A Canonical Transformation in the Theory of Particles of Arbitrary Spin

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Canonical transformation between the β-operators and certain components of the spin-tensor yields explicit expressions for the latter. The transformation is used to deduce general consequences that simplify the derivation of commutation relations. It can also be applied with advantage to construct free particle solutions of the wave equation.

BHABHA'S theories of the equations of particles of arbitrary spin[1,2] all rest on a basic assumption about the explicit expression for the relativistic skew-symmetric spin-tensor $t_{\mu\nu}$. A possible set of "five-dimensional" theories is based on the assumption that $t_{\mu\nu}=\beta_\mu\beta_\nu-\beta_\nu\beta_\mu$. In the following paper details of a different approach[3] are presented. It is shown that a certain symmetry in the equations expressing the relativistic covariance of the linear first-order wave equation suggests the existence of a canonical transformation between three components $t_{\mu\nu}$ and three β_λ. An almost obvious specialization of this canonical transformation leads to the above expression for $t_{\mu\nu}$ as a result. Conversely it can be shown that within the frame work of Bhabha's "five-dimensional" theories the canonical transformation always exists; it simply corresponds to a rotation through the angle $\pi/2$ in the (x_5, x_μ) plane. Leaving aside the problem of other possible transformations leading to different expressions for $t_{\mu\nu}$ (which may yield the equations of Dirac, Fierz, and Pauli for higher spins, or the symmetric equation of the meson suggested by Eliezer[4] and considered by Bhabha[2]), the implications and applications of this particular transformation, and others related to it by covariance, are studied. Hitherto awkwardly derived results, such as the equality of the characteristic equations of the spin operators σ_k and the β_λ, or the imitation by two or three σ_k of the algebraic relations of the β_λ, are here deduced as almost immediate consequences. Furthermore, by way of illustration, the equations of the Dirac electron and the Duffin-Kemmer meson are derived in but a few lines. Finally, taking the case of a particle of spin $\frac{3}{2}$ as an example, it is shown that the canonical transformation also provides an elegant method for obtaining free particle solutions of the wave equation.

A. THE CANONICAL TRANSFORMATION

The condition that the linear first-order wave equation be covariant under Lorentz transformations requires that the components of the skew-symmetric tensor $t_{\lambda\nu}$ formed by the generators of the Lorentz group satisfy the commutation relation

$$[\beta_\mu, t_{\lambda\nu}]=\delta_{\lambda\mu}\beta_\nu-\delta_{\nu\mu}\beta_\lambda, \quad (\lambda, \mu, \nu=1, \cdots 4), \tag{1}$$

where $[A, B]\equiv AB-BA$. On the other hand, the generators of the Lorentz group satisfy the commutation relation

$$[t_{\mu\nu}, t_{\rho\sigma}]=-\delta_{\mu\rho}t_{\nu\sigma}-\delta_{\nu\sigma}t_{\mu\rho}. \tag{2}$$

Defining

$$\sigma_k=-it_{lm}, \quad \gamma_k=-it_{4k}, \quad (k, l, m=1, 2, 3 \text{ cyclic}), \tag{3}$$

(1) and (2) give the following two sets of equations:

$$\begin{array}{lll} \text{(a)}\ [\beta_k, \sigma_k]=0, & \text{(d)}\ [\beta_k, \gamma_l]=0, \\ \text{(b)}\ [\beta_k, \sigma_l]=i\beta_m, & \text{(e)}\ [\beta_k, \gamma_k]=i\beta_4, \\ \text{(c)}\ [\beta_4, \sigma_k]=0, & \text{(f)}\ [\beta_4, \gamma_k]=-i\beta_k, \end{array} \tag{4}$$

and

$$\begin{array}{ll} \text{(a)}\ [\gamma_k, \sigma_k]=0, & \text{(b)}\ [\gamma_k, \sigma_l]=i\gamma_m, \\ \text{(c)}\ [\gamma_k, \gamma_l]=i\sigma_m, & \text{(d)}\ [\sigma_k, \sigma_l]=i\sigma_m. \end{array} \tag{5}$$

Furthermore, if the wave equation is to be covariant under reflections, there exist four operators η_μ such that

$$\eta_\mu{}^2=1, \quad \eta_\mu\beta_\nu+\beta_\nu\eta_\mu=0, \quad (\mu\neq\nu), \quad \eta_\mu\beta_\mu=\beta_\mu\eta_\mu, \tag{6}$$

i.e., in particular,

$$\eta_4\beta_k+\beta_k\eta_4=0, \quad \eta_4\beta_4=\beta_4\eta_4. \tag{6a}$$

It is noteworthy that from Eqs. (4) it can be derived (see Appendix I) that the three operators γ_k, too, satisfy

$$\eta_4\gamma_k+\gamma_k\eta_4=0, \tag{7a}$$

and that

$$\eta_4\sigma_k=\sigma_k\eta_4. \tag{7b}$$

Now in addition to this similarity between Eqs. (6a) and (7a), it is remarkable that Eqs. (5a) and (5b), too, differ from (4a) and (4b) merely by having γ_k and γ_m in place of β_k and β_m. This suggests the possibility of a canonical transformation

$$\gamma_k=gS\beta_kS^{-1}, \tag{8}$$

where

$$S\sigma_k=\sigma_kS, \tag{9a}$$

and where g is an ordinary constant. In addition assume that S commutes with β_4:

$$S\beta_4=\beta_4S. \tag{9b}$$

Substituting (8) in (4d) gives

$$S^{-1}\gamma_kSS\beta_lS^{-1}-S\beta_lS^{-1}S^{-1}\gamma_kS=0,$$

[1] H. J. Bhabha, Revs. Modern Phys. 17, 200 (1945).
[2] H. J. Bhabha, Revs. Modern Phys. 21, 451 (1949).
[3] W. A. Hepner, Phys. Rev. 81, 290 (1951); 82, 447 (1951).
[4] C. J. Eliezer, Nature 159, 60 (1947).

whence, multiplying by S^{-1} on the left and by S on the right,

$$(S^{-1})^2\gamma_k S^2\beta_l-\beta_l(S^{-1})^2\gamma_k S^2=0.$$

Adding

$$\gamma_k\beta_l-\beta_l\gamma_k=0, \tag{4d}$$

one obtains

$$\beta_l[(S^{-1})^2\gamma_k S^2+\gamma_k]-[(S^{-1})^2\gamma_k S^2+\gamma_k]\beta_l=0. \tag{10a}$$

Similarly, using (8) and (9b), one obtains from (4e)

$$\beta_k[(S^{-1})^2\gamma_k S^2+\gamma_k]-[(S^{-1})^2\gamma_k S^2+\gamma_k]\beta_k=0. \tag{10b}$$

Finally, multiplying (4f) by $(S^{-1})^2$ on the left and by S^2 on the right, and adding (4f), one obtains

$$\beta_4[(S^{-1})^2\gamma_k S^2+\gamma_k]-[(S^{-1})^2\gamma_k S^2+\gamma_k]\beta_4 = -i[(S^{-1})^2\beta_k S^2+\beta_k]. \tag{10c}$$

It can now easily be seen that a consistent solution is obtained assuming

$$\beta_k S^2+S^2\beta_k=0. \tag{11}$$

For Eqs. (10) then merely say that the operators $(S^{-1})^2\gamma_k S^2+\gamma_k$ commute with all four β_μ, and hence with all the elements of the group or algebra composed of the β_μ and their products. The operators $(S^{-1})^2\gamma_k S^2+\gamma_k$ must therefore be multiples of the unit operator. Since, from (5b),

$$\mathrm{spur}(S^{-1})^2\gamma_k S^2=\mathrm{spur}\gamma_k=0, \tag{12}$$

one obtains

$$(S^{-1})^2\gamma_k S^2+\gamma_k=0, \tag{13}$$

i.e.,

$$S^2\gamma_k+\gamma_k S^2=0, \tag{14}$$

which, according to (8), is in agreement with the assumption (11). From (11) and (9b) it can therefore be deduced that a consistent solution is obtained if S^2 is a numerical multiple of the operator η_4 defined in (6a).

Applying the transformation (8) and (9) to (4f) and (5c), one now obtains

$$\gamma_k=-ig^2(\beta_4\beta_k-\beta_k\beta_4),\quad \sigma_m=-ig^2(\beta_k\beta_l-\beta_l\beta_k), \tag{15a}$$

i.e.,

$$t_{\mu\nu}=g^2(\beta_\mu\beta_\nu-\beta_\nu\beta_\mu). \tag{15b}$$

This expression for $t_{\mu\nu}$ forms the starting point of Bhabha's theory of particles of arbitrary spin; here it is obtained as a result of the canonical transformation (8), (9).

Conversely, it can be shown that within the framework of Bhabha's theory the S-transformation satisfying (8), (9), (11) always exists. For in Bhabha's theory Eqs. (15) together with Eqs. (1) and (2) are interpreted as the commutation relations of the components of an angular momentum tensor I^{KL} in five dimensions (K, $L=1, \cdots 5$), with $g\beta_k=I^{k5}$. This means that to every Lorentz transformation in five dimensions specified by $x_K'=a_K{}^L x_L$ there corresponds an operator T such that

$$TI^{MN}T^{-1}=a_K{}^M a_L{}^N I^{KL}=I'^{MN}. \tag{16}$$

Now consider the particular transformation representing a rotation through the angle $\pi/2$ in the (x_4, x_5) plane:

$$x_k'=x_k,\quad x_4'=x_5,\quad x_5'=-x_4,$$

i.e.,

$$a_1{}^1=a_2{}^2=a_3{}^3=1,\quad a_4{}^5=-a_5{}^4=1.$$

Equation (16) now gives

$$TI^{15}T^{-1}=gT\beta_1T^{-1}=a_K{}^1a_L{}^5I^{KL}=a_1{}^1a_4{}^5I^{14}=\gamma_1,$$
$$TI^{45}T^{-1}=gT\beta_4T^{-1}=a_K{}^4a_L{}^5I^{KL}=a_5{}^4a_4{}^5I^{54}=g\beta_4,$$

and

$$TI^{14}T^{-1}=T\gamma_1T^{-1}=a_K{}^1a_L{}^4I^{KL}=a_1{}^1a_5{}^4I^{15}=-g\beta_1,$$
$$T\sigma_1T^{-1}=\sigma_1,$$

and similarly for β_2, β_3, γ_2, γ_3, and σ_2, σ_3. Hence T is identical with the S-transformation specified by (8), (9), and (11).

It can also be shown that, if the expressions (15) are to hold, then the transformation specified by (8) and (11) is the only one such that S commutes with β_4. For from (4f) one obtains by canonical transformation, using (8) and (9b),

$$-i\gamma_k=g^2\beta_4S^2\beta_k(S^{-1})^2-g^2S^2\beta_k(S^{-1})^2\beta_4.$$

Hence, if (15) is to hold, then

$$-ig^2(\beta_4\beta_k-\beta_k\beta_4)=ig^2[\beta_4S^2\beta_k(S^{-1})^2-S^2\beta_k(S^{-1})^2\beta_4],$$

i.e.,

$$\beta_4[S^2\beta_k(S^{-1})^2+\beta_k]=[S^2\beta_k(S^{-1})^2+\beta_k]\beta_4.$$

Similarly, by canonical transformation of (10a) and (10b) one obtains, respectively,

$$S\beta_lS^{-1}[S^2\beta_k(S^{-1})^2+\beta_k]=[S^2\beta_k(S^{-1})^2+\beta_k]S\beta_lS^{-1},$$
$$S\beta_kS^{-1}[S^2\beta_k(S^{-1})^2+\beta_k]=[S^2\beta_k(S^{-1})^2+\beta_k]S\beta_kS^{-1}.$$

These equations show that the operators $S^2\beta_k(S^{-1})^2+\beta_k$ commute with β_4 and with the three γ_l. But since, in virtue of (4f), β_l can be expressed in terms of β_4 and γ_l, it follows that $S^2\beta_k(S^{-1})^2+\beta_k$ is a constant multiple of the unit operator. Since $\mathrm{spur}\beta_k=0$, it follows that the constant is zero, i.e., (11) must hold.

Assuming, as is compatible with Eqs. (8), (9), and (11), that S depends on β_4 only, an expression for S can easily be given. Since S^2 anticommutes with the three β_k and commutes with β_4, S^4 can be represented by the unit operator. Now it is shown below that the characteristic values of $g\beta_4$ and σ_k must be the same, *viz.*, either the integers $-n, \cdots 0, \cdots +n$, for integral spin, or the half-integers $-n, \cdots -\frac{1}{2}, +\frac{1}{2}, \cdots +n$, for half-integral spin. It is therefore possible to represent S^4 by

$$S^4=\exp(2\pi ig\beta_4), \tag{17a}$$

this operator being equal to $+1$ for integral spin and to -1 for half-integral spin. Hence S and S^{-1} can be represented by

$$S=\exp(\tfrac{1}{2}i\pi g\beta_4), \quad S^{-1}=\exp(-\tfrac{1}{2}i\pi g\beta_4), \quad (17b)$$

showing that S is unitary if β_4 is hermitian. It is clear that S can be expanded into a polynomial of degree $2n$ in $g\beta_4$, the $2n+1$ coefficients being determined by (17b), using the $2n+1$ eigenvalues of $g\beta_4$.

In Appendix II it is shown that, as is to be expected from relativistic covariance, the transformation (8), (9) is not the only one that leads to (15). More generally, there are four transformations S_μ such that $[S_\mu, \beta_\mu]=0$, $S_\mu^2\beta_r+\beta_r S_\mu^2=0$ $(r\neq\mu)$; they correspond, in Bhabha's five-dimensional scheme, to rotations through the angle $\pi/2$ in the (x_μ, x_5) plane. Finally, it is interesting to note another possibility leading yet again to the expressions (15), *viz.*, (see Appendix III)

$$S_\mu\beta_\mu+\beta_\mu S_\mu=0, \quad [S_\mu^2, \beta_\lambda]=0.$$

These four transformations correspond to rotation through the angle $\pi/2$ in the (x_μ, x_5) plane and reflection of x_5.

B. APPLICATIONS OF THE CANONICAL TRANSFORMATION

(i) Some Consequences

The existence of the transformation (8), (9) has numerous immediate consequences that hitherto have been derived only by laborious calculations, using the expressions (15). For example, it follows immediately that the characteristic equation of σ_k, which determines the eigenvalues of the spin components, is also the characteristic equation of $g\beta_\mu$. For owing to the complete parity of the four β_μ, the characteristic equations of γ_k and σ_k are the same, and the characteristic equation of γ_k transforms into that for $g\beta_k$. Furthermore, it is now obvious that the four operators γ_k and $g\beta_4$ satisfy the same algebraic relations as the four β_μ.

Similarly, it is now easy to see why the three operators σ_k satisfy commutation relations of three operators $g\beta_k$. Regarding relations involving only two different σ's this is now almost self-evident; for owing to the complete parity of all four β's and in virtue of the equivalent structure of two σ's on the one hand and two γ's on the other, the relations between two γ's will also be satisfied by two σ's, and the relations between two γ's are satisfied by two $g\beta$'s. Now the relations between three different $g\beta$'s are obtained from (see 5c) (5a)

$$[[\gamma_k, \gamma_l], \gamma_m]=0, \quad (18a)$$

by canonical transformation, after suitable reductions with the help of the characteristic equations and the relations involving only two different γ's. Since, according to (5d), the three σ's, too, satisfy

$$[[\sigma_k, \sigma_l], \sigma_m]=0, \quad (18b)$$

it follows that the resulting equation for three σ's, after similar suitable reductions with the help of the characteristic equations and the relations involving only two different σ's, will be the same as for three $g\beta$'s. The converse is not true; owing to the different structure of the sets of three σ's and three γ's, an equation between three σ's, for example (5d), need not be satisfied by three $g\beta$'s.

(ii) Derivation of Commutation Relations for Spin $\frac{1}{2}$ and Spin 1

Substituting (15) in (1), and using the result that $g\beta_\mu$ and σ_k satisfy the same characteristic equation, the commutation relations of the β's can now be derived in a few lines. Putting $\lambda=\mu\neq\nu$, one obtains

$$\beta_\mu^2\beta_\nu+\beta_\nu\beta_\mu^2-2\beta_\mu\beta_\nu\beta_\mu=(1/g^2)\beta_\nu, \quad (\mu\neq\nu), \quad (19)$$

whatever the value of the spin.

(a) Dirac's Commutation Relations

For spin $\frac{1}{2}$ the characteristic equation of σ_k is given by

$$(\sigma_k-\tfrac{1}{2})(\sigma_k+\tfrac{1}{2})=0, \quad (20)$$

whence also

$$(g\beta_\mu)^2=\tfrac{1}{4}. \quad (21a)$$

Hence (19) becomes

$$\beta_\mu\beta_\nu\beta_\mu=-(1/4g^2)\beta_\nu,$$

whence, multiplying by β_μ and using (21a),

$$\beta_\mu\beta_\nu+\beta_\nu\beta_\mu=0, \quad (\mu\neq\nu). \quad (21b)$$

Since the numerical factor g can be absorbed in the mass constant, its value can be freely adjusted. Without loss of generality one may therefore put $g^2=\frac{1}{4}$, so that

$$\beta_\mu^2=1. \quad (21c)$$

Equations (21) are identical with Dirac's commutation relations for the electron. The expressions (17b) now give

$$S=(1-i\beta_4)/\sqrt{2}, \quad S^{-1}=(1+i\beta_4)/\sqrt{2}; \quad S^2=-i\beta_4=-i\eta_4. \quad (22)$$

(b) The Duffin-Kemmer Relations

For spin 1 the characteristic equation of σ_k is given by

$$(\sigma_k-1)\sigma_k(\sigma_k+1)=0, \quad (23)$$

whence also

$$\beta_\mu^3=(1/g^2)\beta_\mu. \quad (24)$$

Hence, multiplying (19) by β_μ on both sides, one obtains

$$2\beta_\mu^2\beta_\nu\beta_\mu^2=(1/g^2)\beta_\mu\beta_\nu\beta_\mu, \quad (25)$$

whence again

$$(1/g^2)\beta_\mu^2\beta_\nu\beta_\mu^2=(2/g^4)\beta_\mu\beta_\nu\beta_\mu, \quad (26)$$

and therefore

$$\beta_\mu\beta_\nu\beta_\mu=0, \quad (\mu\neq\nu). \tag{27}$$

Taking $\mu\neq\lambda$, $\nu\neq\lambda$, $\mu\neq\nu$, and using (15b), (1) gives

$$\beta_\mu\beta_\lambda\beta_\nu+\beta_\nu\beta_\lambda\beta_\mu=\beta_\mu\beta_\nu\beta_\lambda+\beta_\lambda\beta_\nu\beta_\mu, \tag{28}$$

whence, multiplying by β_λ^2 on the right and using (27) and (19),

$$\beta_\mu\beta_\nu\beta_\lambda+\beta_\lambda\beta_\nu\beta_\mu=0. \tag{29}$$

Putting $g^2=1$, Eqs. (19), (24), (27), (29) are identical with the Duffin-Kemmer commutation relations for the meson of spin 1. The expressions (17b) now give

$$S=1-i\beta_4-\beta_4^2, \quad S^{-1}=1+i\beta_4-\beta_4^2; \tag{30a}$$

$$S^2=1-2\beta_4^2=\eta_4. \tag{30b}$$

(iii) Derivation of Solutions of the Wave Equation by Lorentz Transformation

Apart from making the derivation of commutation relations much simpler, the knowledge of the existence of the canonical transformation (8), (9) is of great help in finding solutions of the wave equation. Instead of solving a system of n simultaneous linear equations, where n is the degree of the matrix representation, the idea is to construct the solutions from the obvious elementary solutions in the rest system (all but one spinor components vanish) by Lorentz transformation, the procedure being greatly simplified by an application of the canonical transformation (8). This method also facilitates the finding of that combination of solutions that is also eigensolution of certain operators, as is frequently desirable. It is best to demonstrate the method for a particular case, say Bhabha's equation[5] for a particle of spin $\frac{3}{2}$. In this case the canonical transformation (8), (9) holds true with

$$S=\frac{1}{2\sqrt{2}}\left(\frac{5}{2}-\frac{13}{3}i\beta_4-2\beta_4^2+\frac{4}{3}i\beta_4^3\right),$$

$$S^{-1}=\frac{1}{2\sqrt{2}}\left(\frac{5}{2}+\frac{13}{3}i\beta_4-2\beta_4^2-\frac{4}{3}i\beta_4^3\right),$$

$$S^2=i\eta_4=-\tfrac{1}{3}i(7\beta_4-4\beta_4^3);$$

but the explicit expressions will not be required in the following. Designating the four-vector of momentum by $\mathbf{p}$, $(p_4=i\omega)$, and the position vector of the world point by $\mathbf{x}$, $(x_4=it)$, consider the plane wave solutions

$$\psi=U\exp(i\mathbf{p}\cdot\mathbf{x}). \tag{31}$$

The (16-component or 20-component) spinor amplitude U satisfies

$$(ip_\mu\beta_\mu+\kappa)U=0. \tag{32}$$

Taking the direction of x_1 in the direction of the three-vector with components p_1, p_2, p_3, and writing $p_1=p$, (32) becomes

$$(ip\beta_1-\omega\beta_4+\kappa)U=0. \tag{33}$$

The solutions U of these equations for given p can be obtained from the elementary solutions U_0 for $p=0$, $(\omega=\omega_0)$, by the Lorentz transformation

$$U=TU_0, \quad U_0=T^{-1}U, \tag{34}$$

where

$$(-\omega_0\beta_4+\kappa)U_0=0, \tag{35a}$$

i.e.,

$$(-\omega_0\beta_4+\kappa)T^{-1}U=0. \tag{35b}$$

Multiplying (35b) on the left by T and using (33), one obtains

$$-\omega_0T\beta_4T^{-1}U+\kappa U$$
$$=-\omega_0T\beta_4T^{-1}U-ip\beta_1U+\omega\beta_4U=0. \tag{36a}$$

Hence T satisfies the equation

$$T\beta_4=(\omega/\omega_0)\beta_4T-i(p/\omega_0)\beta_1T. \tag{36b}$$

It is natural to assume that T can be expressed in terms of powers of the operator γ_1, this being the generator of translations in the x_1-direction. Now the components of the spin satisfy the characteristic equation of fourth order

$$(\sigma_k-\tfrac{3}{2})(\sigma_k-\tfrac{1}{2})(\sigma_k+\tfrac{1}{2})(\sigma_k+\tfrac{3}{2})=0. \tag{37}$$

Since γ_1 satisfies the same equation, T can be represented by

$$T=1+A\gamma_1+B\gamma_1^2+C\gamma_1^3, \tag{38}$$

where A, B, C are ordinary constants.

A considerable simplification can now be effected by substituting (38) not in (36b) but in the canonically transformed equation

$$STS^{-1}\beta_4=(\omega/\omega_0)\beta_4STS^{-1}-i(p/\omega_0)S\beta_1S^{-1}STS^{-1}, \tag{39}$$

where S satisfies (8), (9), (11). Since $S\gamma_1S^{-1}=S^2\beta_1(S^{-1})^2=-\beta_1$, the awkward terms γ_1^2 and γ_1^3 in T are now simply transformed into β_1^2 and $-\beta_1^3$, (39) thus giving

$$\beta_4-A\beta_1\beta_4+B\beta_1^2\beta_4-C\beta_1^3\beta_4$$
$$=(\omega/\omega_0)(\beta_4-A\beta_4\beta_1+B\beta_4\beta_1^2-C\beta_4\beta_1^3)$$
$$-i(p/\omega_0)(\gamma_1-A\gamma_1\beta_1+B\gamma_1\beta_1^2-C\gamma_1\beta_1^3). \tag{40}$$

Using the direct product representation[6] $\beta=\xi\times\eta$, (40) can be written in the form

$$(a_1\xi_4+b_1\xi_1\xi_4+c_1\xi_4\xi_1)\eta_4$$
$$+(a_2\xi_4+b_2\xi_1\xi_4+c_2\xi_4\xi_1)\eta_1\eta_4=0, \tag{41}$$

where the constants a, b, c, involve the coefficients A, B, C. Now η_4 and $\eta_1\eta_4$ are independent operators, and so are ξ_4, $\xi_1\xi_4$, $\xi_4\xi_1$. Hence $a=b=c=0$, which gives six linear equations for the coefficients A, B, C. The solu-

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[6] Madhavarao, Thiruvenkatachar, and Venkatachaliengar, Proc. Roy. Soc. (London) **187**, 385 (1946).

tion is

$$A=\frac{2}{3}\frac{p}{\omega+\omega_0}\frac{\omega-13\omega_0}{5\omega_0-\omega},\quad B=4\frac{\omega-\omega_0}{5\omega_0-\omega},$$
$$C=-\frac{8}{3}\frac{p}{\omega+\omega_0}\frac{\omega-\omega_0}{5\omega_0-\omega},\tag{42a}$$

providing that

$$\omega^2=\omega_0^2+p^2.\tag{42b}$$

In the general case, for motion in any direction with momentum components p_1, p_2, p_3, the transformation T is obtained by simply replacing in (38) γ_1 by $(p_1\gamma_1+p_2\gamma_2+p_3\gamma_3)/p$, where $p^2=p_1^2+p_2^2+p_3^2$.

The solution U can now easily be constructed from (34), (38), (42). This method has the particular advantage that if, as is often desirable, the solutions are to be eigensolutions of the component of the spin-vector in the direction of p (σ_1 in the above case) it is easier to construct first the corresponding eigensolutions U_0 in the rest system. Since γ_1 commutes with σ_1, the transformed solutions $U=TU_0$ are then still eigensolutions of σ_1.

APPENDIX I

Multiplying Eqs. (4d), (e), (f) by η_4 on the left and right, and using (6a), one obtains, adding respectively (4d), (e), (f),

$$\beta_\lambda(\eta_4\gamma_k\eta_4+\gamma_k)-(\eta_4\gamma_k\eta_4+\gamma_k)\beta_\lambda=0,$$

valid for all four β_λ. The operator $\eta_4\gamma_k\eta_4+\gamma_k$ thus commutes with all the elements of the algebra and must therefore be a multiple of the unit operator, say $a1$. But since $\eta_4=\eta_4^{-1}$, and since, from (5b), spur $\gamma_k=0$, it follows that $a\,\mathrm{spur}1=0$, whence $a=0$.

Similarly, it follows from (4a), (b), (c) that $\eta_4\sigma_k=\sigma_k\eta_4$.

APPENDIX II

In the transformation (8), (9) the role of β_4 differs from that of the three other β_k in that S commutes with β_4 and with the three components $t_{\mu\nu}$ for which both $\mu\neq4$ and $\nu\neq4$, while the three t_{k4} are transformed into β_k. Owing to the relativistic covariance, there is of course no real distinction of β_4, and in fact any other β can play the same role. It is instructive to see how, despite the altered correspondence between Eqs. (4) and (5), the expressions (15) are again obtained if, say, β_3 is chosen to play that role. Thus, in place of (8), (9), consider the transformation characterized by

$$[S,\beta_3]=0,\quad [S,t_{\mu\nu}]=0\quad\text{if both } \mu\neq3 \text{ and } \nu\neq3,$$
$$t_{13}\rightarrow\beta_1,\quad t_{23}\rightarrow\beta_2,\quad t_{43}\rightarrow\beta_4;\tag{A1}$$

i.e.,

$$[S,\beta_3]=[S,\gamma_1]=[S,\gamma_2]=[S,\sigma_3]=0,$$
$$\sigma_2=S\beta_1S^{-1},\quad \sigma_1=-S\beta_2S^{-1},\quad \gamma_3=-S\beta_4S^{-1}.\tag{A2}$$

Now from (4b)

$$\beta_1\sigma_2-\sigma_2\beta_1=i\beta_3,$$

whence, multiplying by S^{-1} on the left and S on the right, and using (A2),

$$\beta_1(S^{-1})^2\sigma_2S^2-(S^{-1})^2\sigma_2S^2\beta_1=-i\beta_3.$$

Hence

$$\beta_1[\sigma_2+(S^{-1})^2\sigma_2S^2]-[\sigma_2+(S^{-1})^2\sigma_2S^2]\beta_1=0.\tag{A3}$$

Similarly, canonical transformation of

$$\beta_1\sigma_1-\sigma_1\beta_1=0$$

gives

$$\beta_2(S^{-1})^2\sigma_2S^2-(S^{-1})^2\sigma_2S^2\beta_2=0.$$

Adding

$$\beta_2\sigma_2-\sigma_2\beta_2=0,$$

one obtains

$$\beta_2[\sigma_2+(S^{-1})^2\sigma_2S^2]-[\sigma_2+(S^{-1})^2\sigma_2S^2]\beta_2=0.\tag{A4}$$

Again, by canonical transformation of

$$\beta_1\gamma_3-\gamma_3\beta_1=0$$

one has

$$\beta_4(S^{-1})^2\sigma_2S^2-(S^{-1})^2\sigma_2S^2\beta_4=0.$$

Adding

$$\beta_4\sigma_2-\sigma_2\beta_4=0,$$

one obtains

$$\beta_4[\sigma_2+(S^{-1})^2\sigma_2S^2]-[\sigma_2+(S^{-1})^2\sigma_2S^2]\beta_4=0.\tag{A5}$$

Finally, by canonical transformation of

$$\beta_3\sigma_2-\sigma_2\beta_3=-i\beta_1$$

one obtains

$$\beta_3[\sigma_2+(S^{-1})^2\sigma_2S^2]-[\sigma_2+(S^{-1})^2\sigma_2S^2]\beta_3=-i[\beta_1+(S^{-1})^2\beta_1S^2].\tag{A6}$$

Hence, assuming that β_1 anticommutes with S^2, one now derives, since $\mathrm{spur}\,\sigma_2=0$,

$$\sigma_2S^2+S^2\sigma_2=0,\tag{A7}$$

which, because of $\sigma_2=S\beta_1S^{-1}$, is consistent with the assumption.

Similarly one finds that, if β_2 and β_4 anticommute with S^2, then

$$\sigma_1S^2+S^2\sigma_1=0,\quad \gamma_3S^2+S^2\gamma_3=0,\tag{A8}$$

in agreement with (A2). It can therefore be deduced that, apart from a constant factor, S^2 is identical with the operator η_3 defined in (6). The expression for S can be obtained from (17) by replacing β_4 by β_3.

It now remains to show that the expressions (15) still hold true. Indeed from

$$i\sigma_3=\sigma_1\sigma_2-\sigma_2\sigma_1$$

one now obtains by the canonical transformation (A2)

$$i\sigma_3=\beta_1\beta_2-\beta_2\beta_1.\tag{A9a}$$

Similarly, from

$$\begin{aligned}
&(b)\quad -i\beta_2=\sigma_1\beta_3-\beta_3\sigma_1,\\
&(c)\quad i\beta_1=\sigma_2\beta_3-\beta_3\sigma_2,\\
&(d)\quad i\gamma_1=\sigma_2\gamma_3-\gamma_3\sigma_2,\\
&(e)\quad i\gamma_2=\gamma_3\sigma_1-\sigma_1\gamma_3,\\
&(f)\quad -i\beta_4=\gamma_3\beta_3-\beta_3\gamma_3,
\end{aligned}\tag{A9}$$

the old expressions (15) for σ_1, σ_2, γ_1, γ_2, γ_3 are obtained, respectively, by the canonical transformation (A2).

APPENDIX III

Consider the transformation

$$\gamma_k=S\beta_kS^{-1},\quad S\beta_4=-\beta_4S.\tag{B1}$$

From Eqs. (4d), (e), (f) one derives by a now familiar procedure

$$[\beta_k,\Gamma_l]=0,\quad [\beta_k,\Gamma_k]=0,$$
$$[\beta_4,\Gamma_k]=-i[\beta_k-(S^{-1})^2\beta_kS^2],\tag{B2}$$

where

$$\Gamma_k\equiv\gamma_k-(S^{-1})^2\gamma_kS^2.$$

A consistent solution can be obtained assuming

$$S^2\beta_k=\beta_kS^2,\quad\text{i.e.,}\quad S^2\gamma_k=\gamma_kS^2.\tag{B3}$$

Multiplying (4f) by S on the left and by S^{-1} on the right, and using (B1), one obtains

$$\gamma_k=-i(\beta_4\beta_k-\beta_k\beta_4).$$

To obtain the expression for σ_m note that from (4a), (b), (c) one now derives by canonical transformation

$$[\gamma_k,\Sigma_k]=0,\quad [\gamma_k,\Sigma_l]=0,\quad [\beta_4,\Sigma_k]=0,\tag{B4}$$

where

$$\Sigma_k \equiv \sigma_k - S\sigma_k S^{-1}.$$

Owing to (4f) this means that the operator Σ_k commutes with the four β_μ; it is therefore a multiple of the unit operator. Since, from (5d), spur$\sigma_k=0$, it follows that

$$S\sigma_k=\sigma_k S. \tag{B5}$$

Hence by canonical transformation of (5c) one obtains

$$i\sigma_m=\beta_k\beta_l-\beta_l\beta_k.$$

The canonical transformation under consideration is characterized by

$$\beta_k\rightarrow\gamma_k,\quad \gamma_k\rightarrow\beta_k,\quad \sigma_k\rightarrow\sigma_k,\quad \beta_4\rightarrow-\beta_4. \tag{B6}$$

In Bhabha's five-dimensional scheme, using (16), this corresponds to the transformation

$$x_k'=x_k,\quad x_4'=x_5,\quad x_5'=x_4,$$

i.e., (B7)

$$a_1{}^1=a_2{}^2=a_3{}^3=1,\quad a_4{}^5=a_5{}^4=1.$$

This describes a rotation through the angle $\pi/2$ in the (x_4, x_5) plane, and reflection of x_5.

Owing to the relativistic covariance it is clear that any other β_k can play the role of β_4. Hence there are four transformations S_μ such that

$$S_\mu\beta_\mu+\beta_\mu S_\mu=0,\quad [S_\mu{}^2, \beta_\lambda]=0. \tag{B8}$$

It is interesting to note that $S_\mu{}^2$ is now a multiple of the unit operator.

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Energy Release in the Disintegration of Be^8

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Thin beryllium targets were bombarded with 400-kev protons and the energy spectra of the particles given off at 90 degrees to the beam direction were observed with a cylindrical electrostatic analyzer. The beryllium was evaporated onto a nickel backing which was thin enough to confine the elastically scattered protons to a narrow energy range. At energies below that of the elastically scattered protons, peaks were observed in the energy spectra which corresponded to the maximum alpha-particle energy in the continuous energy distribution of alpha-particles resulting from the breakup of Be^8. The position of these maxima give a value for the energy release in the disintegration of 77.5 ± 4 kev.

INTRODUCTION

THE nucleus Be^8 occurs as a compound state, or an intermediate product, in a large number of nuclear reactions.[1] In cases where the ground state is involved, there is evidence that alpha-decay occurs.[2] The result of the early work on this problem was the conclusion that the ground state of Be^8 was unstable against alpha-decay by about 125 kev. The conclusion, as to the instability of Be^8, is bolstered by the fact that naturally occurring beryllium contains no detectable amount of mass eight isotope.[3] Recently, a measurement of the half-life for this decay was made by measuring the track lengths of fragments of oxygen nuclei in an emulsion when the emulsion had been exposed to energetic gamma-radiation.[4] Some of these fragments were identified as Be^8 nuclei. The half-life was found to be $(5\pm1)\times10^{-14}$ second. This value corresponds to an energy of the order of 100 kev which is available for decay into two alpha-particles, assuming the latter have zero angular momentum.[5] Two recent direct measurements of the energy release give values of 103 ± 10 kev,[6] and 89 ± 4 kev.[7]

The method used in the present experiment, consisted in bombarding an evaporated beryllium target with protons. This results in the reactions,

$$Be^9+H^1\rightarrow Li^6+He^4+Q_1, \tag{1}$$

$$Be^9+H^1\rightarrow Be^8+He^2+Q_2, \tag{2}$$

$$Be^8\rightarrow He^4+He^4+Q_3, \tag{3}$$

where Q_1, Q_2, and Q_3 refer to the energy releases. Previous work has shown that the alpha-particles from reaction (3) have less energy than the elastically scattered protons at bombardment energies above 240 kev, where sufficient yields for our measurements may be expected.[7,8] In the present work, a backing of nickel foil was used, which was thin enough to confine the elastically scattered protons to a narrow range of energies, thus permitting alpha-particles to be observed without serious interference. An electrostatic analyzer was used to separate particles of different energy-to-charge ratio. In previous work with these reactions,

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