bismuth, small amounts of nonferromagnetic impurities do not appreciably effect the field dependence.

In Fig. 1 $\Delta\chi$ is plotted against the component of the magnetic field strength in the direction of the hexagonal axis for the five temperatures investigated. It is seen that the field dependence persists up to at least 85°K. The variation of χ_3 with field strength in the region investigated is of order 0.01×10^{-6} while it is of order 0.05×10^{-6} at 20°K and 0.25×10^{-6} at 4°K. At 85°K χ_3 is of order 0.20×10^{-6} and $\Delta\chi$ is of order 0.05×10^{-6}. Thus at 85°K the variation is χ_3 in about 5 percent while the variation in $\Delta\chi$ is about 20 percent. As previous work at this temperature was done using the body force method, which gives a direct measurement of χ_3, it is not surprising that field dependence was not observed.

It would be desirable to have more data at higher field strengths to see if χ_3 performs the oscillations characteristic of the de Haas-van Alphen effect. Note that, in contrast to the effect at hydrogen and helium temperatures, the maxima and minima do not occur at the same field strengths for different temperatures. It would also be desirable to have more data between 85°K and 300°K in order to determine the point at which the field dependence disappears.

At helium temperatures χ_3 oscillates about the value which it approaches for low field strengths. Table I lists the average values of χ_3 found by other investigators and the low field (2.7 kilogauss) values found in the present work. It is seen that the low field value of χ_3 is approximately constant up to at least 63°K.

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A Note on the Quantum Rule of the Harmonic Oscillator

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The purpose of the present note is to show, with the aid of an elementary example, that the commutation rules, which are usually given the rule of postulates in quantum mechanics, are in fact not arbitrary, provided that a more stringent definition of the Hilbert space and a strict expansion theorem are adopted.

I. INTRODUCTION

IN this note the following problem, which has been recently treated by Wigner,[1] is considered again. In the formulation of quantum mechanics, one often wonders whether it is possible to reverse the ordinary procedure, in which the commutation relations are first postulated as a generalization of the classical poisson bracket and then the equations of motion are deduced. The problem is whether one can derive the commutation rules from the equation of motion taken over from the classical theory, together with the postulate that the energy is a time displacement operator, i.e.,

$$\dot{f}=[f,H],\tag{1}$$

where f is any dynamical variable of a given system and H the total hamiltonian. By way of illustration, we shall consider the case of an harmonic oscillator where $H=\frac{1}{2}(x^2+\dot{x}^2)$ with $\hbar=\omega=1$. Here one notices that H is itself a function of x and $\dot{x}$, where $\dot{x}$ is defined by $\dot{x}=[x,H]$. Thus a relation of the type (1), with $f=f(x,\dot{x})$, is a complicated relation between the commutators. The conclusion which Wigner arrived at in this example is negative; i.e., the correct solution $[x,\dot{x}]=1$ does not follow uniquely. It will be shown in the present note that by properly formulating the conditions, including a more stringent definition of Hilbert space and a strict expansion theorem, the commutation rule will follow uniquely, though with less stringent definitions other solutions cannot be excluded.

It must be stressed that whether or not a state is physically permissible, cannot be seen clearly without referring to a special representation. Hence, it is the suitable boundary conditions in a special representation, laid down on physical grounds and in general being different for different systems, that serves to restrict wave functions to a certain special type, to include at the same time all permissible ones, and thereby to mark precisely the complete Hilbert space in which the state of the system is depicted and its operators apply.

For the oscillator one requires that the eigenvalues of x and $\dot{x}$ form continuous spectra and extend from $-\infty$ to $+\infty$, and that H be positive definite.[2] These restrictions do not suffice to mark completely the appropriate Hilbert space. For this purpose one has to impose restrictions on the energy eigenfunctions $\psi_n(x)$. Here we have, as is obvious on physical grounds, a natural boundary condition, i.e., $\psi_n(x)\rightarrow0$ as $x\rightarrow\pm\infty$ for all n.

[1] E. P. Wigner, Phys. Rev. 77, 711 (1950).

[2] It is then sufficient to deduce that H is discrete [see (10)].

We summarize the conditions for the deduction of the commutation rule in the case of the harmonic oscillator.

(a) Hamiltonian $H=\frac{1}{2}(x^2+\dot{x}^2)$

(b) Equation of motion $\ddot{x}+x=0$

(c) The complete Hilbert space for the system defined by the complete set of energy eigenfunctions $\psi_n(x)$ satisfying the boundary condition that $\psi_n(x)\to 0$ as $x\to\pm\infty$ $(-\infty<x<+\infty)$

(d) Superposition principle in the Hilbert space defined in (c). By (c) and (d) we require that any physically admissible state represented by a wave function satisfying the boundary condition in (c) shall be expansible in terms of the set of energy eigenfunctions. Here we need the stringent definition of the expansion theorem; for an arbitrary admissible wave function $f(x)$, we require that the expansion

$$\sum_{n=0}^{\infty} a_n\psi_n(x)$$

converges absolutely and uniformly to $f(x)$. If a less stringent definition is adopted, namely,

$$\sum_{n=0}^{\infty} a_n\psi_n(x)$$

converges to $f(x)$ only in the mean

$$\lim_{N\to\infty}\int_{-\infty}^{+\infty}|f(x)-\sum_{n=0}^{N} a_n\psi_n(x)|^2dx=0,$$

then we cannot rule out other possibilities than $[x,\dot{x}]=1$. These conclusions, however, cannot be reached without referring to the x-representation where a natural boundary condition can be laid down.

II. DEDUCTION OF THE COMMUTATION RULE

From (1) and (a) with $f=x$, it follows that

$$\dot{x}=[x,\tfrac{1}{2}\dot{x}^2]=\tfrac{1}{2}(\dot{x}[x,\dot{x}]+[x,\dot{x}]\dot{x}).$$

Introducing $S=[x,\dot{x}]-1$, one has

$$\{S,\dot{x}\}=0, \tag{2}$$

where the curly bracket is the anti-commutator. From (1) and (b), one has similarly

$$\{S,x\}=0. \tag{3}$$

From (2) and (3), it can easily be shown that

$$[S^2,x]=[S^2,\dot{x}]=0,\quad [S,H]=0$$

which shows that S is a constant of motion, and that S^2 is a real numerical constant.

In the x-representation, (2) becomes $(x'+x'')\times\langle x'|S|x''\rangle=0$. Hence it follows that

$$\langle x'|S|x''\rangle=c(x')\delta(x'+x'') \tag{4}$$

where $c(x')$ is an arbitrary function of x', and the hermitian property of S requires that $c(x')=c^*(-x')$. Hence in the x-representation one can write

$$S=c(x)R, \tag{5}$$

where R is the reflection operator defined by

$$R|x\rangle=|-x\rangle. \tag{6}$$

From this representation of S, one obtains the explicit operational form of $\dot{x}$;

$$\dot{x}=-i\frac{d}{dx}+g(x)+i\frac{c(x)}{2x}R, \tag{7}$$

where $g(x)$ is real. It can be shown that the term $g(x)$ can be removed by properly choosing the phase factor in the x-representation. Using a star to denote the operator in the new representation, one has

$$\left(\frac{d}{dx}\right)^*=e^{-iy}\frac{d}{dx}e^{iy}=\frac{d}{dx}+i\frac{dy}{dx}$$

$$R^*=e^{-iy}Re^{iy}=e^{-2iy_-}R$$

where y is a real function of x, and y_- is the odd part of y. If for y one chooses $y=\int^x g(x)dx$, (7) becomes

$$\dot{x}=-i\left(\frac{d}{dx}\right)^*+i\frac{c'(x)}{2x}R^*,\quad c'(x)=c(x)e^{2iy_-}.$$

Dropping the stars and the dash, and with the help of (2), one can show that $c(x)$ is a numerical constant, thus obtaining

$$\dot{x}=-i\frac{d}{dx}+\frac{ic}{2x}R. \tag{8}$$

The next step is to set up the symbolic energy eigenstates by using the variables $\eta=(\dot{x}+ix)/\sqrt{2}$ and $\eta^*=(\dot{x}-ix)/\sqrt{2}$.[3] Following a similar argument to that usually used in the treatment of the harmonic oscillator, it can be shown that starting from the lowest state $|0\rangle$ characterized by

$$\eta^*|0\rangle=0 \tag{9}$$

one can obtain the excited states by successively multiplying on the left of $|0\rangle$, the energy difference of two neighboring states being always 1. The scheme of energy eigenstates and their corresponding eigenvalues are

$$\left.\begin{array}{cccc}|0\rangle, & \eta|0\rangle, & \eta^2|0\rangle, & \cdots\\ \frac{1}{2}(1+cR_0), & \frac{1}{2}(3+cR_0), & \frac{1}{2}(5+cR_0), & \cdots\end{array}\right\} \tag{10}$$

where c is a real constant so far undetermined and R_0 is the eigenvalue of R in the ground state, being either 1 or -1. The condition that H is positive definite

[3] See P. A. M. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1947), 3rd edition.

requires that

$$cR_0 \geq -1. \tag{11}$$

By using the result of (8) and (9), it is possible to investigate the ground state of the oscillator. For

$$\left\langle x\left|\frac{1}{\sqrt{2}}(\dot{x}-ix)\right|0\right\rangle = \left\langle x\left|\frac{i}{\sqrt{2}}\left(-\frac{d}{dx}+\frac{c}{2x}R-x\right)\right|0\right\rangle$$

or, writing $\psi_0(x) = \langle x|0\rangle$

$$\left(\frac{d}{dx}+x\right)\psi_0(x) = \frac{c}{2x}\psi_0(-x). \tag{12}$$

Here the condition (c) is used. It follows that the admissible solutions can be written in the form

$$\psi_0^{(\tau)}(x) = b_\tau x^\tau \exp(-x^2/2) \tag{13}$$

where $\tau = (-1)^{c/2}c/2$ and $b_\tau = (2^{\tau+1}/1\cdot3\cdot5\cdot(2\tau-1)\sqrt{\pi})^{\frac{1}{2}}$ is a normalizing constant. It should be noted that (12) is an eigenvalue problem; there exist solutions of (12) only for certain values of c. The spectrum of c and the corresponding ones of cR_0 and τ allowed by (11) are

$$\left.\begin{array}{rrrrrl} c=0 & -2 & 4 & -6 & 8 & \cdots \\ cR_0=0 & 2 & 4 & 6 & +8 & \cdots \\ \tau=0 & 1 & 2 & 3 & 4 & \cdots \end{array}\right\} \tag{14}$$

The wave function of the nth excited state for any τ follows from (8) and (10)

$$\begin{aligned}\psi_n^{(\tau)}(x) &= \langle x|\eta^n|0\rangle \\ &= \left(\frac{i}{\sqrt{2}}\right)^n\left(-\frac{d}{dx}+x+\frac{c}{2x}R\right)^n\psi_0^{(\tau)}(x)N_n^{(\tau)}\end{aligned} \tag{15}$$

where $N_n^{(\tau)}$ is a normalizing factor

$$N_n^{(\tau)} = \underbrace{\{(1+2\tau)2(3+2\tau)4\cdots\}}_{n\text{ factors}}{}^{-\frac{1}{2}}.$$

Substituting (13) in (15) one obtains

$$\begin{aligned}\psi_n^{(\tau)}(x) = &\left(\frac{i}{\sqrt{2}}\right)^n N_n^{(\tau)}b_\tau x^\tau e^{x^2/2} \\ &\times\underbrace{\{\cdots(D+2\tau/x)D(D+2\tau/x)D\}}_{n\text{ factors}}e^{-x^2}.\end{aligned} \tag{16}$$

It becomes clear that for even n the lowest power in x in the expression (16) is always x^τ, and for odd n it is $x^{\tau+1}$, irrespective of the value of τ. The set $\psi_n^{(\tau)}(x)$ with $\tau \neq 0$ will vanish at $x=0$ for all n. That these sets with $\tau \neq 0$ do not meet our requirement (d) is obvious, when one considers the expansion of a permissible wave function that is finite at the origin, such as $\exp(-x^2/2)$.

$$\exp(-x^2/2) = \sum_{n=0}^{\infty} a_n^{(\tau)}\psi_n^{(\tau)} \tag{17}$$

where

$$\begin{aligned}a_n^{(\tau)} = &\left(\frac{i}{\sqrt{2}}\right)^n N_n^{(\tau)}b_\tau \\ &\times\int_{\infty}^{\infty} x^\tau \underbrace{\{\cdots(D+2\tau/x)D(D+2\tau/x)D\}}_{n\text{ factors}}\exp(-x^2dx),\end{aligned}$$

which is zero when $(n+\tau)$ is odd, and finite when $(n+\tau)$ is even. The expansion in (17), therefore, ceases to hold at $x=0$, though they can be shown to converge in the mean. We have seen now that only the set $\psi_n^{(0)}$ meets all four requirements and that S must be a null operator.

III. DISCUSSION

It is seen that the deduction of the commutation rule in the case of the harmonic oscillator requires in particular "a suitable boundary condition to mark the appropriate Hilbert space" and "the strict expansion theorem." It is difficult to say that such a stringent expansion theorem should be rigidly followed in quantum mechanics. But once this is adopted, it has been found that this formulation of the quantum rule seems to hold for any nonrelativistic particle with a reasonable potential function expressible as a power series in x. A possible application of the present formulation is its connection with the second quantization of Bose particles. If the lagrangian of the field contains derivatives of the field variables higher than the second, it is no longer possible to derive the hamiltonian from the lagrangian as an explicit function of pairs of conjugate variables, but the former can only be brought into the form $\sum_\mu \psi_\mu^* F(D', D)\psi_\mu$ where ψ_μ are the field variables, D and D' operate on ψ_μ and ψ_μ^*, respectively, such that $D\psi_\mu = \dot{\psi}_\mu$ and $\psi_\mu^* D' = \dot{\psi}_\mu^*$, and F is an arbitrary function of D' and D. The definition $\dot{\psi}_\mu = [\psi_\mu, H]$ together with the equation of motion derived from the lagrangian is all that is needed to obtain quantization; any guess of the commutation rules among the field variables and their derivatives is redundant and may be quite wrong.

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