CONCLUSIONS

The decay of Se^{73} is complex. With the help of the Fermi theory the positron distribution has been resolved into four groups with end-point energies and relative intensities as given in Table I. Four gamma-rays have been observed. Their energies and relative intensities are given in Table I. The 67.1-kev gamma-ray has been found to be a magnetic octipole transition in the parent selenium. The 361-kev gamma-ray follows the 1.318-Mev positron group and is magnetic quadrupole radiation. The other gamma-rays and positron groups have energies and intensities compatible with the decay scheme proposed in Fig. 7.

Estimates of K-capture—positron branching ratios have been made for several of the positron transitions. For the intense 1.318-Mev positron group this ratio is 0.45 as compared to the theoretical value of 0.42. The measured ratio for the weak 0.750-Mev positron group is 1.6 and the theoretical value is 2.5, and for the very weak 0.250-Mev positron transition the measured K-capture—positron ratio is estimated as 6 and the corresponding theoretical value is 100. From the relative intensity of the K-auger electrons, a value of 0.59 was obtained from the gross ratio of K-capture to positrons.

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A Canonical Field Theory with Spinors*

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In this paper, we have constructed a general field theory in covariant form which incorporates the usual covariant generalization of Dirac matrices. The field equations are derived from a lagrangian that is a second-order differential covariant (a scalar density of weight 1) constructed in the covariant spinor formalism by the same method that in riemannian geometry leads to the curvature tensor. It is possible to show that, in spite of the apparently greater wealth of geometrical elements, this theory is completely equivalent to the general theory of relativity. The field equations satisfy the usual differential identities and, in addition, "spin" identities'; there are four "strong" conservation laws which can be used to obtain equations of motion for singularities. Since we do not known at present whether the equivalence with the theory of relativity may not be lost in the process of quantization, we consider eventual quantization desirable and have, in this paper, converted the theory into the canonical form.

INTRODUCTION

IN a series of papers, Bergmann and co-workers[1–3] developed the theory of canonization of covariant field theories, in the hope that some of the difficulties in quantum field physics might be overcome by the adoption of the "best" classical (nonquantized) field theory and its subsequent quantization.

The question now arises which theory is to be considered the "best" one. We could consider Einstein's theory of gravitation (with the electromagnetic field included). In that theory the laws of physics are generally covariant, i.e., unchanged by all types of coordinate transformations for which the jacobian of the transformation is non-zero. There are other possibilities, for example, the recent theory of Einstein,[4] in which he attempts to unify the gravitational and elecmagnetic fields. This theory also assumes the basic laws of physics to be generally covariant.

There are, however, indications that the basic laws of physics contain spinors as well as tensors. In the Dirac theory of the electron, anticommuting quantities arise with transformation laws different from those of tensors. The success of Dirac's theory of the electron and, among others, the ample evidence of atomic spectra, furnish strong indications that the basic laws of physics contain spinors.

In this paper we shall develop a classical field theory which is generally covariant and contains spinors. We

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[1] P. G. Bergmann, Phys. Rev. **75**, 680 (1949).

[2] P. G. Bergmann and J. H. M. Brunings, Revs. Modern Phys. **21**, 480 (1949).

[3] Bergmann, Penfield, Schiller, and Zatzkis, Phys. Rev. **80**, 81 (1950).

[4] A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, New Jersey, 1950), third edition.

shall follow the spin algebra and analysis developed by Schrödinger,[5] Bargmann,[6] van der Waerden and Infeld,[7] Schouten,[8] Pauli,[9] and others, and then develop a lagrangian from the spin quantities. This lagrangian is constructed from a spin tensor discovered by Schrödinger.[5] Because of its relation to the Riemann-Christoffel curvature tensor, the spin lagrangian and the lagrangian of the general theory of relativity turn out to be proportional. Even though the spin vectors satisfy algebraic conditions which must be introduced into the lagrangian with undetermined multipliers, the field equations in spin form are equivalent to the field equations of the general theory of relativity. The identities due to the coordinate covariance are the same as those obtained from the general theory of relativity (the Bianchi identities). Besides, in the spin theory there is another set of identities which reflects the spin covariance. Finally, as a preliminary to quantization, we cast the theory into the canonical form.[2]

1. SPIN ALGEBRA AND ANALYSIS

We shall first summarize those results of the spin algebra and analysis necessary for our needs. Inasmuch as the spinors of Dirac and the other early workers were introduced as a half-odd representation of the lorentz group, a representation that has no analog among the representations of the full linear or even the unimodular group, we cannot expect to develop a general covariant spin algebra without sacrificing some of the formal beauty of Dirac's original theory. The loss, it turns out, consists of the complete separation of the spin transformation group from the coordinate transformation group. We write the transformation law of a spin vector γ^μ (the Dirac matrices)

$$\gamma^{\mu\prime}=(\partial x^{\mu\prime}/\partial x^\rho)S\gamma^\rho S^{-1}, \tag{1.1}$$

suppressing all spin indices. The spin transformation matrix S is invariant with respect to coordinate transformations and is composed of arbitrary scalar functions of the coordinates. We take as the fundamental relation between the covariant and contravariant spin vectors the anticommutation relation

$$\gamma_\mu\gamma^\nu+\gamma^\nu\gamma_\mu=2\delta_\mu{}^\nu E, \tag{1.2}$$

where E is the four-rowed unit matrix and γ_μ and γ^μ are two sets of four matrices each. In reliance on the "correspondence principle" with the lorentzian case, we shall also assume that each set by itself, together with all its products, forms the base of an algebra. Now we can prove, with the help of Eq. (1.2) alone, that the anticommutators of the covariant γ's as well as the anticommutators of the contravariant γ's are "c" numbers. We have

$$(\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu)\gamma^\rho-\gamma^\rho(\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu)=0, \tag{1.3}$$

i.e., the covariant anticommutators commute with every γ^ρ. Naturally, if we reverse the role of subscripts and superscripts, nothing is changed, and our assertion above is proved. We shall call one-half of each anticommutator $g_{\mu\nu}E$ and $g^{\mu\nu}E$, respectively, for example,

$$2g_{\mu\nu}E=\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu. \tag{1.4}$$

We need to make one further assumption in order to be able to derive all of the usual relationships, and that is

$$\gamma^\rho\gamma_\rho=4E.$$

Then, very simple calculations (not reproduced here) are needed to show that

$$g_{\mu\rho}g^{\rho\nu}=\delta_\mu{}^\nu$$

and

$$\gamma^\nu=g^{\nu\rho}\gamma_\rho,\quad \gamma_\nu=g_{\nu\rho}\gamma^\rho.$$

Since the spin vectors form a linear algebra, we can represent them as matrices. The irreducible representation is four-rowed. Schouten has shown[8] that the matrix $\gamma^1\gamma^2\gamma^3\gamma^4$, which with respect to coordinate transformations is a scalar density of weight $+1$, has as eigenvalues $i(-g)^{\frac{1}{2}}$, $-i(-g)^{\frac{1}{2}}$, and we shall, therefore, introduce a "special" spin frame in which

$$\gamma^1\gamma^2\gamma^3\gamma^4=i(-g)^{\frac{1}{2}}\mu,$$

with

$$\mu=\begin{pmatrix} I, & 0 \\ 0, & -I \end{pmatrix},\qquad I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.5}$$

(I stands for the two-rowed unit matrix).

In any four-rowed representation, the spin vectors, their distinct products two at a time, three at a time, four at a time, and the unit matrix form a complete base, i.e., any four-rowed matrix can be represented as a linear combination of these 16 matrices. In the "special" representation, they take the form

$$\gamma^\mu=\begin{pmatrix} 0, & \beta^\mu \\ \bar{\beta}^\mu, & 0 \end{pmatrix}, \tag{1.6}$$

where the β^μ and $\bar{\beta}^\mu$ are 2×2 matrices. From now on, we shall restrict the spin transformation matrices S to a form that will maintain the validity of Eqs. (1.5) and (1.6). The spin transformations that do this are

$$S=\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}, \tag{1.7}$$

where u and v are arbitrary 2×2 matrices. We can restrict the spin transformations further by noting that in the Lorentz case the spin vectors are "self-adjoint." We define self-adjointness in the following way:[6] for a four-rowed matrix A its "adjoint" shall be

$$A^a=\eta A^\dagger\eta, \tag{1.8}$$

[5] E. Schrödinger, Berl. Ber. 105 (1932).
[6] V. Bargmann, Preuss, Akad. Wiss. Berlin, Ber. **25**, 346 (1932).
[7] L. Infeld and B. L. van der Waerden, Preuss. Akad. Wiss. Berlin, Ber. **9**, 380 (1933).
[8] J. A. Schouten, J. Phys. Math., 331 (1933).
[9] W. Pauli, Ann. Physik **18**, 337 (1933).

where

$$\eta=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$$

and the dagger † denotes the hermitian adjoint. If $A^a=A$, we call A self-adjoint. From Eqs. (1.6) and (1.8) we find that

$$\beta^{\mu\dagger}=\beta^\mu,\quad \bar{\beta}^{\mu\dagger}=\bar{\beta}^\mu. \tag{1.9}$$

Since the self-adjointness of γ^μ must hold in all spin systems, we find from Eq. (1.8), subjecting the spin vectors to a spin transformation, that

$$\begin{aligned}S\gamma^\mu S^{-1}&=\eta(S\gamma^\mu S^{-1})^\dagger\eta\\ &=\eta S^{\dagger-1}\gamma^{\mu\dagger}S^\dagger\eta\\ &=\eta S^{\dagger-1}\eta\gamma^\mu\eta S^\dagger\eta.\end{aligned}$$

Thus, we must have

$$S=\eta S^{\dagger-1}\eta.$$

This can be true only if

$$S=\begin{pmatrix}s & 0\\ 0 & s^{\dagger-1}\end{pmatrix}, \tag{1.10}$$

where s is an arbitrary 2×2 matrix composed of complex scalar functions of the coordinates.

All the interesting matrices arising in the spin theory are composed either of 2×2 matrices along the diagonal or of 2×2 matrices off the diagonal.

To develop spinor analysis, we define the covariant derivative of a spin quantity in analogy to the covariant derivative of a tensor in differential geometry. If χ^Γ is a quantity with one spin index, its covariant derivative is

$$\chi^\Gamma{}_{;\tau}=\chi^\Gamma{}_{,\tau}+\Gamma_{\Sigma\tau}{}^\Gamma\chi^\Sigma$$

(all capital Greek indices are spin indices), or in matrix form

$$\chi_{;\tau}=\chi_{,\tau}+\Gamma_\tau\chi.$$

We refer to Γ_τ as the spin connection coefficients. The spin vector γ^α has two spin indices and one coordinate index. Its covariant derivative is

$$\gamma^\alpha{}_{;\tau}=\gamma^\alpha{}_{,\tau}+\Gamma_{\rho\tau}{}^\alpha\gamma^\rho+\Gamma_\tau\gamma^\alpha-\gamma^\alpha\Gamma_\tau. \tag{1.11}$$

Pauli[9] has shown that solutions of the spin connections Γ_τ exist irrespective of the choice of affine connections $\Gamma_{\rho\tau}{}^\alpha$. Assuming the affine connections to be symmetric in their lower indices, we find with the help of Eq. (1.4) that they are the Christoffel symbols of the second kind. We can solve for the spin connections Γ_τ, but for our present purpose these explicit expressions will not be needed. However, since the spin connection appears as a commutator in its determining Eq. (1.11), we can add to its solution, say Λ_τ, a "c" number vector $i\phi_\tau E$,

$$\Gamma_\tau=\Lambda_\tau+i\phi_\tau E. \tag{1.12}$$

Because the covariant derivative of a spin vector is spin- as well as coordinate-covariant, we find that the spin connections must satisfy a transformation law of the form

$$\Gamma_\tau'=(\partial x^\rho/\partial x^\tau)'(S\Gamma_\rho S^{-1}-S_{,\rho}S^{-1}). \tag{1.13}$$

Thus we see that Γ_τ transforms as a vector but not as a spinor.

To complete that portion of the analysis that interests us, we form with Schrödinger a spin tensor by considering the alternator of the second covariant derivative of a spin vector. We find

$$\gamma^\alpha{}_{;\mu\nu}-\gamma^\alpha{}_{;\nu\mu}=R_{\mu\nu\lambda}{}^\alpha\gamma^\lambda+\Phi_{\mu\nu}\gamma^\alpha-\gamma^\alpha\Phi_{\mu\nu}=0. \tag{1.14}$$

$R_{\mu\nu\lambda}{}^\alpha$ is the Riemann-Christoffel curvature tensor, and

$$\Phi_{\mu\nu}=\Gamma_{\mu,\nu}-\Gamma_{\nu,\mu}-\Gamma_\mu\Gamma_\nu+\Gamma_\nu\Gamma_\mu \tag{1.15}$$

is a spin tensor, i.e.,

$$\Phi_{\mu\nu}'=\frac{\partial x^\rho}{\partial x^{\mu'}}\frac{\partial x^\sigma}{\partial x^{\nu'}}S\Phi_{\rho\sigma}S^{-1}.$$

This spin tensor is of the second differential order in the γ^α.

2. LAGRANGIAN, FIELD EQUATIONS, AND IDENTITIES

We shall construct the field equations from a four-dimensional variational principle, where we choose as the lagrangian a linear combination of second-order scalar densities constructed from quantities that arise in the spin algebra and analysis. The resulting field equations, we shall show, are the same as those of the general theory of gravitation. Finally, we shall determine the differential identities which exist because of the coordinate and spin transformations.

The following building blocks are available for the construction of scalar densities of weight one of the second differential order: a scalar density of weight one

$$4(-g)^{\frac{1}{2}}=i\,\mathrm{tr}\{\mu\gamma^1\gamma^2\gamma^3\gamma^4\}, \tag{2.1}$$

the spin tensor $\Phi_{\mu\nu}$ (1.15), and the spin vector γ^μ. From the spin tensor and spin vectors we can form the two second-order scalar matrices

$$(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)\Phi_{\mu\nu},\quad \gamma^\mu\Phi_{\mu\nu}\gamma^\nu. \tag{2.2}$$

Taking the trace of either expression (2.2) (i.e., contracting on the spin indices) and multiplying by the scalar density (2.1), we obtain to within a numerical factor the scalar density

$$L_{(\mathrm{s})}=(-g)^{\frac{1}{2}}\,\mathrm{tr}\{\gamma^\mu\Phi_{\mu\nu}\gamma^\nu\}, \tag{2.3}$$

which we shall adopt as the spin lagrangian. It is possible to show that with the elements available, $L_{(\mathrm{s})}$ is the only scalar density of second differential order in existence. The proof will not be presented here, though. The arbitrary vector that can be added to the spin connection [see Eq. (1.12)] gives no contribution to the scalar density (2.3). In order to introduce the electromagnetic field, we must assume an electromagnetic four vector ϕ_μ and construct a term to be added to the spin lagrangian. This term is

$$L_{(\mathrm{em})}=(-g)^{\frac{1}{2}}g^{\mu\alpha}g^{\nu\beta}(\phi_{\mu,\nu}-\phi_{\nu,\mu})(\phi_{\alpha,\beta}-\phi_{\beta,\alpha}). \quad (2.4)$$

Its treatment in what follows is the usual one and will be omitted.[10]

We shall now relate the spin lagrangian to the general theory of gravitation. By pre- and post-multiplying Eq. (1.14) by γ^μ, contracting on α and μ, adding, taking the trace, and finally multiplying through by $(-g)^{\frac{1}{2}}$, we find that

$$L_{(\mathrm{s})}=2(-g)^{\frac{1}{2}}R \quad \text{or} \quad L_{(\mathrm{s})}=2L_{(\mathrm{m})}, \quad (2.5)$$

where R is the riemann scalar curvature. Both the metric and the spin lagrangian contain second derivatives of the field variables only linearly. These terms can be converted into complete divergences, plus terms quadratic in the first derivatives of the field variables. The divergences do not contribute to the field equations, and the lagrangians in either metric or spin form may be considered as depending on the field variables and their first derivatives only. This consideration will be used to simplify the later computations.

Since $L_{(\mathrm{m})}=(-g)^{\frac{1}{2}}R$ is the lagrangian of the general theory of gravitation, the spin lagrangian (2.3) would obviously yield the same field equations were it not for the restrictions (1.3) and (1.4) on the variations of the spin vectors. These conditions can be taken into account in the hamiltonian principle by the method of undetermined multipliers.

The variational principle is, therefore,

$$\delta\int[L_{(\mathrm{s})}+\mathrm{tr}\{M_\nu{}^\mu(\gamma_\mu\gamma^\nu+\gamma^\nu\gamma_\mu)+N\gamma_\mu\gamma^\mu\}]d^4x=0, \quad (2.6)$$

where the matrices $M_\nu{}^\mu$ and N are the undetermined multipliers. The first term of Eq. (2.6) can be related to the gravitational field equations after variation with respect to the field variables, the elements of the spin vectors γ^μ. Using Eq. (2.5) and the variation of Eq. (1.4), we find, apart from discarded divergences,

$$\begin{aligned}\delta L_{(\mathrm{s})}&=\delta(2(-g)^{\frac{1}{2}}R)\\&=\tfrac{1}{2}(-g)^{\frac{1}{2}}(R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R)\ \mathrm{tr}\{\gamma^\mu\delta\gamma^\nu+\delta\gamma^\mu\gamma^\nu\}\\&=(-g)^{\frac{1}{2}}G_{\mu\nu}\ \mathrm{tr}\{\gamma^\mu\delta\gamma^\nu\}. \quad (2.7)\end{aligned}$$

If we denote the variations of those terms in Eqs. (2.6) which contain the undetermined multipliers by the symbol $\mathrm{tr}\{U_\mu\delta\gamma^\mu\}$, then the differential equations satisfied by solutions of the variational principle are

$$G_{\mu\nu}\gamma^\nu+U_\mu=0. \quad (2.8)$$

We shall now show that the term U_μ actually vanishes and may be omitted. Consider a solution of the field equations that would result if the variations were not subject to the algebraic restrictions (1.3), (1.4). In that case, the field equations would reduce to $G_{\mu\nu}\gamma^\nu=0$. But we know that we can construct fields of γ^μ which satisfy the algebraic restrictions (1.3), (1.4) as well as these stronger equations; in other words, we find that there are true solutions of the unrestricted variational problem which, nevertheless, are consistent with the algebraic restrictions. Hence, we can assert that among the solutions of Eqs. (2.8), with the multipliers determined subsequently by standard methods, are at least some for which the additional terms U_μ vanish. From now on, those are the only solutions we shall consider. We have not ascertained whether they are the only solutions that exist.

In the general theory of relativity we have the option of introducing a cosmological term or considering the field equations to be

$$G_{\mu\nu}=0.$$

We choose the latter, for we do not consider the cosmological term of importance, at any rate in atomic phenomena. The field equations are then

$$L_\mu\equiv G_{\mu\nu}\gamma^\nu=0. \quad (2.9)$$

In the presence of matter, the field equations are not satisfied; but rather the left-hand sides of the field equations equal expressions which represent charge, current, and mass densities, momentum densities, and stress components. Nevertheless, the left-hand sides will satisfy identities because of the following consideration. If we vary an arbitrary set of functions for the field variables infinitesimally, the lagrangian will, as a result, undergo an infinitesimal transformation of the form

$$\bar{\delta}L_{(s)}=\mathrm{tr}\left\{\frac{\partial L_{(s)}}{\partial\gamma^\mu}\bar{\delta}\gamma^\mu+\frac{\partial L_{(s)}}{\partial\gamma^\mu{}_{,\rho}}\bar{\delta}\gamma^\mu{}_{,\rho}\right\}. \quad (2.10)$$

If, in particular, the variation of the field variables is the result of an infinitesimal coordinate-plus-spin transformation,

$$x^{\rho\prime}=x^\rho+\xi^\rho, \quad (2.11)$$

$$S=E+\Sigma, \quad (2.12)$$

then the integral over the lagrangian should not change at all, except possibly as the result of the variations on the surface of the domain of integration. In other words, the variation of the lagrangian itself should be a complete divergence,

$$\bar{\delta}L_{(\mathrm{s})}=Q^\rho{}_{,\rho}.$$

If we carry out this idea, we find, first, that

$$\bar{\delta}\gamma^\mu=\xi^\mu{}_{,\rho}\gamma^\rho-\gamma^\mu{}_{,\rho}\xi^\rho+\Sigma\gamma^\mu-\gamma^\mu\Sigma$$

and, therefore,

$$Q^\rho{}_{,\rho}=\mathrm{tr}\{L_\mu(\xi^\mu{}_{,\rho}\gamma^\rho-\gamma^\mu{}_{,\rho}\xi^\rho+\Sigma\gamma^\mu-\gamma^\mu\Sigma)\}+[\mathrm{tr}\{(\partial L_{(s)}/\partial\gamma^\mu{}_{,\sigma})\delta\gamma^\mu\}]_{,\sigma},$$

[10] P. G. Bergmann, *Introduction to the Theory of Relativity* (Prentice-Hall, Inc., New York, 1942).

to be satisfied identically even though the generators of the infinitesimal transformations are completely arbitrary. This condition can be satisfied only if both

$$\mathrm{tr}\{L_\mu \gamma^\mu{}_{,\nu} + (L_\nu \gamma^\mu)_{,\nu}\} \equiv 0, \qquad (2.13)$$

$$L_\mu \gamma^\mu - \gamma^\mu L_\mu \equiv 0. \qquad (2.14)$$

These two distinct sets of identities must hold for any lagrangian that yields convariant Euler-Lagrange equations.

The first set of identities are a result of the covariance of the field equations with respect to coordinate transformations. They are the spin form of the contracted Bianchi identities. The second set of identities are the result of the covariance of the field equations with respect to the spin transformations.

Because of the identities (2.13), we can construct a quantity whose ordinary divergence is zero. This is the "strong" form of the conservation laws. If the field equations are satisfied, we have[1]

$$t_\iota{}^\rho{}_{,\rho} = 0, \quad t_\iota{}^\rho = \delta_\iota{}^\rho L_{(s)} - \mathrm{tr}\{(\partial L_{(s)}/\partial \gamma^\mu{}_{,\rho})\gamma^\mu{}_{,\iota}\}, \qquad (2.15)$$

which can be verified to be equal to the relativistic expressions by means of Eq. (1.4). In the presence of matter the field equations are not satisfied, but we have instead

$$L_\mu = P_\mu.$$

The divergence of Eq. (2.15) does not vanish but is equal to $\mathrm{tr}\{P_\mu \gamma^\mu{}_{,\iota}\}$. Because of the identities (2.13), we can still form 16 quantities whose ordinary divergence is zero. We have

$$T_\iota{}^\rho{}_{,\rho} \equiv 0, \quad T_\iota{}^\rho = t_\iota{}^\rho - \mathrm{tr}\{L_\iota \gamma^\rho\}. \qquad (2.16)$$

3. SPIN CONSTRAINTS AND HAMILTONIAN

In this section we construct the constraints and hamiltonian in spin form. The coordinate and parameter constraints can be formed in the same manner as they are formed in the metric case.[3] However, in the present theory there is a transformation group—the spin group—which is not present in the metric case. This group gives rise, as we have seen, to a spin identity (2.14). From this identity we shall construct the spin constraint in a manner analogous to the construction of the coordinate constraints from the contracted Bianchi identities. Knowledge of all these constraints is required for the construction of the hamiltonian.[3]

To construct the spin constraint, we consider the highest derivatives of the spin vectors in the spin identity (2.14). They are

$$\left(\frac{\partial^2 L_{(s)}}{\partial \gamma^\mu{}_{,\rho} \partial \gamma_\Sigma{}^{\Gamma\mu}{}_{,\sigma}} \gamma^\mu - \gamma^\mu \frac{\partial^2 L_{(s)}}{\partial \gamma^\mu{}_{,\rho} \partial \gamma_\Sigma{}^{\Gamma\nu}{}_{,\sigma}}\right) \gamma_\Sigma{}^{\Gamma\nu}{}_{,\rho\sigma},$$

where the capital Greek letters again represent spin indices. These terms must vanish for they appear by themselves. After symmetrizing with respect to ρ and σ, we have

$$\frac{\partial^2 L_{(s)}}{\partial \gamma^\mu{}_{,\rho} \partial \gamma_\Sigma{}^{\Gamma\nu}{}_{,\sigma}} \gamma^\mu - \gamma^\mu \frac{\partial^2 L_{(s)}}{\partial \gamma^\mu{}_{,\rho} \partial \gamma_\Sigma{}^{\Gamma\nu}{}_{,\sigma}} + \frac{\partial^2 L_{(s)}}{\partial \gamma^\mu{}_{,\sigma} \partial \gamma_\Sigma{}^{\Gamma\nu}{}_{,\rho}} \gamma^\mu - \gamma^\mu \frac{\partial^2 L_{(s)}}{\partial \gamma^\mu{}_{,\sigma} \partial \gamma_\Sigma{}^{\Gamma\nu}{}_{,\rho}} = 0. \qquad (3.1)$$

The momentum densities conjugate to the spin vectors are

$$\pi_\mu = Jt_{,\rho}(\partial L_{(s)}/\partial \gamma^\mu{}_{,\rho}). \qquad (3.2)$$

Multiplying Eq. (3.1) by $Jt_{,\rho}Jt_{,\sigma}$ and introducing Eq. (3.2), we obtain

$$\frac{\partial \pi_\mu}{\partial \dot\gamma_\Sigma{}^{\Gamma\nu}} \gamma^\mu - \gamma^\mu \frac{\partial \pi_\mu}{\partial \dot\gamma_\Sigma{}^{\Gamma\nu}} = 0.$$

Since the γ^μ are independent of $\dot\gamma^\nu$, we have

$$\pi_\mu \gamma^\mu - \gamma^\mu \pi_\mu = K, \qquad (3.3)$$

where K is independent of "dotted" quantities. In order to evaluate K, we introduce the expression for π_μ into Eq. (3.3). From Eqs. (3.2), (2.5), and the trace of Eq. (1.4), we obtain

$$\pi_\mu = Jt_{,\rho}(\partial L_{(m)}/\partial g^{\mu\beta}{}_{,\rho})\gamma^\beta. \qquad (3.4)$$

Substituting this expression into Eq. (3.3), we find

$$K = Jt_{,\rho}(\partial L_{(m)}/\partial g^{\mu\beta}{}_{,\rho})(\gamma^\beta \gamma^\mu - \gamma^\mu \gamma^\beta) = 0, \qquad (3.5)$$

since $\partial L_{(m)}/\partial g^{\mu\beta}{}_{,\rho}$ is a "c" number symmetric in μ and β.

In a similar manner we calculate the coordinate constraint from the contracted Bianchi identities (2.13). We find

$$Jt_{,\rho}\, \mathrm{tr}\{\gamma^\rho \pi_\mu\} = K_\mu, \qquad (3.6)$$

where K_μ is independent of $\dot\gamma^\nu$. From Eq. (3.4), we obtain

$$K_\mu = Jt_{,\rho}Jt_{,\sigma}(\partial L_{(m)}/\partial g^{\mu\beta})\, \mathrm{tr}\{\gamma^\beta \gamma^\sigma\} = 4Jt_{,\rho}Jt_{,\sigma} g^{\beta\sigma}(\partial L_{(m)}/\partial g^{\mu\beta}{}_{,\rho}). \qquad (3.7)$$

The metric lagrangian $L_{(m)}$, given by Eq. (2.5), contains only first derivatives of field variables. Using Eq. (2.5), we finally obtain

$$K_\mu = -\frac{(-g)^{\frac{1}{2}}}{8\pi\kappa} Jt_{,\rho}Jt_{,\sigma}\left(\delta_\mu{}^\rho g^{\alpha\beta}\begin{Bmatrix}\sigma\\ \alpha\beta\end{Bmatrix} - \delta_\mu{}^\rho g^{\alpha\sigma}\begin{Bmatrix}\beta\\ \alpha\beta\end{Bmatrix} - 2g^{\alpha\rho}\begin{Bmatrix}\sigma\\ \mu\alpha\end{Bmatrix} + \delta_\mu{}^\sigma g^{\alpha\rho}\begin{Bmatrix}\alpha\\ \alpha\beta\end{Bmatrix} + g^{\rho\sigma}\begin{Bmatrix}\alpha\\ \mu\alpha\end{Bmatrix}\right).$$

In this and the following expressions, the numerical coefficients agree with those of a recent paper concerned with purely metric space.[3] The parameter constraint[2] is

$$\mathrm{tr}\{\gamma_{\mu|r}\pi^\mu\} + x^\rho{}_{|r}\lambda_\rho = 0, \qquad (3.8)$$

where

$$\pi^{\mu}=g^{\mu\alpha}\pi_{\alpha}$$

and

$$\lambda_{\rho}=Jt_{,\sigma}t_{\rho}{}^{\sigma}. \quad (3.9)$$

The electromagnetic constraint is the same as in the metric theory:[3]

$$Jt_{,\rho}\psi^{\rho}=0,$$

where

$$\psi^{\rho}=Jt_{,\sigma}\frac{\partial L_{(em)}}{\partial\phi_{\rho,\sigma}}.$$

To construct the hamiltonian constraint, we could proceed by the straightforward method developed earlier.[3] But we can save ourselves the work of repeated transformations in the symbolic vector space by simply introducing the spin vector γ^{μ} and its canonically conjugate momentum density π_{μ} into the hamiltonian of the general theory of relativity.[3, 11] In view of the fact that in a parametrized theory the hamiltonian is an expression that vanishes identically, a constraint, and since the constraints are associated with the transformation group of the theory considered, we are assured that if we can discover a constraint independent of those already enumerated in this section, then that new constraint is the hamiltonian. And naturally, the hamiltonian constraint of the general theory of relativity is an expression that will continue to vanish identically if we introduce into it the spin quantities.

The metric tensor is related to the spin vector via Eq. (1.4). We shall now calculate the metric momentum density conjugate to the metric tensor in terms of the spin momentum density conjugate to the spin vectors. The spin momentum density is

$$\pi^{\mu}=Jt_{,\rho}\frac{\partial L_{(s)}}{\partial\gamma_{\mu,\rho}}=2Jt_{,\rho}\frac{\partial L_{(m)}}{\partial g_{\alpha\beta,\sigma}}\frac{\partial g_{\alpha\beta,\sigma}}{\partial\gamma_{\mu,\rho}},$$

using Eq. (2.5) and the chain rule of differentiation. Using the trace of Eq. (1.4) we obtain

$$\pi^{\mu}=It_{,\rho}(\partial L_{(m)}/\partial g_{\mu\beta,\rho})\gamma_{\beta}=\pi^{\mu\beta}\gamma_{\beta}. \quad (3.10)$$

We solve for $\pi^{\mu\beta}$ in the usual way by pre- and post-multiplying by γ^{σ}, adding, and using Eq. (1.2). Doing this, we find

$$E\pi^{\mu\rho}=\tfrac{1}{2}(\pi^{\mu}\gamma^{\rho}+\gamma^{\rho}\pi^{\mu}). \quad (3.11)$$

Multiplying Eq. (3.11) by $g_{\mu\rho}$ and using the spin constraint (3.3), we find the relation

$$Eg_{\mu\rho}\pi^{\mu\rho}=\gamma_{\mu}\pi^{\mu}=\pi^{\mu}\gamma_{\mu}, \quad (3.12)$$

useful in the construction of the hamiltonian.

It is now a straightforward computation to arrive at the spin hamiltonian. Substituting the spin quantities for their corresponding metric quantities, we find

$$\begin{aligned}H=v^{-1}v^{\rho}v^{\sigma}\{&\tfrac{1}{2}(\lambda_{\rho}Jt_{,\sigma}+\lambda_{\sigma}Jt_{,\rho})-G_{\rho\sigma}\\&+[8\pi\kappa/(-g)^{\frac{1}{2}}X](g_{\alpha\gamma}g_{\beta\delta}+g_{\alpha\delta}g_{\beta\gamma}-g_{\alpha\beta}g_{\gamma\delta})\\&\times[\tfrac{1}{4}\operatorname{tr}\{\pi^{\alpha}\gamma^{\beta}\}Jt_{,\rho}-2G^{(\alpha\beta)}{}_{\rho}]\\&\times[\tfrac{1}{4}\operatorname{tr}\{\pi^{\gamma}\gamma^{\delta}\}Jt_{,\sigma}-2G^{(\gamma\delta)}{}_{\sigma}]-[2\pi/(-g)^{\frac{1}{2}}X]\\&\quad\times g_{\mu\nu}(\psi^{\mu}Jt_{,\rho}-2G^{\mu}{}_{\rho})(\psi^{\nu}Jt_{,\sigma}-2G^{\nu}{}_{\sigma})\}. \quad (3.13)\end{aligned}$$

In this expression, we have used all the abbreviations introduced previously.[3]

CONCLUSION

With the setting up of the hamiltonian density (3.13), we have completed the program of this paper. We have succeeded in formulating a theory covariant both with respect to coordinate and with respect to spin transformations, in which the field equations appear as the Euler-Lagrange equations of a four-dimensional variational principle. We have furthermore found the appropriate hamiltonian form of that theory. We have omitted the explicit derivation of the secondary constraints, but the general results reported in a paper by Anderson and Bergmann[12] are applicable to this theory.

We found that this theory is completely equivalent to the general theory of relativity, in spite of the apparently greater wealth of geometric objects. In actual fact, because of the additional transformation group associated with the spinors, the supply of true geometric objects has not been increased. This result is not quite as damning as it may appear at first, for two reasons. One is that subsequent quantization of the theory may lead to an inequivalence not now apparent, just as the difference between the Dirac equations and the Klein-Gordon equation is quantum theoretical, and disappears in the classical (WBK) limit. The other reason is that the theory with spinors permits the introduction of additional geometric objects, ψ-functions, that may assume physical meaning only in the second quantization. In other words, the interaction between particles and field may be such that it can be described more adequately in terms of spinors.

We intend to continue our investigation along these lines.

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[11] F. A. E. Pirani, and A. Schild, Phys. Rev. **79**, 986 (1950).

[12] J. L. Anderson and P. G. Bergmann, Phys. Rev. **83**, 1018 (1951).