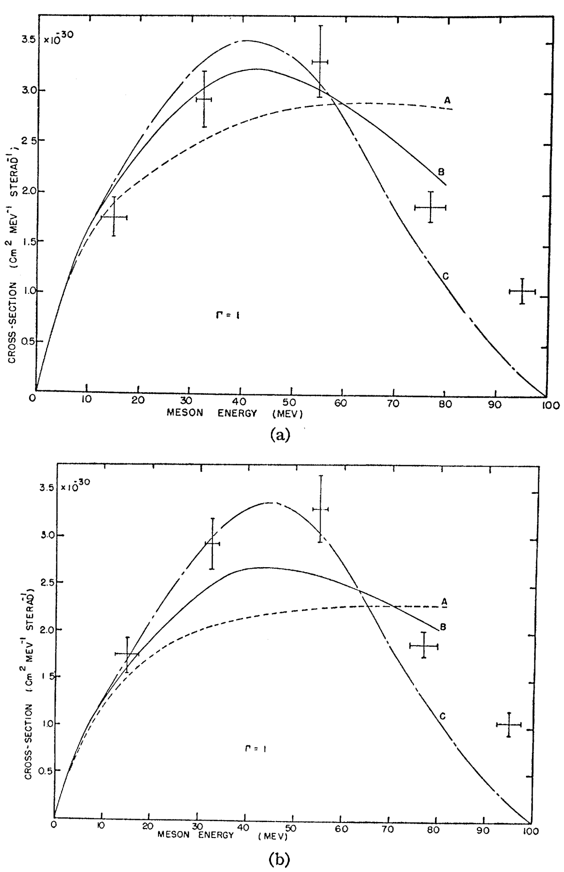

FIG. 4. 90° π^+ spectrum for combined meson $P-\cos^2\theta$, P symmetrical, and S-states, corresponding to $b/a=0.06$ and $e/a=\frac{1}{8}$ in Eq. (2a). Exclusion results are neglected in (a) and their approximate maximum effect is shown in (b).

Figure 3 represents the cross section for mesons emitted into S-states, normalized to $e/a=1$ in Eq. (2a). The remarkable fact here is the close correspondence to the $P-\cos^2\theta$ spectrum. In fact, experimentally, except for the difference in Γ, it would be difficult to distinguish between them. For this reason, the combination of $P-\cos^2\theta$ and $\frac{1}{8}$ S-state is not shown. It is obvious, however, that the main effect of the S-state will be in the renormalization constant, Γ, which becomes approximately equal to 1.6.

Figures 4(a) and 4(b) show the spectrum for the case of a combination of meson $P-\cos^2\theta$, symmetrical P, and S-states with $e/a=\frac{1}{8}$ and $b/a=0.06$. The chief result of the 6 percent symmetrical P-state, as expected from its q^2 dependence (this dependence is suppressed by the $\cos^2\theta$ in the $P-\cos^2\theta$ term), is to shift the spectrum peak to higher meson energies, more in agreement with experiment. It also gives a correctly normalized spectrum; that is $\Gamma=1$. On the basis of the 90° experimental spectrum of production, the above combination-type coupling gives the best fit of those considered. It must be remembered, though, that $\cos\theta$ interference terms have not been treated, and these would tend to have a similar, but lesser, effect on the spectrum, as the q^2 term.

In Fig. 5 the calculated spectrum for combined meson $P-\cos^2\theta$ and S-states with $e/a=\frac{1}{8}$ is compared to the experimental data of Richman and Wilcox (corrected for nuclear absorption) for π^- production at 90°. It is true that the statistical errors are much larger for π^- than for π^+ production, but it is also noted that the experimental spectrum increases more rapidly at low energies, in better agreement with the calculated cross section. Aside from this, no new features appear. The Coulomb effect has been neglected here, as throughout, but it should be small for all but very low energy mesons. The same value of Γ is used as for the equivalent π^+ spectrum.

C. 0° Meson Spectrum

Preliminary experimental results have been obtained by Cartwright,[31] Merrit, Schulz, and Heinz[32] for the production of mesons in carbon, in the direction of an incident 341-Mev proton beam. Their findings can shed considerable light on the matrix element for meson production in $P-N$ collisions. The experimental points together with the calculated spectrum for meson $P\cos^2\theta$ terms is shown in Fig. 6, and for S-state ($e/a=1$) terms in Fig. 7. In each case, the same value of Γ is used as for the corresponding 90° curve. The

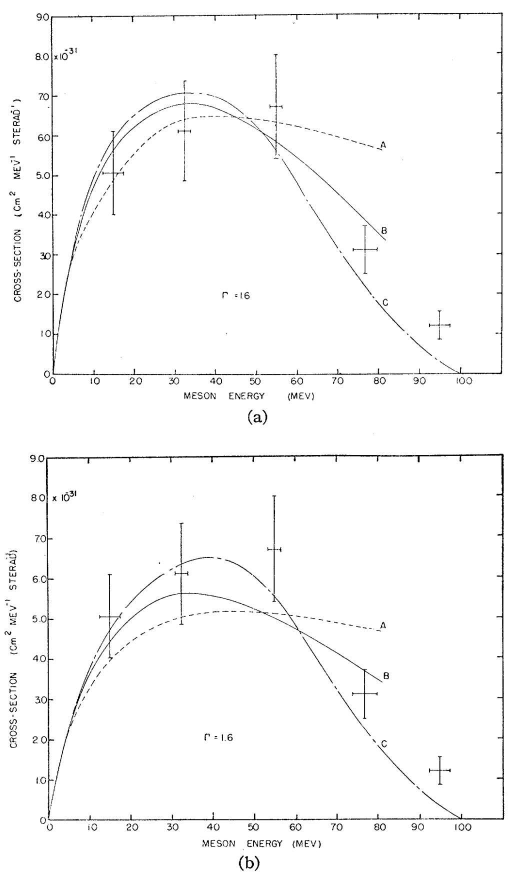

FIG. 5. 90° π^- spectrum for combined meson $P-\cos^2\theta$ and S-states, corresponding to $b=0$, $e/a=\frac{1}{8}$ in Eq. (2a). Exclusion effects are neglected in (a) and considered in (b).

symmetrical P-wave spectrum was not calculated, but for 0° mesons it should be quite similar to the $P-\cos^2\theta$ one. If $b/a=0.06$, the main effect of the q^2 term will merely be a renormalization of Fig. 6(a) by a factor of about 1.06.

The cross section at 0°, unlike that at 90°, is noticed not to depend greatly on the nucleon momentum distribution. A plausible argument for this result can be

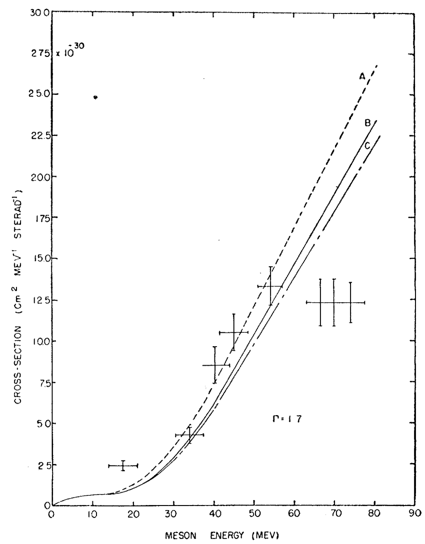

FIG. 6. 0° π^+ spectrum for a meson $P-\cos^2\theta$ state. Exclusion effects are neglected.

made as follows. At 90°, the meson spectrum in nucleon-nucleon collisions is cut off at about 9 Mev by energy-momentum conservation. Hence, in calculating the broadening of this spectrum to 80 Mev by the momentum distribution, the form chosen for this density is expected to influence the spectrum to a large extent. At 0°, however, the free nucleon cutoff does not occur till about 70 Mev, so that in considering the meson spectrum up to 80 Mev, the free-nucleon transition matrix will be of paramount importance.

The $P-\cos^2\theta$ coupling (Fig. 6) cross section is too small at low meson energies, and increases too rapidly at high ones. Both these facts, together with the rapid turning of the experimental spectrum at about 55 Mev, suggest that there is an extensive admixture of S-states and probably of $\cos\theta$ interference terms. Comparison of Figs. 6 and 7 shows that a fairly good fit is obtained if S- and P-states are used in a ratio of 1:1 ($e/a=1$) or more.

The effect of the exclusion terms is not shown in any of these latter figures, but has been calculated. For the

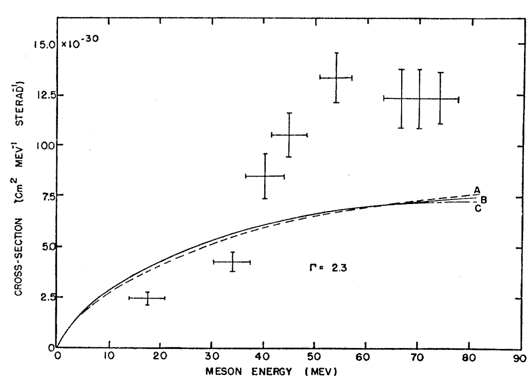

FIG. 7. 0° π^+ spectrum for a meson S-state, normalized to $e/a=1$ in Eq. (2a). Exclusion effects are neglected.

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[31] W. F. Cartwright, University of California thesis (April 16, 1951).

[32] Merrit, Schulz, and Heinz (private communication).

maximum exclusion effect, calculated similarly to the example shown in Appendix B, a 50 percent reduction in the cross section was obtained, almost uniformly, so that the shapes of the curves are hardly altered. The computed decrease in the spectra was actually somewhat less than 50 percent at small energies and somewhat higher at 80 Mev. The approximations used in these calculations are not thought strictly justifiable for this large a correction. On the other hand, it is doubted that a more careful computation will give a result which will change the shape of the spectrum greatly, and certainly not sufficiently to explain the experimental one.

IV. CONCLUSIONS

We have seen that it is possible to analyze the meson production process in nuclei in terms of the free nucleon-nucleon production cross sections. We have shown, furthermore that the exclusion effect is not expected to play a very large role in determining the meson spectrum, at the experimental incident proton energies considered (341 Mev), except possibly for mesons emitted in the direction of the beam. Even here, however, these terms will probably not affect the shape of the spectrum to a great extent. The main result of the meson interaction with nuclear matter after creation is an almost uniform reduction of the cross section for the range of meson energies considered. This decrease is chiefly due to meson reabsorption.

The theoretical development was applied to carbon bombarded by 341-Mev protons. Three different momentum distributions were used for a nucleon in C^{12} in an attempt to fit the experimental meson production spectra at 90° and 0°. These were a modified Chew-Goldberger, a Gaussian, and a Fermi distribution with average kinetic energies of 48.1, 19.3, and 12.8 Mev, respectively. The calculated results are still preliminary, due to a lack of knowledge of the fundamental transition matrices. When this information becomes available, it should be possible to decide which of the momentum densities chosen approximates conditions inside the nucleus.

The calculated spectrum at 90° did not depend strongly on the free-nucleon transition matrices involved, but varied considerably with the momentum distribution chosen. The experimental spectrum could be fitted best with the Gaussian momentum density. For π^+ mesons emitted at 0°, with energies up to 80 Mev, the cross section was found to be relatively independent of the momentum distribution, but quite sensitive to the free nucleon transition matrices. The preliminary experimental data is consistent with a considerable amount of meson S-state (possibly as much as P-state). This shows that the meson production matrices in proton-neutron and proton-proton collisions are not equal. Some experimental evidence for this can be derived from the π^+ to π^- ratio, which is 5.1 ± 0.8, at 345 Mev and 11 ± 3:1 at 381 Mev. Initial results obtained for the π^- spectrum at 0° in C^{12} offer further evidence. Its shape seems to be quite different from the π^+ one at the same angle.[33] In this respect, it may be pointed out that, whereas $\cos\theta$ terms (θ is the meson angle in the interaction center-of-mass system) are forbidden by the exclusion effect for meson production in proton-proton collisions, these terms may very well occur in the case of proton-neutron interactions, but have not been considered.

It is thus seen that experiments, presently in progress at Berkeley, on the production of mesons in carbon at 0° are of great interest, as they can shed considerable light on the free proton-neutron meson production cross section. The π^- meson is only produced in such collisions for incident protons. On the other hand, since the exclusion effects are especially small at 90°, and the meson cross section is relatively independent of the free-nucleon transition matrices at this angle, the experimental spectrum can be used as a useful tool to probe the nucleon momentum distribution in a nucleus.

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APPENDIX A

Evaluation of 90° Meson Spectrum

The specific case of π^+ meson production at 90° to an incident 341-Mev proton beam is treated here. Correlation effects are neglected, and only mesons emitted into $\cos^2\theta-P$-states by proton-proton collisions are considered. For this example, the free particle production cross section in the center-of-mass system of the interacting nucleons, can be written as:

$$d\sigma_{P-P}{}^{(c)}=[(2\pi)^4/v_R{}^{(c)}]\,|(2\pi)^{\frac{3}{2}}\chi_0(p'^{(c)})|^2 \times a(q^{2(c)}/\mu^2)\cos^2\theta\, d\mathbf{J}^{(c)}. \quad \text{(A1)}$$

The superscript (c) is used here to indicate center-of-mass variables. Aside from this, the notation is the same as that used in the main body of the paper. Thus $d\mathbf{J}^{(c)}$ is the phase space volume available

$$\begin{aligned} d\mathbf{J}&=2\pi M^{\frac{3}{2}}\mu q^{(c)}(T_{\max}{}^{(c)}-fT^{(c)})^{\frac{1}{2}}dT^{(c)}d\Omega_q{}^{(c)},\\ &=2\pi M^{\frac{3}{2}}\mu(T_{\max}{}^{(c)}-fT^{(c)})^{\frac{1}{2}}dq^{(c)}/q_0, \end{aligned} \quad \text{(A2)}$$

where $f=1+\mu/2M$, $T^{(c)}$ is the meson energy, and $T_{\max}{}^{(c)}$ the initial kinetic energy of the nucleons minus

[33] Walter Dudziak, private communication.

μ in the center-of-mass system, computed from Eq. (6), with $B_I - B_F^{(c)}$ neglected. This result also neglects proton-neutron mass differences.

The factor a has been defined in Eq. (2a), and its numerical value is adjusted to fit the experimental spectra at various meson angles. Deuteron formation is taken into account in obtaining the magnitude of a but is neglected in the case of meson production in nuclei.

In order to perform the integration over the momentum distribution of the struck nucleon [see Eq. (17)], the cross section must be transformed to the laboratory frame. This is performed relativistically by making use of the generalized relative velocity definition of Møller,[9] which makes $v_R^{(c)} k_0^{(c)} P_0^{(c)}$ an invariant, where $k_0^{(c)}$, $P_0^{(c)}$ are the initial energies of the interacting nucleons in the center-of-mass system. Since, furthermore $d\mathbf{q}/q_0$ is a relativistic invariant as well as the cross section, we obtain in the laboratory frame

$$d\sigma_{P-P} = \frac{(2\pi)^5}{v_R P_0 k_0} \frac{M^{\frac{3}{2}}}{\mu} a \left\{ P_0^{(c)} k_0^{(c)} \left| (2\pi)^{\frac{3}{2}} \chi_0(p'^{(c)}) \right|^2 \times \frac{(\mathbf{P}^{(c)} \cdot \mathbf{q}^{(c)})^2}{(P^{(c)})^2} (T_{\max}{}^{(c)} - f T^{(c)})^{\frac{1}{2}} \right\}_{\text{lab}} \frac{d\mathbf{q}}{q_0}, \quad \text{(A3)}$$

where $p'^{(c)}$ is the relative momentum of the interacting nucleons in the final state and the quantities in the brace are to be expressed in terms of laboratory variables. This can be accomplished relativistically, but it must be kept in mind that $\mathbf{k}$ and $\mathbf{P}$ need not have the same line of action. It thus turns out that the relative velocity of the two systems,

$$\beta = [(P + k_{\parallel})^2 + k_{\perp}^2]^{\frac{1}{2}} / (P_0 + k_0), \quad \text{(A4)}$$

where $k_{\parallel}$ and $k_{\perp}$ are the components of $\mathbf{k}$ parallel and perpendicular to $\mathbf{P}$, respectively. Then, referring to Eq. (17) we obtain for the contribution of $P-P$ collisions to the cross section

$$\frac{d\sigma_A{}^0}{Z} = \frac{d\mathbf{q}}{q_0} \int \frac{(2\pi)^5 a M^{\frac{3}{2}}}{\mu v_0 P_0 k_0} \left\{ \frac{P_0^{(c)} k_0^{(c)}}{(P^{(c)})^2} \left| (2\pi)^{\frac{3}{2}} \chi_0(p'^{(c)}) \right|^2 \right\}_{\text{lax}} \times \left(U - \frac{\gamma \mathbf{k}}{P_0 + k_0} (2\mathbf{P} - f\mathbf{q}) \right)^{\frac{1}{2}} \frac{[\mathbf{q} \cdot \mathbf{k} + q_0 (P_0 - k_0)]^2}{4} \rho(\mathbf{k}) d\mathbf{k},$$

where $\gamma = (1-\beta^2)^{-\frac{1}{2}}$ and

$$U = 2\gamma [P_0 - P^2/(P_0 + k_0)] - 2M + (f-1)\mu - \gamma q_0 f. \quad \text{(A5)}$$

Over the range of integration of k of importance $\beta \sim 0.3$, so that $\gamma \sim 1$. Furthermore, as seen from Eq. (A4), the effect of $k_{\perp}$ on β is quite small, since when $k_{\perp}$ is large, the momentum distribution has cut off, and since $P \gtrsim 2k$ over the integration range. For these two reasons, $k_{\perp}$ is neglected in computing γ. If we take $(2\mathbf{P} - f\mathbf{q})$ to define the z-axis of a cylindrical coordinate system, all but the z-integration can be performed analytically for the momentum distributions chosen. The latter integral is then performed numerically for various values of q_0.

APPENDIX B

Evaluation of Correlation Effects

The effect of the exclusion principle on the cross section, as given by Eqs. (22) and (24), can be calculated on the basis of a single particle nuclear model. The computation involves a double integration over $\mathbf{k}$ and either $\mathbf{p}_2$ or $\mathbf{p}_i$, the other one being taken out by the δ-function in Eq. (22). It was not felt worthwhile carrying out the actual integrals at the present state of experiments. As discussed in the text, an approximate upper limit to the exclusion is obtained if one of the final particles taking part in the reaction has most of the available momentum. In this case, an average value is assumed for $\mathbf{p}_2$ and $\mathbf{p}_i$, consistent with momentum and energy conservation, so that only one integration has to be performed. It suffices to specify the direction of either $\mathbf{p}_2$ or $\mathbf{p}_i$. They are assumed to be along the same line of action. Since $\mathbf{p}_2 + \mathbf{p}_i$ is fixed by momentum conservation, this hypothesis leaves the least momentum to p_i, for a given value of p_2, and the magnitude of these momenta is then entirely determined by momentum and energy conservation. M' can thus be taken to correspond to a small value of p_i, in which case p_2 is sufficiently large that $M'' \approx 0$.

The integration over k is further simplified in this computation by taking $k_{\perp} = 0$ [see Eq. (A4) for the definition of this quantity]. With this assumption, the integration is carried out as that in Appendix A.