The Energy Density Tensor in Gauge-Independent Quantum Electrodynamics

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In Heisenberg representation two different definitions of an energy density tensor are given for gauge-independent quantum electrodynamics, by Eqs. (1)–(4) and Eqs. (20)–(21), respectively. Both tensors lead to the same total energy and momentum, if we assume the interaction to vanish at $t=-\infty$. They both satisfy conservation laws. The tensor character of the first one is proved, and the tensor character of the second one is manifest. The first tensor is obtained by analogy with the result of a derivation of the energy density tensor as source of the gravitational field from general-relativistic considerations in manifestly covariant quantum electrodynamics, with subsequent omission of the "phantom terms" containing the redundant variables of this theory. The second tensor has the advantage of admitting a simple covariant subtraction of its vacuum value, and of simplifying even further by use of the new covariant auxiliary condition proposed recently by the author. Its disadvantage, though, is the impossibility of direct physical interpretation, as in Heisenberg representation it is not expressed in terms of field variables in Heisenberg representation. The inconclusiveness of an argument for possible equality of the two tensors is discussed. Both tensors contain the usual self-interaction effects, and the problem is posed of how to eliminate these effects.

I. DEFINITION OF AN ENERGY DENSITY TENSOR

In a recent paper[1] the author has developed the formalism of a gauge-independent quantum electrodynamics, in interaction representation as well as in Heisenberg representation. It is plausible to identify in such theory the energy density with the sum of the ordinary free-particle energy density and the interaction operator used in the generalized Schrödinger equation.[2]

In manifestly covariant quantum electrodynamics, on the other hand, a scalar first-order[3,4] lagrangian is known,[5] so that the total[6] energy density tensor $T^{\lambda\mu}$ can simply be derived for it by the principles of general relativity theory as the source of the gravitational field.[3] The result is Eq. (46) of reference 5. In order to make the best use of the Lorentz auxiliary condition,[5] it is then convenient to separate the redundant longitudinal and scalar photon variables by a transformation introducing new field variables,[7] and by expressing $T^{\lambda\mu}$ in terms of them. If from the q-number expression for the energy density T^{00} thus obtained[8] one completely omits all so-called "phantom" terms[5] (which have vanishing value due to the Lorentz condition), one simply finds the expression just suggested for the energy density in the gauge-independent theory.[2]

This makes it seems appropriate to try a definition of the other components of $T^{\lambda\mu}$ in a similar way. This leads to the following expressions[8,9] for the energy density T^{00}, for $c\times$ the momentum density $=(1/c)\times$ the energy current density $=T^{k0}=T^{0k}$, and for the stress tensor $T^{kl}=T^{lk}$:

$$T^{\lambda\mu}=T_F{}^{\lambda\mu}+T_{\rm m}{}^{\lambda\mu};\quad T_{\rm m}{}^{\lambda\mu}=T_{\rm e}{}^{\lambda\mu}+T_I{}^{\lambda\mu}, \tag{1}$$

with

$$\left.\begin{aligned} T_F{}^{00}&=\{\mathfrak{B}^2+\mathbf{E}^2\}/8\pi,\\ T_F{}^{0k}&=\textstyle\sum_l\{E_l\mathfrak{B}_{kl}+\mathfrak{B}_{kl}E_l\}/8\pi,\\ T_F{}^{kl}&=\delta_{kl}\{\mathfrak{B}^2+\mathbf{E}^2\}/8\pi-\{\mathfrak{B}_k\mathfrak{B}_l+E_kE_l\}/4\pi;\end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned} T_I{}^{00}&=-\mathbf{j}\cdot\mathfrak{A},\quad T_I{}^{0k}=-\rho\mathfrak{A}_k,\\ T_I{}^{kl}&=-\tfrac{1}{2}\{j_k\mathfrak{A}_l+j_l\mathfrak{A}_k\};\end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} T_{\rm e}{}^{00}&=\mathfrak{R}\{\psi^\dagger H_0\psi\},\\ T_{\rm e}{}^{kl}&=-\tfrac{1}{2}\mathfrak{R}\{\psi^\dagger i\hbar c(\alpha_k\nabla_l+\alpha_l\nabla_k)\psi\},\\ T_{\rm e}{}^{0k}&=-\mathfrak{R}\{\psi^\dagger i\hbar c\nabla_k\psi\}+\tfrac{1}{4}\hbar c\textstyle\sum_l\nabla_l\mathfrak{R}\{\psi^\dagger\sigma_{kl}\psi\}.\end{aligned}\right\} \tag{4}$$

Here

$$H_0=mc^2\beta-i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla};\quad \sigma_{kl}=-\tfrac{1}{2}i(\alpha_k\alpha_l-\alpha_l\alpha_k); \tag{5}$$

$$\mathfrak{R}\{\psi^\dagger\Omega\varphi\}=\tfrac{1}{4}\{\psi^\dagger\Omega\varphi+\varphi^\dagger\Omega^\dagger\psi-\varphi^{\rm T}\Omega^{\rm T}\psi^*-\psi^{\rm T}\Omega^*\varphi^*\}, \tag{6}$$

where $^{\rm T}$ means transposition in the space of coordinates[10] and undor indices[11] (the latter meaning: interchanging rows and columns of Dirac matrices and Dirac wave functions), while * means hermitian conjugation in Hilbert space (including complex conjugation of c-numbers), and $^\dagger={}^{*\rm T}$. Further, ρ (in esu) and $\mathbf{j}$ (in emu) in Eq. (3) form a four-vector, which according to reference 5 is defined by

$$j^\mu=\mathfrak{R}\{e\psi^\dagger\alpha^\mu\psi\}=\mathfrak{R}\{ie\bar\psi\gamma^\mu\psi\}, \tag{7}$$

[1] F. J. Belinfante and J. S. Lomont, Phys. Rev. 84, 541 (1951). In the following we refer to its formulas by (G-I:).

[2] The real part of the integrand of Eq. (G-I:31), plus the expression (G-I:10); thence, the integrand in (G-I:56).

[3] F. J. Belinfante, Physica 7, 449 (1940).

[4] F. J. Belinfante, Phys. Rev. 74, 779 (1948).

[5] F. J. Belinfante, Physica 12, 17 (1946).

[6] F. J. Belinfante, Physica 6, 887 (1939).

[7] F. J. Belinfante, Physica 12, 1 (1946); in particular Sec. 4.

[8] Reference 5, Eqs. (48), (49), (50).

[9] For the notation used in reference 8, compare reference 5, Eq. (47), and Sec. 1 of reference 7. Italic indices run from 1 to 3, Greek from 0 to 3, with $x^0=-x_0=ct$, and $\nabla_0=-\nabla^0=\partial/c\partial t$. The symmetrization $E\mathfrak{B}+\mathfrak{B}E$ in Eq. (2) is due to a symmetrization in the lagrangian proposed in F. J. Belinfante, Physica 7, 765 (1940), and simplifies the derivation of the first of the Eqs. (18) below, but is further superfluous due to Eq. (G-I:2) with Eq. (A.7) of Appendix A. As in Eq. (G-I:56), we use here the abbreviation $\mathbf{E}=\mathbf{E}_{\rm II}+\mathfrak{E}$ (G-I:44), with $\mathbf{E}_{\rm II}$ given by Eqs. (G-I:6–7) of reference 1.

[10] Thus, $f\nabla_x{}^{\rm T}g$ means $(\partial f/\partial x)g$. Further, $\boldsymbol{\nabla}^*=\boldsymbol{\nabla}$ and $\boldsymbol{\nabla}^\dagger=\boldsymbol{\nabla}^{\rm T}$.

[11] F. J. Belinfante, Physica 6, 849 (1939).

with $\alpha^0=1$, $\gamma^\mu=-i\beta\alpha^\mu$, $\bar{\psi}=\psi^\dagger\beta$. One easily proves the formal equality[12] of this expression (7) first proposed by Dirac,[13] and the expression (G-I:8) of reference 1 first proposed by Furry and Oppenheimer.[14]

For $T_{\rm m}{}^{00}$, another convenient expression is found by use of the field equation for ψ in Heisenberg representation (G-I:57), and of Eqs. (6) and (7):

$$\left.\begin{array}{r} T_{\rm m}{}^{00}=T_{e'}{}^{00}+T_{I'}{}^{00};\quad T_{e'}{}^{00}=\Re\{\psi^\dagger i\hbar\partial_{\rm H}\psi/\partial t\},\\ T_{I'}{}^{00}=-\tfrac{1}{2}\rho V-\tfrac{1}{2}e\Re\{\psi^\dagger V\psi\}.\end{array}\right\}\quad (8)$$

Here, $\partial_{\rm H}/\partial t$ means time differentiation in Heisenberg representation.

II. THE CONSERVATION LAWS

Using Eqs. (2), (G-I:64), and (G-I:66), we find

$$\nabla_\mu\,{}^{\rm H}T_F{}^{0\mu}=-\tfrac{1}{2}(\mathbf{E}\cdot\mathbf{j}+\mathbf{j}\cdot\mathbf{E}).\quad (9)$$

In this and the following equations, all field variables, etc., are tacitly understood to be expressed in Heisenberg representation.

In Appendix A, we prove that

$$\nabla_\mu\,{}^{\rm H}T_{\rm m}{}^{0\mu}=+\tfrac{1}{2}(\mathbf{E}\cdot\mathbf{j}+\mathbf{j}\cdot\mathbf{E}),\quad (10)$$

so that, by Eq. (1), we find the law of conservation of energy

$$\nabla_\mu\,{}^{\rm H}T^{0\mu}=0.\quad (11)$$

Similarly, Eqs. (2), (G-I:64), (G-I:65), (G-I:66), and (G-I:60), yield

$$\begin{aligned}\nabla_\mu\,{}^{\rm H}T_F{}^{k\mu}&=-\{\tfrac{1}{2}(\mathbf{E}\rho+\rho\mathbf{E})+[\mathbf{j}\times\mathfrak{B}]\}_k\\ &=-\tfrac{1}{2}(E_k\rho+\rho E_k)+\mathbf{j}\cdot(\nabla\,\mathfrak{A}_k-\nabla_k\mathfrak{A}).\quad (12)\end{aligned}$$

Equations (3), (G-I:58), and (G-I:59), yield

$$\nabla_\mu\,{}^{\rm H}T_I{}^{k\mu}=\tfrac{1}{2}(\mathrm{div}\,\mathbf{j})\mathfrak{A}_k+\rho\mathfrak{E}_k-\tfrac{1}{2}\nabla\cdot(j_k\mathfrak{A})-\tfrac{1}{2}(\mathbf{j}\cdot\nabla)\mathfrak{A}_k.\quad (13)$$

By Eqs. (4), (G-I:57) and conjugate, (5), (6), (7), (G-I:6), and various reasonings analogous to (A.2)–(A.12) of Appendix A, also using Eq. (A.8) and conjugate, we find after somewhat lengthy calculation

$$\begin{aligned}\nabla_\mu\,{}^{\rm H}T_{\rm e}{}^{k\mu}=\mathbf{j}\cdot\nabla\,{}_k\mathfrak{A}+\tfrac{1}{2}\nabla\cdot(j_k\mathfrak{A}-\mathbf{j}\mathfrak{A}_k)\\ +\tfrac{1}{2}(E_{\rm II\mathit{k}}\rho+\rho E_{\rm II\mathit{k}}),\quad (14)\end{aligned}$$

where we finally used the (not very convincing) Eq. (A.6) of Appendix A. [Otherwise, the expression (A.6) would have appeared added to the right-hand side of (14).] Adding (12), (13), and (14), we find the law of conservation of momentum

$$\nabla_\mu\,{}^{\rm H}T^{k\mu}=0.\quad (15)$$

Equations (11) and (15) together can be written as

$$\nabla_\mu\,{}^{\rm H}T^{\lambda\mu}=0,\quad (16)$$

which equation remains valid after subtraction of the (infinite) vacuum values of $T^{\lambda\mu}$, which are c-numbers not depending on $xyzt$. From the q-number relation (16), a similar equation follows for the expectation values of $T^{\lambda\mu}$, as the state vector in Heisenberg representation is a constant.

III. LORENTZ TRANSFORMATION OF THE ENERGY DENSITY TENSOR

For showing that the expectation values of $T^{\lambda\mu}$ form a tensor, it suffices to prove the tensor character of the q-numbers ${}^{\rm H}T^{\lambda\mu}$ in Heisenberg representation (with its scalar state vector). We shall use the infinitesimal Lorentz transformation formulas for the field variables in Heisenberg representation derived in Chapter 6 of reference 1. Thus we find that, under the infinitesimal Lorentz transformation

$$\delta x^0=-\textstyle\sum_l b_l x^l,\quad \delta x^k=-b_k x^0,\quad (\mathbf{b}=\mathbf{v}/c),\quad (17)$$

${}^{\rm H}T_F{}^{\lambda\mu}$ of Eq. (2) by itself transforms on account of (G-I:45) as a symmetric tensor, according to

$$\begin{aligned}\delta^{\rm H}T^{00}=-2\textstyle\sum_l b_l\,{}^{\rm H}T^{0l},\quad \delta^{\rm H}T^{kl}=-b_k\,{}^{\rm H}T^{0l}-b_l\,{}^{\rm H}T^{k0},\\ \delta^{\rm H}T^{0k}=-b_k\,{}^{\rm H}T^{00}-\textstyle\sum_l b_l\,{}^{\rm H}T^{lk}.\quad (18)\end{aligned}$$

We used here the fact that $E_k\mathfrak{B}_{lm}=\mathfrak{B}_{lm}E_k$ because of (G-I:2) with (A.7).

In Appendix B, we prove (Eqs. (B.10a-b-c)) that also ${}^{\rm H}T_{\rm m}{}^{\lambda\mu}$ is transformed according to (18) as a tensor, so that, by (1), also ${}^{\rm H}T^{\lambda\mu}$ itself is a tensor.

If the "vacuum" can be defined in a covariant way, as a property invariant under Lorentz transformations, then also the c-number vacuum expectation value of $T^{\lambda\mu}$ should form a tensor, and therefore should then be expected to take the form

$$\langle T^{\lambda\mu}\rangle_{\rm vac}=g^{\lambda\mu}T,\quad (19)$$

where T is some (infinite) c-number scalar. In that case, also $T^{\lambda\mu}-\langle T^{\lambda\mu}\rangle_{\rm vac}$ is a tensor, satisfying conservation laws by (16) and following remarks.

IV. AN ALTERNATIVE ENERGY DENSITY TENSOR

Equations (1)–(4) for the energy density tensor of gauge-independent quantum electrodynamics were not

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[12] Expand ψ in terms of eigenfunctions ϕ_n of H_0. Let w_n be the sign of the corresponding eigenvalues of H_0. In Heisenberg as well as in interaction representation, $\psi^{(+)}$ and $\psi^{(-)}$ are defined as the sum of the terms with $w_n=\pm1$ in this expansion. Then apparently $(1/e)\times$the difference between Eq. (G-I:8) and Eq. (7) is given by $:\psi^\dagger\alpha^\mu\psi:-\Re\{\psi^\dagger\alpha^\mu\psi\}=\tfrac{1}{2}\sum_n w_n(\phi_n{}^\dagger\alpha^\mu\phi_n)$. However, this infinite sum over n formally vanishes, as the c-number functions ϕ_n come in charge-conjugate pairs (see reference 11) for which $(\phi_n{}^\dagger\alpha^\mu\phi_n)$ has the same value, but w_n has opposite values.

[13] P. A. M. Dirac, "Théorie du Positron," *Structure et Propriétés des Noyaux Atomiques*, Rapports et Discussions du 7me Conseil de Physique tenu à Bruxelles du 22 au 29 octobre 1933. Institut International de Physique Solvay. (Gauthier Villars, Paris 1934), pp. 203–212 (Library of Congress QC1-I6-1933). See also P. A. M. Dirac, Proc. Cambridge Phil. Soc. **30**, 150 (1934).

[14] W. H. Furry and J. R. Oppenheimer, Phys. Rev. **45**, 245 (1934).

derived, but were based on an analogy. We could have made a different guess at an energy density tensor by writing

$$^{\mathrm{H}}\mathfrak{T}^{\lambda\mu}={}^{\mathrm{H}}\mathfrak{T}_F{}^{\lambda\mu}+\tfrac{1}{2}\hbar c\{{}^{\circ}\bar{\psi}(\gamma^\lambda\nabla^\mu+\gamma^\mu\nabla^\lambda){}^{\circ}\psi\} \tag{20}$$

with

$$^{\mathrm{H}}\mathfrak{T}_F{}^{\lambda\mu}=(4\pi)^{-1}\{{}^{\circ}\mathfrak{F}^{\nu\mu}\,{}^{\circ}\mathfrak{F}_\nu{}^{\cdot\lambda}-\tfrac{1}{4}g^{\lambda\mu}\,{}^{\circ}\mathfrak{F}_{\nu\sigma}{}^{\circ}\mathfrak{F}^{\nu\sigma}\}. \tag{21}$$

We have expressed $\mathfrak{T}^{\lambda\mu}$ here as a q-number in Heisenberg representation with the help of the variables $\mathfrak{F}^{\lambda\mu}$ (transverse electromagnetic field, with $\mathfrak{F}_{10}=\mathfrak{F}^{01}=\mathfrak{E}_x$, $\mathfrak{F}_{23}=\mathfrak{B}_x$, etc.) and ψ (matter field) in interaction representation. Since the transformation (G-I:37) with (G-I:34–35) between both representations, or

$$^{\mathrm{H}}Q(t)={}^{\circ}Q(t)+(i\hbar)^{-1}\int_{-\infty}^{t}dt_1[{}^{\circ}Q(t);{}^{\circ}\mathcal{W}(t_1)]$$
$$+(i\hbar)^{-2}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2[[{}^{\circ}Q(t);{}^{\circ}\mathcal{W}(t_1)];$$
$$\times{}^{\circ}\mathcal{W}(t_2)]+\cdots, \tag{22}$$

is expressed by (G-I:10) with (G-I:3, 6, 7) in terms of gauge-independent variables, Eqs. (20)–(21) define a gauge-independent quantity, which has the following properties:

1. This gauge-independent quantity $^{\mathrm{H}}\mathfrak{T}^{\lambda\mu}$ is a manifestly covariant symmetric tensor.

2. Due to the field equations (G-I:29) in interaction representation,

$$\left.\begin{aligned}&\nabla_\mu{}^{\circ}\mathfrak{F}^{\nu\mu}=0,\quad \nabla_\mu{}^{\circ}\mathfrak{F}_{\nu\lambda}+\nabla_\nu{}^{\circ}\mathfrak{F}_{\lambda\mu}+\nabla_\lambda{}^{\circ}\mathfrak{F}_{\mu\nu}=0,\\&\gamma^\mu\nabla_\mu{}^{\circ}\psi=-\kappa\,{}^{\circ}\psi,\quad \nabla_\mu{}^{\circ}\bar{\psi}\gamma^\mu=+\kappa\,{}^{\circ}\bar{\psi},\\&\Box\,{}^{\circ}\psi=\kappa^2\,{}^{\circ}\psi,\end{aligned}\right\} \tag{23}$$

with

$$\mathfrak{F}^{\lambda\mu}=-\mathfrak{F}^{\mu\lambda};\quad \mathfrak{R}\{\bar{\psi}\gamma^\lambda\psi\}=0=\mathfrak{R}\{(\nabla_\mu\bar{\psi})\gamma^\lambda(\nabla^\mu\psi)\}, \tag{24}$$

the tensor $\mathfrak{T}^{\lambda\mu}$ satisfies conservation laws

$$\nabla_\mu{}^{\mathrm{H}}\mathfrak{T}^{\lambda\mu}=0,\quad\text{thence}\quad \nabla_\mu\langle\mathfrak{T}^{\lambda\mu}\rangle_{\mathrm{Av}}=0. \tag{25}$$

3. Because of (25), the integrals of the density components $\mathfrak{T}^{\lambda 0}$ over space give constants of motion:

$$c\,{}^{\mathrm{H}}\mathcal{P}^\lambda=\int{}^{\mathrm{H}}\mathfrak{T}^{\lambda 0}d^3\mathbf{x},\quad\text{with}\quad \partial{}^{\mathrm{H}}\mathcal{P}^\lambda/\partial t=0. \tag{26}$$

4. The energy $c\,{}^{\mathrm{H}}\mathcal{P}^0$ in Heisenberg representation thus defined is, by (20)–(21), (G-I:29), and (G-I:31), equal to ${}^{\circ}\mathcal{H}_0$, and therefore, by (G-I:54) with (G-I:56) and (1)–(4), equal to the energy as calculated from $^{\mathrm{H}}T^{00}$. Remember that (G-I:54) was derived under the tacit assumption of vanishing of the interaction at $t=-\infty$ (compare footnote 13 of reference 1). In fact, by (G-I:52) and (G-I:37) with (G-I:33), and by the fact that ${}^{\circ}\mathcal{H}_0$ and $^{\mathrm{H}}\mathcal{H}$ both are integrals of motion, we have

$$^{\mathrm{H}}\mathcal{H}(t)={}^{\mathrm{H}}\mathcal{H}(-\infty)={}^{\circ}\mathcal{H}(-\infty)={}^{\circ}\mathcal{H}_0(-\infty)+{}^{\circ}\mathcal{W}(-\infty)$$
$$«=»\ {}^{\circ}\mathcal{H}_0(-\infty)={}^{\circ}\mathcal{H}_0(t). \tag{27}$$

5. By a similar argument, all components of $^{\mathrm{H}}T^{\lambda\mu}(x)$ and of $^{\mathrm{H}}\mathfrak{T}^{\lambda\mu}(x)$ $[\equiv{}^{\circ}T_{\mathrm{free}}{}^{\lambda\mu}(x)]$ are completely identical *at* $t=-\infty$, if we assume that the interaction constant e vanishes at $t=-\infty$. ("Interaction switched on adiabatically since the infinite past.") Here, "free" refers to omission of all interaction terms, including the difference between $\mathbf{E}$ and $\mathfrak{E}$.

6. Thence, also the momentum vectors $^{\mathrm{H}}\mathcal{P}^k$ calculated according to (26) from $^{\mathrm{H}}T^{k0}$ and from $^{\mathrm{H}}\mathfrak{T}^{k0}$ will then be equal at $t=-\infty$, and therefore always, as they both are constants of motion:

$$c\,{}^{\mathrm{H}}\mathcal{P}^\lambda=\int{}^{\mathrm{H}}\mathfrak{T}^{\lambda 0}d^3\mathbf{x}=\int{}^{\mathrm{H}}T^{\lambda 0}d^3\mathbf{x}. \tag{28}$$

Because of the above properties, $^{\mathrm{H}}\mathfrak{T}^{\lambda\mu}$ serves most purposes of a total energy density tensor, better than for instance the "canonical" energy density tensor.[6] However, it does not prove equality of $\mathfrak{T}^{\lambda\mu}$ and $T^{\lambda\mu}$.

The physical interpretation of $\mathfrak{T}^{\lambda\mu}$ is complicated by the fact that in Heisenberg representation it was expressed in terms of field variables (and therefore of creation and annihilation matrices) in interaction representation.[15] While we leave unanswered the question whether $\mathfrak{T}^{\lambda\mu}$ or $T^{\lambda\mu}$ is to be regarded as source of the gravitational field[3] (although the correctness of $\mathfrak{T}^{\lambda\mu}$ in this regard seems somewhat dubious), it must be admitted that the definition (20)–(21) of $\mathfrak{T}^{\lambda\mu}$ has the advantage of permitting an extremely simple subtraction of its covariantly defined vacuum value, by a simple rearrangement of factors as suggested by Furry and Oppenheimer[14] after splitting all field variables in interaction representation into positive and negative frequency parts according to Schwinger. After this has been done, the photon part $^{\mathrm{H}}\mathfrak{T}_F{}^{\lambda\mu}$ (21) will contain only products ${}^{\circ}\mathfrak{F}^{(+)}\,{}^{\circ}\mathfrak{F}^{(+)}$, ${}^{\circ}\mathfrak{F}^{(-)}\,{}^{\circ}\mathfrak{F}^{(+)}$, and ${}^{\circ}\mathfrak{F}^{(-)}\,{}^{\circ}\mathfrak{F}^{(-)}$. Then, the expectation value of $\mathfrak{T}_F{}^{\lambda\mu}$ after this subtraction is supposed to vanish completely by the new covariant auxiliary condition in Heisenberg representation ${}^{\circ}\mathfrak{F}_{\lambda\mu}{}^{(+)}\,\Phi=0=\Phi^*\,{}^{\circ}\mathfrak{F}_{\lambda\mu}{}^{(-)}$ proposed recently by the author,[16] and all energy and momentum is then due to the "matter" terms in (20). It should be kept in mind, though, that these "matter" terms would still depend on photon variables when expressed, by Eq. (22) or

[15] Re-interpretation of the q-number occupation numbers derived from field variables in interaction representation, as representing the numbers of particles in Heisenberg representation, was shown by F. J. Dyson, Phys. Rev. **75**, 486 (1949), to oversimplify the theory such as to completely eliminate all interactions. (Compare p. 489, same article.)

[16] F. J. Belinfante, Phys. Rev. **84**, 644 (1951).

by its inverse

$$^{\circ}q(t)={}^{\mathrm{H}}q(t)+\sum_{n=1}^{\infty}\left(\frac{i}{\hbar}\right)^n\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\cdots\int_{-\infty}^{t_{n-1}}dt_n$$
$$\times[[\cdots[{}^{\mathrm{H}}q(t);{}^{\mathrm{H}}\mathcal{W}(t_1)];{}^{\mathrm{H}}\mathcal{W}(t_2)];\cdots{}^{\mathrm{H}}\mathcal{W}(t_n)],\quad(29)$$

in terms of variables in the Heisenberg representation used for describing the physical state by Φ. Moreover, as discussed earlier,[16] it is often convenient for practical reasons to ignore the new auxiliary condition notwithstanding its fundamental significance, and to do as if the photon terms (21) describe the energy density tensor of incident free photons.

Perhaps one may want to argue that the tensors $T^{\lambda\mu}$ and $\mathfrak{T}^{\lambda\mu}$ should be expected to be identical. If we call their difference $\mathfrak{X}^{\lambda\mu}$, then ${}^{\mathrm{H}}\mathfrak{X}^{\lambda\mu}(-\infty)\equiv 0$ by the property (5) above in case of vanishing of interaction at $t=-\infty$. On the other hand, its time derivative

$$\partial\,{}^{\mathrm{H}}\mathfrak{X}^{\lambda\mu}(x)/\partial t=(i\hbar)^{-1}[{}^{\mathrm{H}}\mathfrak{X}^{\lambda\mu}(x)\,{}^{\mathrm{H}}\mathcal{H}-{}^{\mathrm{H}}\mathcal{H}\,{}^{\mathrm{H}}\mathfrak{X}^{\lambda\mu}(x)]\quad(30)$$

vanishes wherever ${}^{\mathrm{H}}\mathfrak{X}^{\lambda\mu}(x)$ itself vanishes. From this, one might be inclined to conclude the equality of $T^{\lambda\mu}$ and $\mathfrak{T}^{\lambda\mu}$ throughout space-time.[17]

It would have been more convincing, however, if equality of $T^{\lambda\mu}$ and $\mathfrak{T}^{\lambda\mu}$ could have been shown by explicit calculation, using Eq. (22) or (29), since the vanishing of $\mathfrak{X}^{\lambda\mu}$ at $t=-\infty$ was based on the vanishing of e there, and if yet e can differ from zero at later time, then also $\mathfrak{X}^{\lambda\mu}$ can, with some discontinuity in some (higher order) derivative of $\mathfrak{X}^{\lambda\mu}$. Moreover, one may well prefer, in defining the interaction representation by (G-I:33), to postulate the vanishing of the interaction terms in the energy density in the limit $t\to-\infty$ only, without $\mathfrak{X}^{\lambda\mu}$ being zero for any finite time, so that the above argument is not conclusive.

V. DISCUSSION

While the validity of conservation laws and of covariance for $T^{\lambda\mu}$ and $\mathfrak{T}^{\lambda\mu}$ seems to be satisfactory, they both have the disadvantage of taking infinite values (even after subtraction of their vacuum value) in states, in which a finite number of occupation numbers have nonvanishing values. For $T^{\lambda\mu}$ this is easily seen by calculating the expectation value of the term $\mathbf{E}_{\mathrm{II}}{}^2/8\pi$ in Eq. (2) (with (G-I:44)) in state with one free electron present. If $\Psi(\mathbf{x})$ is the c-number one-electron wave function of such electron, Fock's rules of second quantization[18] give, for one-electron theory and *without* the symmetrization (7) of the charge density,

$$\langle\rho(\mathbf{x}')\rho(\mathbf{x}'')\rangle_{\mathrm{Av}}=e^2\Psi^{\dagger}(\mathbf{x}')\delta(\mathbf{x}'-\mathbf{x}'')\Psi(\mathbf{x}''),\quad(31)$$

so that

$$\left\langle\frac{\mathbf{E}_{\mathrm{II}}(\mathbf{x})^2}{8\pi}\right\rangle_{\mathrm{Av}}=\frac{1}{8\pi}\int'\int''\left[\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right]\cdot\left[\nabla\frac{1}{|\mathbf{x}-\mathbf{x}''|}\right]\langle\rho(\mathbf{x}')\rho(\mathbf{x}'')\rangle_{\mathrm{Av}}\quad(32)$$
$$=(e^2/8\pi)\int'|\Psi(\mathbf{x}')|^2r^{-4}d^3\mathbf{x}'.$$

In a point where $|\Psi(\mathbf{x})|^2\neq 0$, this integral diverges as $\frac{1}{2}e^2|\Psi(\mathbf{x})|^2\int_0 r^{-2}dr$.

This one-electron theory is of course meaningless, but a similar slightly more involved treatment in position theory *with* use of Eq. (7) leads to the logarithmically divergent electrostatic self-energy calculated by Weisskopf.[19]

For $\mathfrak{T}^{\lambda\mu}$, the divergence of $\langle\mathfrak{T}^{\lambda\mu}\rangle_{\mathrm{Av}}$ is due to the commutators of ${}^{\mathrm{H}}T_{\mathrm{free}}{}^{\lambda\mu}(t)$ with ${}^{\mathrm{H}}\mathcal{W}(t_1)$, when Eq. (29) is used for expressing ${}^{\mathrm{H}}\mathfrak{T}^{\lambda\mu}(={}^{\circ}T_{\mathrm{free}}{}^{\lambda\mu})$ in terms of field variables in Heisenberg representation.

To avoid the electrostatic self-energy difficulties, one would be tempted to rearrange ${}^{\circ}\psi^{(+)}$ and ${}^{\circ}\psi^{(-)}$ factors in the expression $({}^{\circ}\mathbf{E}_{\mathrm{II}}{}^2/8\pi)$ occurring in the interaction operator ${}^{\circ}W$ (G-I:10). However, this alone would disturb all our proofs of the integrability of the generalized Schrödinger equation[1] and of the covariance of the energy density tensor ${}^{\mathrm{H}}T^{\lambda\mu}$. In a following paper, we shall discuss some possibilities of eliminating self-interaction effects without sacrificing covariance or integrability.[20]

APPENDIX A

Using Eqs. (3), (G-I:59), and (G-I:60), we find

$$\nabla_\mu\,{}^{\mathrm{H}}T_I{}^{0\mu}=\mathbf{j}\cdot\mathfrak{E}-(\nabla_0\mathbf{j}+\nabla\rho)\,\mathfrak{A}.\quad(\mathrm{A}.1)$$

(All quantities in Heisenberg representation.)

The anticommutation relations for ψ and $\psi^{\dagger}$ yield, for any Dirac matrix ω, and $t=t'$,

$$\psi^{\dagger}(\mathbf{x})\omega\psi(\mathbf{x}')+\psi^{\mathrm{T}}(\mathbf{x}')\omega^{\mathrm{T}}\psi^*(\mathbf{x})=(\mathrm{spur}\;\omega)\delta(\mathbf{x}-\mathbf{x}'),\quad(\mathrm{A}.2)$$

which we shall assume to vanish even for $\mathbf{x}=\mathbf{x}'$, if spur $\omega=0$. For instance,

$$\psi^{\dagger}\alpha\psi+\psi^{\mathrm{T}}\alpha^{\mathrm{T}}\psi^*=0.\quad(\mathrm{A}.3)$$

Similarly, we assume the vanishing of expressions like

$$(\nabla\psi)^{\dagger}\cdot\alpha\psi+\psi^{\mathrm{T}}\alpha^{\mathrm{T}}\cdot\nabla\psi^*=\{\nabla\,\delta(\mathbf{x}-\mathbf{x}')\cdot\mathrm{spur}\;\alpha\}_{\mathbf{x}'=\mathbf{x}}=0.\quad(\mathrm{A}.4)$$

By a similar argument

$$\tfrac{1}{2}e\mathfrak{R}\{\psi^{\dagger}\mathbf{E}_{\mathrm{II}}\cdot\alpha\psi\}=\tfrac{1}{4}(\mathbf{E}_{\mathrm{II}}\cdot\mathbf{j}+\mathbf{j}\cdot\mathbf{E}_{\mathrm{II}}),\quad(\mathrm{A}.5)$$

[17] A somewhat similar argument is used by G. Wentzel in his book *Einführung in die Quantentheorie der Wellenfelder* (F. Deuticke, Vienna, 1943) in Sec. 18, on page 137, for proving the vanishing of $(\Omega\Phi)$ for all times t_n. See also the translation *Quantum Theory of Fields* (Interscience Publishers, New York, New York, 1949), p. 143.

[18] V. Fock, Z. Physik **75**, 622 (1932); in particular Eq. (11a) of its second part, on page 639.

[19] V. F. Weisskopf, Phys. Rev. **56**, 72 (1939), and A. Pais, Proc. Roy. Acad. Amsterdam (Verhandelingen, 1e sectie) **19**, 5 (1947), in particular page 25, footnote.

[20] See also F. J. Belinfante, Phys. Rev. **82**, 767(A) (1951); Prog. Theor. Phys. **6**, 202 (1951); and F. J. Belinfante and J. S. Lomont, Phys. Rev. **83**, 225(A) (1951).

where we have taken $\mathbf{E}_{\parallel}$ first at a different point $\mathbf{x}'$, then used Eqs. (6), (G-I:48) for $[\mathbf{E}_{\parallel}';\psi]$ and its conjugate for $[\psi^*;\mathbf{E}_{\parallel}']$, and finally Eq. (7) and Eq. (A.3) above, before equating $\mathbf{x}'$ to $\mathbf{x}$. Similarly one shows

$$\tfrac{1}{2}e\Re\{\psi^\dagger\mathbf{E}_{\parallel}\psi\}-\tfrac{1}{4}\{\mathbf{E}_{\parallel}\rho+\rho\mathbf{E}_{\parallel}\}=\tfrac{1}{4}e^2(\psi^\dagger\psi+\psi^{\mathrm{T}}\psi^*)[\nabla'(1/r)]_{\mathbf{x}'=\mathbf{x}}\text{ «=» }0 \quad (\mathrm{A.6})$$

"because of symmetry." Other symmetry arguments yield

$$[d\delta(x)/dx]_{x=0}=0, \quad (\mathrm{A.7})$$

so that

$$\psi^\dagger\nabla\psi+(\nabla\psi)^{\mathrm{T}}\psi^*=0. \quad (\mathrm{A.8})$$

Equation (6) with $\varphi=V\psi+\psi V$ and (A.3) also yields

$$\Re\{\psi^\dagger(i\boldsymbol{\alpha})(V\psi+\psi V)\}=\tfrac{1}{4}i[(\psi^\dagger\boldsymbol{\alpha}\psi+\psi^{\mathrm{T}}\boldsymbol{\alpha}^{\mathrm{T}}\psi^*);V]=0, \quad (\mathrm{A.9})$$

while

$$\Re\{i\psi^\dagger(\boldsymbol{\alpha}\cdot\mathfrak{A})(\mathfrak{A}\cdot\boldsymbol{\alpha}\psi)\}=0 \quad (\mathrm{A.10})$$

follows from (6) alone.

Equation (G-I:57) and conjugate, with Eqs. (A.9), (A.10), (6), and $2e\nabla=-(e/i\hbar c)(\boldsymbol{\alpha}H_0+H_0\boldsymbol{\alpha})$, yield, in Heisenberg representation,

$$\nabla_0\mathbf{j}+\nabla\rho=\Re\{\psi^\dagger(2e/i\hbar c)\boldsymbol{\alpha}H_0\psi\}+\Re\{2e\psi^\dagger\nabla\psi\}=\Re\{\psi^\dagger(e/i\hbar c)[\boldsymbol{\alpha};H_0]\psi\}. \quad (\mathrm{A.11})$$

Finally, from Eqs. (5)–(6) follows

$$\Re\{(H_0\psi)^\dagger(-H_0/i\hbar c)\psi\}=0=\Re\{(\nabla\psi)^\dagger\cdot i\hbar c\nabla\psi\} \quad (\mathrm{A.12})$$

and, with $\triangle\equiv\nabla^2$,

$$\Re\{\psi^\dagger(H_0/i\hbar c)(H_0\psi)\}=\Re\{\psi^\dagger(H_0^2/i\hbar c)\psi\}=\Re\{\psi^\dagger i\hbar c\triangle\psi\}. \quad (\mathrm{A.13})$$

By Eqs. (4), (G-I:57) and conjugate, (A.12) and (A.13), and (6),

$$\begin{aligned}\nabla_\mu{}^{\mathrm{H}}T_{\mathrm{e}}{}^{0\mu}&=\nabla_0\Re\{\psi^\dagger H_0\psi\}+\textstyle\sum_k\nabla_k\Re\{-\psi^\dagger i\hbar c\nabla_k\psi\}\\&=\Re\{\psi^\dagger(e/i\hbar c)[\mathfrak{A}\cdot\boldsymbol{\alpha};H_0]\psi\}\\&\quad+\Re\{\psi^\dagger(e/2i\hbar c)[H_0;V]\psi\}\\&\quad+(e/8i\hbar c)[\{\psi^\dagger(H_0+H_0{}^\dagger)\psi\\&\qquad+\psi^{\mathrm{T}}(H_0{}^{\mathrm{T}}+H_0{}^*)\psi^*\};V].\end{aligned} \quad (\mathrm{A.14})$$

By use of Eq. (5), (A.3) with β replacing $\boldsymbol{\alpha}$, the definitions of $\nabla^\dagger$ and ∇^{T} (footnote 10), and Eq. (A.4) and conjugate, in the last term of Eq. (A.14) only terms with ∇ from H_0 operating on V are left, which yield, by (G-I:6) and by (7), $\frac{1}{4}(\mathbf{E}_{\parallel}\cdot\mathbf{j}+\mathbf{j}\cdot\mathbf{E}_{\parallel})$. The term in (A.14) with $[H_0;V]$ yields by (5), (G-I:6), and (A.5) the same amount. Adding (A.1), with (A.11) substituted for its last terms, to Eq. (A.14), we thus find, by Eq. (1),

$$\nabla_\mu{}^{\mathrm{H}}T_{\mathrm{m}}{}^{0\mu}=\mathbf{j}\cdot\mathfrak{E}+\tfrac{1}{2}(\mathbf{E}_{\parallel}\cdot\mathbf{j}+\mathbf{j}\cdot\mathbf{E}_{\parallel}), \quad (\mathrm{A.15})$$

which gives Eq. (10) by (G-I:44).

APPENDIX B

With the abbreviating notation of reference 1, and by Eqs. (8), (G-I:14), (G-I:22) with note between (G-I:41) and (G-I:42), Eq. (G-I:50) and conjugate, and Eq. (3), we find, all in Heisenberg representation,

$$\begin{aligned}\delta\,{}^{\mathrm{H}}T_{I'}{}^{00}&=\tfrac{1}{2}e\Re\{\psi^\dagger(\mathbf{b}\cdot\boldsymbol{\alpha})(\psi V+V\psi)\}\\&\quad-\tfrac{1}{2}e\textstyle\int'(\mathbf{b}\cdot\mathbf{r}')[\nabla'(1/r)]\\&\qquad\cdot\Re\{\psi^\dagger(\mathbf{j}'\psi+\psi\mathbf{j}')\},\end{aligned} \quad (\mathrm{B.1a})$$

$$\delta\,{}^{\mathrm{H}}T_I{}^{0k}=(\mathbf{b}\cdot\mathbf{j})\mathfrak{A}_k-\rho\delta\mathfrak{A}_k, \quad (\mathrm{B.1b})$$

$$\delta\,{}^{\mathrm{H}}T_I{}^{kl}=\tfrac{1}{2}\rho(b_k\mathfrak{A}_l+b_l\mathfrak{A}_k)-\tfrac{1}{2}(j_k\delta\mathfrak{A}_l+j_l\delta\mathfrak{A}_k). \quad (\mathrm{B.1c})$$

Temporarily putting $\mathbf{j}=\mathbf{j}_\perp+\nabla\xi$ (with div $\mathbf{j}_\perp=0$), we find by some integrations by parts, with use of (G-I:20),

$$\begin{aligned}\textstyle\int'(\mathbf{b}\cdot\mathbf{r}')(\mathbf{j}'\cdot\nabla')(1/r)&=-\textstyle\int'\xi'(\mathbf{b}\cdot\nabla')(1/r)-\int'\mathbf{b}\cdot\mathbf{j}_\perp'/r\\&=\textstyle\int'(\mathbf{j}_{\parallel}'-\mathbf{j}_\perp')\cdot\mathbf{b}/r.\end{aligned} \quad (\mathrm{B.2})$$

Further, by Eqs. (5)–(6),

$$-\Re\{\psi^\dagger\boldsymbol{\alpha}H_0\psi\}=\Re\{\psi^\dagger i\hbar c\nabla\psi\}-\mathrm{curl}\,\tfrac{1}{2}\hbar c\Re\{\psi^\dagger\boldsymbol{\sigma}\psi\}. \quad (\mathrm{B.3})$$

Then, by Eqs. (8), (G-I:15), (G-I:50), and (G-I:57) and their conjugates, (G-I:58), (G-I:63), (G-I:60), (G-I:20), and Eqs. (B.2), (B.3), (4), (3), and (1), one finds

$$\begin{aligned}\delta\,{}^{\mathrm{H}}T_{\mathrm{e}'}{}^{00}&=-2\textstyle\sum_k b_k\,{}^{\mathrm{H}}T_{\mathrm{m}}{}^{0k}-\tfrac{1}{2}e\Re\{\psi^\dagger(\mathbf{b}\cdot\boldsymbol{\alpha})(V\psi+\psi V)\}\\&\quad+e\textstyle\int'(\mathbf{b}\cdot\mathbf{r}')[\nabla'(1/r)]\cdot\Re\{\psi^\dagger\mathbf{j}'\psi\}.\end{aligned} \quad (\mathrm{B.4a})$$

By Eqs. (4), (G-I:15), (G-I:50), (G-I:51), (5), (6), (7), by relations like $\alpha_k\alpha_l+\alpha_l\alpha_k=2\delta_{kl}$ and $\sigma_{kl}\alpha_m-\alpha_m\sigma_{kl}=2i(\delta_{mk}\alpha_l-\delta_{ml}\alpha_k)=\alpha_l\sigma_{km}+\sigma_{km}\alpha_l+2\sigma_{kl}\alpha_m$, and by Eqs. (3), (4), and (1), we find

$$\begin{aligned}\delta\,{}^{\mathrm{H}}T_{\mathrm{e}}{}^{0k}&=-b_k\,{}^{\mathrm{H}}T_{\mathrm{m}}{}^{00}-\textstyle\sum_l b_l\,{}^{\mathrm{H}}T_{\mathrm{e}}{}^{kl}-\tfrac{1}{2}b_k\rho V\\&\quad+\rho\textstyle\int'(\mathbf{b}\cdot\mathfrak{E}')\nabla_k'(1/4\pi r)\\&\quad+\tfrac{1}{2}e\Re\{\psi^\dagger\textstyle\int'\rho'(\mathbf{b}\cdot\mathbf{r}')[\nabla_k'(1/r)]\psi\}\\&\quad+\tfrac{1}{2}\textstyle\sum_l b_l(\mathfrak{A}_lj_k-\mathfrak{A}_kj_l).\end{aligned} \quad (\mathrm{B.4b})$$

By Eqs. (4), (G-I:15), (G-I:50), and (G-I:57) and conjugate equations, by $\alpha_k\alpha_l=\delta_{kl}+i\sigma_{kl}$, and by Eqs. (6), (7), and (G-I:7), we find

$$\begin{aligned}\delta\,{}^{\mathrm{H}}T_{\mathrm{e}}{}^{kl}&=[\![-b_l\,{}^{\mathrm{H}}T_{\mathrm{e}}{}^{0k}+\tfrac{1}{2}b_l\rho\mathfrak{A}_k\\&\quad+\tfrac{1}{2}j_l\textstyle\int'(\mathbf{b}\cdot\mathfrak{E}')\nabla_k'(1/4\pi r)-\tfrac{1}{4}j_lb_kV\\&\quad+\tfrac{1}{4}e\textstyle\int'(\mathbf{b}\cdot\mathbf{r}')[\nabla_k'(1/r)]\Re\{\psi^\dagger\rho'\alpha_l\psi\}]\!]\\&\quad+[\![\text{same with } k,l \text{ interchange}]\!].\end{aligned} \quad (\mathrm{B.4c})$$

Now, by (G-I:47), in Heisenberg representation

$$\begin{aligned}\delta\mathfrak{A}_k&=-\textstyle\int'(\mathbf{b}\cdot\mathbf{r}')(\mathbf{E}'\cdot\nabla')\nabla_k'(1/4\pi r)\\&=\textstyle\int'(\mathbf{b}\cdot\mathbf{E}')\nabla_k'(1/4\pi r)+\int'(\mathbf{b}\cdot\mathbf{r}')\rho'\nabla_k'(1/r).\end{aligned} \quad (\mathrm{B.5})$$

By (G-I:44) and (G-I:6), the first term of the last member of (B.5) equals

$$\textstyle\int'(\mathbf{b}\cdot\mathfrak{E}')\nabla_k'(1/4\pi r)+\int'V'(\mathbf{b}\cdot\nabla')\nabla_k'(1/4\pi r). \quad (\mathrm{B.6})$$

By (G-I:7) and (G-I:65),

$$\int' r_l' \rho' \nabla_k'(1/r) = -\delta_{kl} \int' \rho'/r - \int' (r_l'/r) \nabla_k' \rho'$$
$$= -\delta_{kl} V + \int' (r_l'/4\pi r) \triangle' \nabla_k' V'$$
$$= -\delta_{kl} V + 2\int' [\nabla_k' V'] \nabla_l'(1/4\pi r). \quad \text{(B.7)}$$

Thence

$$\int' V' \nabla_k' \nabla_l'(1/4\pi r) = -\tfrac{1}{2}\delta_{kl} V - \tfrac{1}{2}\int' r_l' \rho' \nabla_k'(1/r), \quad \text{(B.8)}$$

so that, by (B.5) with (B.6) and (B.8),

$$\delta \mathfrak{A}_k = \int' (\mathbf{b} \cdot \mathfrak{E}') \nabla_k'(1/4\pi r) - \tfrac{1}{2} b_k V + \tfrac{1}{2} \int' (\mathbf{b} \cdot \mathbf{r}') \rho' \nabla_k'(1/r). \quad \text{(B.9)}$$

Now adding (B.1a, b, c) to (B.4a, b, c) and making use of Eqs. (8), (1), (3), (B.9), and finally of Eqs. (7), (G-I:48), and (G-I:20), we find

$$\delta\, {}^{\mathrm{H}}T_{\mathrm{m}}^{00} + 2 \textstyle\sum_k b_k\, {}^{\mathrm{H}}T_{\mathrm{m}}^{0k}$$
$$= \tfrac{1}{2} e \int' (\mathbf{b} \cdot \mathbf{r}') [\boldsymbol{\nabla}'(1/r)] \cdot \mathfrak{R}\{\psi^\dagger [\mathbf{j}'; \psi]\}$$
$$= -\tfrac{1}{2} e \int' (\mathbf{b} \cdot \mathbf{r}')\, \delta(\mathbf{r}')\, \mathbf{j} \cdot \boldsymbol{\nabla}'(1/r) = 0; \quad \text{(B.10a)}$$

$$\delta\, {}^{\mathrm{H}}T_{\mathrm{m}}^{0k} + b_k\, {}^{\mathrm{H}}T_{\mathrm{m}}^{00} + \textstyle\sum_l b_l\, {}^{\mathrm{H}}T_{\mathrm{m}}^{kl}$$
$$= \tfrac{1}{2} e \int' (\mathbf{b} \cdot \mathbf{r}') [\nabla_k'(1/r)] \mathfrak{R}\{\psi^\dagger [\rho'; \psi]\}$$
$$= -\tfrac{1}{2} e \int' (\mathbf{b} \cdot \mathbf{r}')\, \delta(\mathbf{r}')\, \rho\, \nabla_k'(1/r) = 0; \quad \text{(B.10b)}$$

$$\delta\, {}^{\mathrm{H}}T_{\mathrm{m}}^{kl} + b_l\, {}^{\mathrm{H}}T_{\mathrm{m}}^{0k} + b_k\, {}^{\mathrm{H}}T_{\mathrm{m}}^{0l}$$
$$= [\![\tfrac{1}{4} e \int' (\mathbf{b} \cdot \mathbf{r}') [\nabla_k'(1/r)] \mathfrak{R}\{\psi^\dagger \alpha_l [\rho'; \psi]\}]\!] + [\![k, l \text{ interchanged}]\!]$$
$$= -\tfrac{1}{4} e \int' (\mathbf{b} \cdot \mathbf{r}')\, \delta(\mathbf{r}')\, \{ j_l \nabla_k' + j_k \nabla_l' \} (1/r) = 0. \quad \text{(B.10c)}$$

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Energy Distribution, Drift Velocity, and Temperature of Slow Electrons in Helium and Argon

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The energy distribution, and the drift velocity and electron temperature which are closely dependent on the distribution, were calculated for slow electrons in He and A under fields ranging from $X/p=1$ to 4 volts/cm mm. The calculations were based on the theory developed by Holstein. The diffusion cross sections of the gases for electrons which account for the effect of elastic collisions were computed from the scattering data of Ramsauer and Kollath and of Normand. The effect of excitation collisions was taken into account by comparing the results for three representative values of excitation cross section: $Q_{ex}=0$ and $Q_{ex}=\infty$ which form the limiting boundaries and $Q_{ex}=$a constant obtained from Maier-Leibnitz. Although a considerable percentage of the electrons were to be found in the excitation region for $Q_{ex}=0$, the results for $Q_{ex}=$constant were nearly the same as those for $Q_{ex}=\infty$. In the case of small fields, therefore, the electron energy distribution in the elastic region and related quantities may be calculated with Q_{ex} assumed infinite.

A complete set of curves are given below illustrating some of the properties of the calculated quantities. The agreement between experiment and the curves obtained with the scattering data of Ramsauer and Kollath is good, while the curves obtained from the corresponding data of Normand indicate that his cross-section values are too low.

I. INTRODUCTION

BECAUSE of the lack of a complete theory and sufficient cross-section data, the energy distribution and related parameters for electrons in gases have heretofore been calculated under various simplifying assumptions. These approximations have resulted in considerable discrepancies between the theoretical results and experiment. In the discussion below, the energy distribution, drift velocity, and electron temperature of slow electrons in helium and argon are more exactly calculated for fields up to $X/p=4$ volts/cm mm. The following factors are considered, some or all of which have not been taken into account in previous calculations: (a) an electron loses a small fraction of its energy by recoil in elastic collisions with molecules of finite mass, (b) the effect of excitation collisions cannot be neglected, (c) the elastic cross section of gases for electrons is a function of the electron energy, (d) the type of elastic cross section that should be used in computing the energy distribution is not the Ramsauer cross section but, more exactly, the diffusion cross section. The latter cross section is also referred to as the cross section for momentum transfer.

The calculations below are based on the theoretical investigations of Holstein,[1] whose work represents the most inclusive theory to date on the energy distribution of electrons. Values for the diffusion cross section could not be found in the literature and were therefore calculated from existing angular scattering data. Energy distribution, drift velocity, and electron temperature curves were obtained for three values of excitation cross section: $Q_{ex}=0$, $Q_{ex}=\infty$ and $Q_{ex}=$a reasonable constant. For the small fields considered here, the calculated curves for the latter two cases were found to be nearly coincident. As a result, for the calculation of gas parameters depending on the electron energy distribution in the non-excitation region, Q_{ex} may be

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[1] T. Holstein, Phys. Rev. 70, 367 (1946).