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Hamiltonians without Parametrization*

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Theories which are covariant with respect to transformation groups involving arbitrary functions cannot be of the Cauchy-Kowalewski type. Therefore, it would appear that such theories cannot be cast into a hamiltonian formalism. However, a hamiltonian function can be formed although it is, to a certain extent, arbitrary. This lack of uniqueness is an immediate consequence of the covariance requirements. The adoption of a particular hamiltonian function destroys the covariance of the theory. This situation is similar to the introduction of "cordinate conditions" in the theory of gravitation or the use of the Lorentz gauge condition in electrodynamics.

I. INTRODUCTION

THE development of a covariant field theory in the hamiltonian form has to overcome the difficulty that the usual (lagrangian) field equations do not permit the unique determination of a particular solution by suitable initial conditions on a (space-like) hypersurface, inasmuch as from any one solution formally different ones may be obtained by a coordinate transformation confined to a bounded domain, but with arbitrary generating functions, while on the other hand, in the typical hamiltonian theory the equations are solved with respect to the time derivatives of all the variables, and as a result, the equations are a typical Cauchy-Kowalewski system; if the initial conditions are properly set, the solution is unique. The first successful attempt to solve the problem was made by Rosenfeld[1] although the Lagrangian considered by him has a much more simple transformation law than those to be considered in this paper. Dirac[2] and after him Schild and Pirani[3] have solved this apparent contradiction by retaining in the expression for the hamiltonian some of the velocities, so that some of the usual canonical equations reduce to empty relations. The Syracuse group[4–6] has taken a different approach, by constructing a hamiltonian which, while free of velocities, contains a certain number of arbitrary functions. Choice of these functions in a particular calculation amounts to the adoption of so-called coordinate conditions in the usual formalism.

Originally, the Syracuse method included the adoption of a system of supplementary coordinates, the so-called parameters. Their introduction leads to certain homogeneity properties in the lagrangian and in the momentum densities which are utilized to construct the hamiltonian. In the present paper, a different method will be presented. Parameters are not introduced at all, but instead, by means of a transformation of variables, the time derivatives of the variables are separated into two sets, such that one set of "velocities" can be expressed uniquely in terms of the canonical variables ("coordinates" and "momentum densities"), while the velocities of the other set are completely arbitrary. The corresponding momentum densities then turn out to satisfy certain relationships not involving velocities, and these relationships correspond exactly to the "constraints" of the earlier Syracuse papers and will again be called "constraints." When the customary expression for the hamiltonian is written out, the velocities are then multiplied by expressions which vanish because of the constraints, and thus these terms may be omitted. The constraints then form a set of conditions which must be satisfied on the initial hypersurface, while the canonical equations must be satisfied throughout space-time.

As far as subsequent quantization is concerned, it appears now as if the new formalism will be considerably simpler to work with than the parameter formalism, at least in the absence of particles. Whether results of the two approaches will be equivalent in the presence of matter, only future investigations will show.

In the sections that follow the new method will be developed first for a general covariant theory with a quadratic lagrangian, without reference to a particular type of physical field. The general result will then be applied to the case of Einstein's theory of gravitation, for the case of combined gravitational and electromagnetic fields.

II. THE FIELD VARIABLES AND THE LAGRANGIAN

The exact nature of the field variables will be left unspecified and denoted by y_A ($A=1, \cdots, N$), where N is the number of algebraically independent components. It will be assumed that the field equations can

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be derived from a variational principle of the form

$$\delta I=0, \quad I=\int_V L(y_A, y_{A,\rho})d^4x,$$

where

$$L=\Lambda^{A\rho B\sigma}y_{A,\rho}y_{B,\sigma}, \tag{1}$$

and

$$\Lambda^{A\rho B\sigma}=\Lambda^{A\rho B\sigma}(y_C).$$

The coefficients $\Lambda^{A\rho B\sigma}$ have the following symmetry property:

$$\Lambda^{A\rho B\sigma}=\Lambda^{B\sigma A\rho}. \tag{2}$$

The field equations which result from a variation of the Lagrangian (1) will be denoted by L^A,

$$\begin{aligned} L^A &\equiv \partial L/\partial y_A-(\partial L/\partial y_{A,\rho})_{,\rho}=0,\\ L^A &\equiv (\Lambda^{A\rho B\sigma}+\Lambda^{A\sigma B\rho})y_{B,\sigma\rho}+\cdots. \end{aligned} \tag{3}$$

Regardless of whether the field equations are satisfied or not, L^A will be used for the left-hand side of these equations.

With respect to infinitesimal coordinate transformations, the field variables will be assumed to transform according to the law (I-2.3)

$$\bar{\delta}y_A=F_{A\mu}{}^{B\nu}\xi^\mu{}_{,\nu}y_B-y_{A,\mu}\xi^\mu. \tag{4}$$

The four functions ξ^μ represent the infinitesimal changes of the coordinate values of a fixed world point. The $F_{A\mu}{}^{B\nu}$ are numbers which depend on the type of field variables which represent the field, independent of the coordinate system used and of the coordinate values themselves. Lastly, the $\bar{\delta}y_A$ are the changes produced in the field variables y_A as a function of their arguments as a result of the infinitesimal coordinate transformation.

To assure covariant field equations from the variational principle, it is required that (I-2.5)

$$\bar{\delta}L=Q^\mu{}_{,\mu}. \tag{5}$$

As a consequence of this requirement and the assumed transformation law for the y_A, one obtains four identities (I-3.3)

$$(F_{A\mu}{}^{B\nu}y_BL^A)_{,\nu}+y_{A,\mu}L^A\equiv 0. \tag{6}$$

By substituting L^A, Eq. (3), into the above identities, the terms containing second derivatives will lead to third derivatives, and these must cancel each other. It follows that the coefficients $\Lambda^{A\rho B\sigma}$ must satisfy the identities (I-3.6)

$$(F_{A\mu}{}^{B\nu}L^{AC\rho\sigma}+F_{A\mu}{}^{B\sigma}L^{AC\nu\rho}+F_{A\mu}{}^{B\rho}L^{AC\sigma\nu})y_B\equiv 0, \tag{7}$$

where

$$L^{AC\rho\sigma}=\Lambda^{A\rho C\sigma}+\Lambda^{A\sigma C\rho}.$$

In what follows, $4N$ of these identities are of special interest, those in which ρ, σ, and ν all equal to 4, (I-3.7)

$$F_{A\mu}{}^{B4}y_BL^{AC44}=2F_{A\mu}{}^{B4}y_B\Lambda^{A4C4}\equiv 0. \tag{8}$$

Also, from Eq. (1),

$$\Lambda^{A4C4}=\tfrac{1}{2}\partial^2L/\partial\dot{y}_A\partial\dot{y}_B, \tag{9}$$

where

$$\dot{y}_A=y_{A,4}.$$

Consequently, the matrix Λ^{A4C4} is a singular matrix because it has four nonvanishing, linearly independent null vectors, $F_{A\mu}{}^{B4}y_B$. In other words, its determinant must vanish. Now Λ^{A4B4} is just the matrix of the coefficients of these second "time" derivatives in the field equations. Therefore, the Euler-Lagrange equations cannot be solved with respect to all the second "time" derivatives. The requirement of covariant field equations forces the matrix Λ^{A4C4} to be singular; that is, its determinant is zero. Since this matrix has four linearly independent null vectors, its rank will in general be $N-4$ unless the lagrangian has other special invariance properties, *viz.*, the guage invariance of electromagnetism. In such cases, its rank will be even lower. Each arbitrary function involved in the transformation group with respect to which the field equations are covariant reduces the rank of the singular matrix Λ^{A4C4} by one.

III. THE MOMENTUM DENSITY AND THE CONSTRAINTS

The momentum densities are introduced by the usual definition

$$\pi^A=\partial L/\partial\dot{y}_A. \tag{10}$$

From Eq. (1), it follows that

$$\pi^A=2\Lambda^{A4B\sigma}y_{B,\sigma}.$$

However, the momentum densities thus defined will not be algebraically independent of each other because, from Eqs. (8) and (9), the functional determinant or jacobian between the "new" and "old" variables, in this case between π^A and $\dot{y}_A$, vanishes.

$$J=|\partial\pi^A/\partial\dot{y}_B|=|\partial^2L/\partial\dot{y}_A\partial\dot{y}_B|\equiv 0. \tag{11}$$

Therefore, it would appear impossible that one could solve for the "velocities," $\dot{y}_A$, in terms of the momentum densities, π^A, in the usual manner. Since the momentum densities are not algebraically independent, they must satisfy a number of algebraic relationships which are free of $\dot{y}_A$. As is apparent from above, these relationships or "constraints" are a consequence of our covariance requirement. If the theory contains any other invariance properties in addition to the coordinate invariance, there will be other constraints. The number of constraints will always be equal to the number of arbitrary functions involved in the transformation group with respect to which the field equations are covariant. In the case of coordinate covariance, there are four arbitrary functions in the transformation group, and consequently, there are four constraints among the momenta π^A. In the case of electromagnetism, the gauge group involves one arbitrary function, and one additional constraint arises from this special invariance. In Eq. (10), we can separate derivatives with respect to x^4 from the other derivatives, and get[7]

$$\pi^A = 2\Lambda^{A4Bn} y_{B,n} + 2\Lambda^{A4B4}\dot{y}_B. \tag{12}$$

We have already shown that the matrix Λ^{A4B4} is a singular matrix, with at least four independent null vectors. Since $F_{A\mu}{}^{B4}y_B$ are four null vectors of the matrix Λ^{A4B4}, $(\mu=1, \cdots, 4)$, multiplication of Eq. (12) by $F_{A\mu}{}^{B4}y_B$ yield four relationships which are free of "velocities" and which have the form,

$$F_{A\mu}{}^{B4}y_B(\pi^A - 2\Lambda^{A4Cn}y_{C,n}) = 0. \tag{13}$$

These four constraints we shall call the *coordinate constraints*. As was mentioned before, the existence of constraints is a direct consequence of covariance requirements, since the matrix, Λ^{A4B4}, is singular for this reason. In what follows it will be assumed that there are w arbitrary and consequently w null vectors to the singular matrix Λ^{A4B4} and w constraints, $(w \geqslant 4)$. The null vectors form a linear manifold. That is, any linear combination of null vectors is again a null vector, and any null vector multiplied by a constant is again a null vector.

IV. THE TRANSFORMATION AND THE HAMILTONIAN

We shall now proceed to construct hamiltonians for covariant field theories whose lagrangian density is homogeneous, quadratic function of the first derivatives of the field variables. The form of the hamiltonian for such lagrangians will be obtained explicitly. The method makes use of the fact that the theory can be made invariant under transformations in the $\dot{y}_A$ space which bring the singular matrix Λ^{A4B4} to a form in which its last w rows and columns are filled with zeros. Such a matrix will be termed a "bordered" matrix. The $N-w$ sub-matrix which results will be a nonsingular matrix having a definite inverse. Then, it will be possible to solve for some of the transformed velocities in terms of the transformed momenta. If the hamiltonian is then formed by the usual means, the "velocities" which could not be expressed in terms of momenta can be eliminated by the constraint conditions. The hamiltonian so formed will then depend intimately on the constraints and cannot be freed from the constraints without re-introducing the velocities.

The transformation matrices, which accomplish the bordering of the matrix Λ^{A4B4}, are to a large extent arbitrary. The last w rows of the transformation matrix (D) are filled with the w independent null vectors, while the first $N-w$ rows are arbitrary except that they must be linearly independent of the null vectors in order that the transformation matrix D should be nonsingular.

[7] In this paper Greek indexes run from one to four and Latin indexes from one to three.

The algebraic constrains between the canonical variables must hold for any combination of the field variables consistent with the expression (10) for the momentum densities. Therefore, the $\dot{y}_A$ will be considered as the coordinates of a symbolical "vector space" and the functions, π^A, are then specific first-degree homogeneous functions of the coordinates in that vector space. In any transformation of the y_A into new y_A with nonvanishing jacobian, the "coordinates" of our vector space (the $\dot{y}_A$) will undergo a linear transformation, and the momentum densities will transform contragradiently to them. The $\dot{y}_A$ will be referred to as "coordinates," and all quantities with the same transformation law will be called "contravariant vectors." The π^A form a "covariant vector," the matrix Λ^{A4B4} a "covariant symmetric tensor," and the lagrangian is a "scalar." Therefore, the hamiltonian as it is usually defined,

$$H = -L + \dot{y}_A\pi^A, \quad A = 1, \cdots, N, \tag{14}$$

is also a "scalar." If one considers the non-singular transformation matrix $D^{-1A}{}_B$, then the coordinates $\dot{y}_A$ transform according to the law

$$\dot{y}_B' = \dot{y}_A D^{-1A}{}_B, \tag{15}$$

and other important quantities according to the laws

$$\begin{gathered}(\Lambda^{A4B4})' = \Lambda^{C4D4}D^A{}_C D^B{}_D,\\ (\Lambda^{AnB4}y_{A,n})' = (\Lambda^{AnC4}y_{A,n})D^B{}_C,\\ \pi'^A = D^A{}_B\pi^B, \quad V'_{AS} = V_{BS}D^{-1B}{}_A,\\ S = 1, \cdots, w.\end{gathered} \tag{16}$$

V_{AS} represents the null vectors of Λ^{A4B4}. This transformation procedure is similar to that carried out in III.

As indicated previously, the D matrix must be chosen in such a way that the new matrix, Λ'^{A4B4} of Eq. (16), becomes a matrix in which the last w rows and columns are filled with zeros. That is, the matrix Λ'^{A4B4} must be "bordered" with zeros. The matrix will have this structure if the last w rows of the D matrix are chosen as the w null vectors of Λ'^{A4B4} and if the first $N-w$ rows are any arbitrary set of vectors, not containing velocities, which are linearly independent of the null vectors. A D matrix so determined will be a non-singular matrix which has a unique inverse D^{-1}. The D and D^{-1} matrices have the following structure:

$$D = \begin{array}{c} \\ \mathrm{N}-w \\ \\ w \end{array}\!\!\begin{array}{c} \mathrm{N} \\ \left[\begin{array}{c} D^{A^*}{}_A \\ \hline D^S{}_A \end{array}\right] \end{array}, \quad \begin{array}{l} A = 1, \cdots, N \\ A^* = 1, \cdots, N-w \\ S = (N-w+1), \cdots, N \end{array}$$

$$D^{-1} = \mathrm{N}\begin{array}{cc} \mathrm{N}-w & w \\ \left[\begin{array}{c|c} D^{-1A}{}_{A^*} & D^{-1A}{}_S \end{array}\right. & \left.\right] \end{array}. \tag{17}$$

If the only invariance properties of the theory are coordinate invariance, the rectangular matrix $D^S{}_A$ is given by

$$D^S{}_A = D_{\mu A} = F_{A\mu}{}^{B4} y_B. \tag{18}$$

From Eq. (16),

$$\Lambda'^{S4B4} = \Lambda^{C4D4} D^S{}_C D^B{}_D = 0 \tag{19}$$

because $D^S{}_C$ is a rectangular array of null vectors of Λ^{C4D4}. Moreover, the matrix

$$\Lambda'^{A^*4B^*4} = \Lambda^{C4D4} D^{A^*}{}_C D^{B^*}{}_D, \tag{20}$$

is a nonsingular matrix because $D^{A^*}{}_C$ is a rectangular array of vectors which are linearly independent of the null vectors. However, $\Lambda'^{A^*4B^*4}$ is not determined uniquely until the rectangular matrix $D^{A^*}{}_C$ has been chosen. If the inverse of $\Lambda'^{A^*4B^*4}$ is denoted by $G_{A^*4B^*4}$, then

$$\Lambda'^{A^*4B^*4} G_{B^*4C^*4} = \delta^{A^*}{}_{C^*}. \tag{21}$$

Therefore, the Λ'^{A4B4} matrix has the form

$$\Lambda'^{A4B4} = \begin{array}{c} \\ N-w \\ w \end{array} \begin{array}{c} \begin{array}{cc} N-w & \quad w \end{array} \\ \left[\begin{array}{c|c} \Lambda'^{A^*4B^*4} & 0 \\ \hline 0 & 0 \end{array}\right] \end{array}, \tag{22}$$

and the new lagrangian density has the form,

$$L = L' = (\Lambda^{AnBm} y_{A,n} y_{B,m})' + 2(\Lambda^{AnB4} y_{A,n})' \dot{y}'_B + \Lambda'^{A4B4} \dot{y}'_A \dot{y}'_B. \tag{23}$$

The new momentum densities, calculated from either the new lagrangian density or from the transformation Eq. (16), are

$$\pi'^A = 2(\Lambda^{A4Bn} y_{B,n})' + 2\Lambda'^{A4B4} \dot{y}'_B. \tag{24}$$

Because of the special form of Λ'^{A4B4},

$$\Lambda'^{A4B4} \dot{y}'_B = \Lambda'^{A4B^*4} \dot{y}'_{B^*}. \tag{25}$$

If the components of π'^A in the null space are separated from the other components,

$$\pi'^{A^*} = 2(\Lambda^{A^*4Bn} y_{B,n})' + 2\Lambda'^{A^*4B^*4} \dot{y}'_{B^*}, \tag{26}$$

and,

$$\pi'^S = 2(\Lambda^{S4Bn} y_{B,n})'. \tag{27}$$

Equation (27) is the expression for the constraints in the new coordinate system.

Equation (26) can be solved for the $\dot{y}'_{B^*}$ by multiplication by $G_{A^*4B^*4}$ to obtain

$$\dot{y}'_{A^*} = G_{A^*4B^*4}(\tfrac{1}{2}\pi'^{B^*} - (\Lambda^{B^*4Cn} y_{C,n})'), \tag{28}$$

by using condition (21). The matrix $G_{A^*4B^*4}$ is arbitrary until the form of the rectangular matrix $D^{A^*}{}_C$ has been chosen.

It will be shown, by properly using the constraints [Eq. (27)], that a hamiltonian density can be found in the new space by the usual methods. However, the hamiltonian density so determined is not unique, and, in fact, an infinity of hamiltonians can be found. The hamiltonian density in the original space can then be found by use of the transformation equations for the momentum densities and "tensors." The fact that the hamiltonian is not uniquely determined is not surprising because, the covariant Euler-Lagrange equations cannot be solved with respect to the highest "time" derivatives. Therefore, the continuation of the solutions in "time" is not unique. However, the canonical equations, by their very nature, are solved with respect to the highest "time" derivatives. Yet the hamiltonian formalism is equivalent to that of Lagrange. The apparent contradiction finds its explanation in the fact that the hamiltonian is not uniquely determined. Therefore, the continuation in "time" would not be unique with given initial conditions, and the apparent contradiction is thereby resolved.

The hamiltonian density is usually defined as,

$$H = -L + \dot{y}_A \pi^A. \tag{29}$$

Since the hamiltonian density is a scalar under the "coordinate" transformation described above in the $\dot{y}_A$ vector space,

$$H = H' = -L' + \dot{y}'_A \pi'^A. \tag{30}$$

If the expression for L' [Eq. (23)] is substituted in the above expression for H', one obtains, after separating derivatives with respect to x^4 from the other derivatives and using the fact that Λ'^{A4B4} is a "bordered" matrix,

$$H' = -(\Lambda^{AnBm} y_{A,n} y_{B,m})' - \Lambda'^{A^*4B^*4} \dot{y}'_{A^*} \dot{y}'_{B^*} + (\pi'^{A^*} - 2(\Lambda^{A^*4Bm} y_{B,m})') \dot{y}'_{A^*} + (\pi'^S - 2(\Lambda^{S4Bm} y_{B,m})') \dot{y}'_S, \tag{31}$$

where A^*, A, and S have the same meaning as before. The coefficient of the $\dot{y}'_S$ term vanishes because of the constraints [Eq. (27)]. The fact that the "velocities" $\dot{y}'_S$ drop our of the hamiltonian is very fortunate since we cannot express them in terms of the canonical variables π^A, y_A, and $y_{A,n}$. The "velocities" $\dot{y}'_{A^*}$, on the other hand, can be expressed in terms of the canonical variables [Eq. (28)], and thus we can form a hamiltonian altogether free of "velocities." However, this hamiltonian depends on the constraints in such a manner that it cannot exist unless the constraints are satisfied. From the expression for $\dot{y}'_{A^*}$ [Eq. (28)], the hamiltonian becomes

$$H' = -(\Lambda^{AnBm} y_{A,n} y_{B,m})' + G_{A^*4B^*4}(\tfrac{1}{2}\pi'^{A^*} - (\Lambda^{A^*4Am} y_{A,m})') \times (\tfrac{1}{2}\pi'^{B^*} - (\Lambda^{B^*4Bn} y_{B,n})') \tag{32}$$

after using Eq. (21). However, this expression is not the only hamiltonian which can be obtained. We can add to the above hamiltonian any algebraic combination of the constraints [Eq. (27)] with unknown coefficients which are free of "velocities." Therefore, the hamiltonian can be written in the form

$$H'=H'_1+K_S(y'_A, y'_{A,n}, \pi'^{A^*})(\pi'^S-2(\Lambda^{S4Bm}y_{B,m})'), \quad (33)$$

where H'_1 is the hamiltonian of Eq. (32).

If one expresses the primed quantities in the hamiltonian in terms of the unprimed quantities by means of Eqs. (16), the hamiltonian

$$H=-\Lambda^{AnBm}y_{A,n}y_{B,m}+G_{A^*4B^*4}D^{A^*}{}_A D^{B^*}{}_B \times(\tfrac{1}{2}\pi^A-\Lambda^{A4Cn}y_{C,n})(\tfrac{1}{2}\pi^B-\Lambda^{B4Dm}y_{D,m}), \quad (34)$$

which results contains the original canonical variables π^A, y_A, and $y_{A,n}$. The choice of the rectangular matrix $D^{A^*}{}_A$ has no effect on the final form of H except to permit the addition of linear and quadratic combinations of the coordinate constraints (13). Therefore, the hamiltonian (34) possesses exactly the degree of arbitrariness required by the general theory.

In III a similar hamiltonian to the expression (34) was obtained which we shall call the p-hamiltonian. We can make the p-hamiltonian identical with the expression (34). First we must set the arbitrary function v^ρ in III equal to $\delta_4{}^\rho$; then, remembering that the integral over all space of the momentum density λ_4 (III-7.5) is a constant of the motion,—the "time" derivatives of λ_4 can be expressed as a surface integral which vanishes when the surface boundaries are at infinity—we shall drop the term containing λ-quantities. The remaining terms in (III-7.5) are then identical with the expression (34).

The canonical field equations take the form

$$\dot{y}_A=\partial H/\partial\pi^A-(\partial H/\partial\pi^A{}_{,n})_{,n}\equiv\delta H/\delta\pi^A,$$

and,

$$\dot{\pi}^A=-\partial H/\partial y_A+(\partial H/\partial y_{A,n})_{,n}\equiv-\delta H/\delta y_A. \quad (35)$$

The time derivative of any functional of the canonical variables

$$\mathfrak{F}=\int_V F(y_A, y_{A,n}, \pi^B, \pi^B{}_{,n})d^3x,$$

can be given the form,

$$\dot{\mathfrak{F}}=\int_V\left(\frac{\partial F}{\partial y_A}\frac{\delta H}{\delta\pi^A}+\frac{\partial F}{\partial y_{A,n}}\left(\frac{\delta H}{\delta\pi^A}\right)_{,n} -\frac{\partial F}{\partial\pi^A}\frac{\delta H}{\delta y_A}-\frac{\partial F}{\partial\pi^A{}_{,n}}\left(\frac{\delta H}{\delta y_A}\right)_{,n}\right)d^3x.$$

If a total divergence term is separated from this expression, this term makes no contribution since it can be converted by Gauss theorem into a surface integral which vanishes when surface integral which vanishes when surface boundaries are at infinity. The remaining terms have the form

$$\dot{\mathfrak{F}}=\int_V(F, H)d^3x,$$

where (F, H) is the "Poisson bracket" between F and H and stands for

$$(F, H)=\frac{\delta F}{\delta y_A}\frac{\delta H}{\delta\pi^A}-\frac{\delta F}{\delta\pi^A}\frac{\delta H}{\delta y_A}.$$

This "Poisson bracket" is very useful when passage to the quantum theory is desired.

V. THE LAGRANGIAN OF RELATIVITY WITH ELECTROMAGNETIC FIELD

The lagrangian density of the general theory of relativity has the following structure if the electromagnetic field is included

$$L=L_{\mathrm{grav}}+L_{\mathrm{el}},$$

where L_{grav} and L_{el} have the same form as in III.

The appropriate values for the constant coefficient $F^{B\nu}{}_{A\mu}$, if one considers as the basic field variables the $g_{\mu\nu}$ and ϕ_ν, are given in III. The $g_{\mu\nu}$ are the symmetric components of the covariant metric tensor and the ϕ_μ are the four electromagnetic potentials. This lagrangian density contains the field variables and their first derivatives only. Moreover, the lagrangian is also homogeneous quadratic in the first derivatives of the field variables. Therefore, the theory of the earlier parts of this paper applies to the case of gravitation with electromagnetic field.

The lagrangian density can be rewritten to demonstrate this quadratic structure. Renaming dummy indexes and factoring we get,

$$L=\Lambda^{(\alpha\beta)\rho(\gamma\delta)\sigma}g_{\alpha\beta,\rho}g_{\gamma\delta,\sigma}+Y^{\mu\rho\nu\sigma}\phi_{\mu,\rho}\phi_{\nu,\sigma}, \quad (36)$$

where $\Lambda^{(\alpha\beta)\rho(\gamma\delta)\sigma}$ and $Y^{\mu\rho\nu\sigma}$ with their symmetry properties are given in III. From Eq. (8) it can be shown that

$$\Lambda^{\beta\nu\rho\omega\tau\sigma}+\Lambda^{\beta\nu\sigma\omega\tau\rho}+\Lambda^{\beta\sigma\nu\omega\tau\rho}+\Lambda^{\beta\sigma\rho\omega\tau\nu} +\Lambda^{\beta\rho\sigma\omega\tau\nu}+\Lambda^{\beta\rho\nu\omega\tau\sigma}\equiv 0. \quad (37)$$

If $\rho=\sigma=\nu=4$ in Eq. (37), one sees that

$$\Lambda^{4\beta4\gamma\delta4}=\Lambda^{\gamma\delta44\beta4}\equiv 0, \quad (38)$$

and

$$Y^{\mu444}=Y^{44\mu4}\equiv 0, \quad (39)$$

hold. Therefore, the singular matrices $\Lambda^{\alpha\beta4\gamma\delta4}$ and $Y^{\mu4\nu4}$ have the important property of being "bordered"; that is, they have their zeros in the last rows and columns. Consequently, it is not necessary to make any transformations to bring them into this form. The submatrices Λ^{ab4mn4} and Y^{r4s4} are nonsingular matrices and have unique inverses.

From Eq. (9) the null vectors of this matrix are

$$-(g_{\mu\alpha}\delta_\beta{}^4+g_{\mu\beta}\delta_\alpha{}^4)-\phi_\mu\delta_\alpha{}^4, \qquad \begin{matrix}\mu=1,\cdots,4\\ \alpha,\beta=1,\cdots,4\end{matrix} \tag{40}$$

when substitution for $F_{(\alpha\beta)\mu}{}^{(\gamma\delta)4}$ and $F_{\alpha\mu}{}^{\gamma4}$ is made. Since the null vectors form a linear manifold, linear combinations of these null vectors are again null vectors. Therefore, multiplication of Eq. (40) by $g^{\mu\gamma}$ will give null vectors

$$-(\delta_\alpha{}^\gamma\delta_\beta{}^4+\delta_\beta{}^\gamma\delta_\alpha{}^4)-\phi^\gamma\delta_\alpha{}^4, \qquad \begin{matrix}\gamma=1,\cdots,4\\ \alpha,\beta=1,\cdots,4\end{matrix} \tag{41}$$

where γ indicates the null vector in question and $(\alpha\beta)$ and α indicate the components of the null vector. Written out they become,

$$\begin{matrix}(0\;0\;0\;0\;0\;0, & -1 & 0 & 0 & 0, & 0\;0\;0\;-\phi^1)\\ (0\;0\;0\;0\;0\;0, & 0 & -1 & 0 & 0, & 0\;0\;0\;-\phi^2)\\ (0\;0\;0\;0\;0\;0, & 0 & 0 & -1 & 0, & 0\;0\;0\;-\phi^3)\\ (0\;0\;0\;0\;0\;0, & 0 & 0 & 0 & -1, & 0\;0\;0\;-\phi^4)\end{matrix} \tag{42}$$

A fifth null vector of Λ^{A4B4} independent of those above is,

$$(0\;0\;0\;0\;0\;0,\;0\;0\;0\;0,\;0\;0\;0\;x), \tag{43}$$

where x is any nonvanishing quantity.

Because of the special structure of the Λ^{A4B4} matrix in the case of gravitation with electromagnetic field, the whole development from this point on becomes identical with the primed vector spaced introduced in the general theory of Sec. IV.

The inverses to the nonsingular matrices Λ^{ab4mn4} and Y^{r4s4} must be calculated. The inverse matrices are defined by the equations

$$G_{kl4ab4}\Lambda^{ab4mn4}=\tfrac{1}{2}(\delta_k{}^m\delta_l{}^n+\delta_l{}^m\delta_k{}^n), \tag{44}$$

and,

$$G_{k4a4}Y^{a4n4}=\delta_k{}^n.$$

These inverses were found by the method of "building blocks" similar to that in III. They are

$$G_{kl4mn4}=[2/(-g)^{\frac{1}{2}}g^{44}](g_{km}g_{ln}+g_{kn}g_{lm}-g_{kl}g_{mn}), \tag{45}$$

and

$$G_{k4n4}=-[2/(-g)^{\frac{1}{2}}g^{44}]g_{kn}.$$

VI. THE MOMENTUM DENSITIES AND CONSTRAINTS

The momentum densities are defined by the usual definition which, in this case, become

$$\pi^{\alpha\beta}=\partial L/\partial\dot{g}_{\alpha\beta} \quad\text{and}\quad \psi^\mu=\partial L/\partial\phi_\mu,$$

where

$$\pi^{\alpha\beta}=\text{gravitational momentum densities}$$

and

$$\psi^\mu=\text{electromagnetic momentum densities.}$$

If the lagrangian is supplied with Eq. (36), we find

$$\pi^{\alpha\beta}=2\Lambda^{\alpha\beta4\gamma\delta n}g_{\gamma\delta,n}+2\Lambda^{\alpha\beta4\gamma\delta4}\dot{g}_{\gamma\delta}, \tag{46}$$

and,

$$\psi^\mu=2Y^{\mu4\nu n}\phi_{\nu,n}+2Y^{\mu4\nu4}\dot{\phi}_\nu.$$

Because of the special structure of the matrices $\Lambda^{\alpha\beta4\gamma\delta4}$ and $Y^{\mu4\nu4}$ the constraints can be written down at once. They are the coordinate constraints

$$\pi^{4\beta}=2\Lambda^{4\beta4\gamma\delta n}g_{\gamma\delta,n}, \tag{47}$$

and the gauge constraint,

$$\psi^4=0. \tag{48}$$

Substitution for $\Lambda^{4\beta4\gamma\delta n}$ gives,

$$\pi^{4\beta}=\tfrac{1}{4}(-g)^{\frac{1}{2}}[2g^{4\gamma}(g^{4\beta}g^{\delta n}-g^{\beta n}g^{4\delta})+g^{\gamma\delta}(g^{44}g^{\beta n}-g^{4n}g^{4\beta})]g_{\gamma\delta,n}. \tag{49}$$

As for the remaining momentum densities, we have,

$$\pi^{ab}=2\Lambda^{ab4\gamma\delta n}g_{\gamma\delta,n}+2\Lambda^{ab4mn4}\dot{g}_{mn}, \tag{50}$$

and,

$$\psi^m=2Y^{m4\nu n}\phi_{\nu,n}+2Y^{m4n4}\dot{\phi}_n. \tag{51}$$

If the $g^{\mu\nu}$ are replaced by the $n^{\mu\nu}$, the electric momenta reduce to the expressions usually given for these momenta.[8]

Multiplication of Eqs. (50) and (51) by G_{ab4mn4} and G_{n4m4}, respectively, gives

$$\dot{g}_{mn}=G_{mn4ab4}(\tfrac{1}{2}\pi^{ab}-\Lambda^{ab4\gamma\delta n}g_{\gamma\delta,n}), \tag{52}$$

and

$$\dot{\phi}_n=G_{n4m4}(\tfrac{1}{2}\psi^m-Y^{m4\nu r}\phi_{\nu,r}). \tag{53}$$

VII. THE HAMILTONIAN DENSITY OF RELATIVITY WITH ELECTROMAGNETIC FIELD

The hamiltonian is formed according to the expression (41) which was derived in Sec. IV. If the gravitational part is separated from the electric part,

$$\begin{aligned}H'=H=&-\Lambda^{\alpha\beta n\gamma\delta m}g_{\alpha\beta,n}g_{\gamma\delta,m}\\ &+G_{ab4kl4}(\tfrac{1}{2}\pi^{ab}-\Lambda^{ab4\gamma\delta m}g_{\gamma\delta,m})(\tfrac{1}{2}\pi^{kl}-\Lambda^{kl4\omega\tau n}g_{\omega\tau,n})\\ &-Y^{\mu n\nu m}\phi_{\mu,n}\phi_{\nu,m}+G_{k4l4}(\tfrac{1}{2}\psi^k-Y^{k4\nu m}\phi_{\nu,m})\\ &\times(\tfrac{1}{2}\psi^l-Y^{l4\mu n}\phi_{\mu,n}).\end{aligned} \tag{54}$$

This expression may appear unduly lengthy. But the reader should bear in mind that the ordinary concise expressions for such quantities as the Riemann-Christoffel curvature tensor in the ordinary theory of relativity appear brief only because of the adoption of abbreviating notations—the Christoffel symbols and Einstein's summation convention. We have not used here Christoffel symbols, and in indicating sums, we had in many cases separated the index 4 from the spatial indices. Thus, actually, the present expression

[8] See for instance, L. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), p. 404, Eq. (48.4).

is probably not longer than those current in general relativity.

In the case of special relativity ($g_{\mu\nu}=n_{\mu\nu}$) this hamiltonian reduces to the hamiltonian usually given for the electromagnetic field.

VIII. CONCLUSION

A hamiltonian formalism in nonrelativistic dynamics determines uniquely the state of a dynamical system for all instants $-\infty \leqslant t \leqslant \infty$ if the values of the momenta and coordinates are given at one instant t_0. An "instant" is here defined according to Dirac[1] to be a space-like hypersurface at a given time t. A hamiltonian formalism which satisfies the principle of general covariance with respect to arbitrary coordinate transformations must still show how the dynamical variables vary from instant to instant. However, in a covariant theory the instant can be varied arbitrarily in four different ways, and the hamilton equations of motion must always apply. Therefore, the field variables cannot be determined uniquely as functions of the independent variables, and the arbitrariness in their determination is exactly given by the four arbitrary coefficients of the constraints (13). This arbitrariness is then incorporated in the arbitrariness of the rectangular matrices $D^{A*}{}_{A}$. If the hamiltonian formalism is covariant with respect to other transformations like the guage transformation of electromagnetism, other arbitrary functions will occur in the hamiltonian to allow for the freedom of adding an arbitrary gradient to the field variables after their initial values have been fixed. This arbitrary function, in the case of electromagnetism, will be the arbitrary coefficient of the gauge constraint (46). If the arbitrary coefficient of all the constraints are chosen, this is equivalent to the adoption of particular gauge and coordinate conditions.

These comments apply equally to the parameter-hamiltonian developed in III and the formalism presented in this paper. The chief difference between the two methods of constructing a hamiltonian lies in the use of "parameters" in III. The parameters are an auxiliary coordinate system which is introduced alongside the usual coordinates x^ρ. The purpose of the parameters is twofold. Prior to quantization, the parameters enable us to prescribe the motion of any singularities at will in terms of the parameters and, thus, to restrict the domain of integration of the lagrangian density to regions free of singularities. In the quantized theory, then, the coordinates, including the coordinates of any singularities of the field (particles) appear as q-numbers, while the parameters remain c-numbers. In this formalism, on the other hand, there is only one system of coordinates used, and we succeed in constructing the hamiltonian function without recourse to the homogeneity properties characteristic of the parameter formalism. In the unquantized theory and in the absence of singularities, the two systems of equations must lead to identical results; the treatment of singularities in this formalism must await subsequent investigations. In the quantized theory, we cannot expect the coordinates to appear as q-numbers, they will remain in the c-number-field on which the field variables are defined.

While the parameter formalism may possibly afford a more direct access to the operators representing the dynamical variables of particles, the present formalism is simpler in every other respect. It should, therefore, facilitate the comparison of the covariant formalism with other theories (such às quantum electrodynamics) which lack full covariance, but which have been investigated more fully in the past. Thus, for many purposes the present formalism constitutes a bridge between the covariant and more conventional theories, and we expect that the two formalisms now available will supplement each other in future investigations.

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