A New Covariant Auxiliary Condition for Quantum Electrodynamics

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The covariant auxiliary condition Eq. (3) or (15) is assumed to be valid for all states occurring in nature. Such special assumptions as the condition for a "photon vacuum" of the type of the noncovariant Eq. (18) then become superfluous. The meaning of the new auxiliary condition (15) is that the expectation value of the electromagnetic field is equal to that of the retarded field from charged matter: "All light has once been emitted." If in some discussion of properties of a particular photon one wants to ignore the source of that particular photon (without denying its existence), one can ignore Eqs. (3) and (15) in such a special case, replacing them by an appropriate "practical" boundary condition. In self-energy calculations one sometimes uses a "photon vacuum" condition of the type of Eq. (20). Such approximation to (15) gives correct results up to second but not to fourth powers of e.

I. AUXILIARY CONDITION FOR TRANSVERSE PHOTONS IN HEISENBERG REPRESENTATION

IN Fermi's quantum electrodynamics, the Lorentz condition ensures equality of the values of div $\mathbf{E}_{\shortparallel}$ and $4\pi\rho$, and thus guarantees that all charges are surrounded by their longitudinal electric fields. In "gauge-independent" (G-I) quantum electrodynamics[1,2] the same is guaranteed by the definition of the longitudinal field.

It is however a common experience that there exists a second tie between matter and the Maxwell field, which may be expressed by saying that also each transverse wave that is observed was once emitted by a source. Although the most general solution of Maxwell's equations may be described by

$$A^{\mu}(\mathbf{x},t)=A_{\mathrm{ret}}{}^{\mu}(\mathbf{x},t)+A_{\mathrm{o}}{}^{\mu}(\mathbf{x},t), \tag{1}$$

where $A_{\mathrm{o}}{}^{\mu}$ is a solution of the homogeneous equations, we use to assume that in reality $A_{\mathrm{o}}{}^{\mu}$ or at least its physically significant transverse part $\mathcal{A}_{\mathrm{o}}{}^{\mu}$ (with spatial part $\mathfrak{A}_{\mathrm{o}}$) vanishes. Thus, light spots in the sky at night are interpreted as stars rather than as solutions of the homogeneous equations.

One may object that the vanishing of $A_{\mathrm{o}}{}^{\mu}$ or[3] of $\mathcal{A}_{\mathrm{o}}{}^{\mu}$ is contrary to "the fact that" nature is symmetric in time. Yet, even if this symmetry of nature with respect to the sign of t would exist, there is the other fact that physics (which is our knowledge of nature) is not symmetric in time. In knowledge there is, for instance, a distinct difference between memory and prediction. If we say that $\mathcal{A}_{\mathrm{o}}{}^{\mu}$ vanishes, this is perhaps not a property of objective nature, but rather a property of supposed knowledge. Let us express this in the form of an equation.

The q-number $\mathcal{A}_{\mathrm{o}}{}^{\mu}$ is a part of the description of nature; there are arguments why it probably cannot vanish.[4] Our supposed knowledge of nature is described by the state vector, say Φ in Heisenberg representation. Our statement is then in the first place that the mean expectation value of $\mathfrak{A}_{\mathrm{o}}$ vanishes:

$$(\Phi^*,\ {}^{\mathrm{H}}\mathfrak{A}_{\mathrm{o}}(\mathbf{x},t)\ \Phi)=0. \tag{2}$$

Here, ${}^{\mathrm{H}}$ denotes Heisenberg representation. Equation (2) together with the equality of the mean expectation values of the longitudinal field $\mathbf{E}_{\shortparallel}$ and the coulomb field $-\nabla\int d^3\mathbf{x}'\rho(\mathbf{x}')/r$, tells us how to interpret the dualism between the field concept and the action-at-a-distance concept in quantum electrodynamics.

It is impossible to postulate ${}^{\mathrm{H}}\mathfrak{A}_{\mathrm{o}}(\mathbf{x},t)\Phi=0$, since the commutator Eq. (II.3) of the appendix does not vanish. But we want to state only that in the $\mathfrak{A}_{\mathrm{o}}$-field "no photons are present." This we formulate by assuming[3,5,6]

$$^{\mathrm{H}}\mathcal{A}_{\mathrm{o}}{}^{(+)\mu}(\mathbf{x},t)\ \Phi=0, \tag{3}$$

where $\mathcal{A}_{\mathrm{o}}{}^{(+)}$ is Schwinger's "positive frequency" part of the $\mathcal{A}_{\mathrm{o}}$-field, which describes annihilation of photons only. In fact, Eq. (2) will now follow directly from Eq. (3) and its hermitian conjugate

$$\Phi^*\ {}^{\mathrm{H}}\mathcal{A}_{\mathrm{o}}{}^{(-)\mu}(\mathbf{x},t)=0. \tag{4}$$

Note that (4) follows from (3); it is impossible to replace (4) by a similar formula with $(\mathcal{A}-\mathcal{A}_{\mathrm{adv}})^{(-)}$ replacing $\mathcal{A}_{\mathrm{o}}{}^{(-)}\equiv(\mathcal{A}-\mathcal{A}_{\mathrm{ret}})^{(-)}$, if we want to stick to (3).

It should be well understood that it was our assumption that in principle, for all states of nature considered in physics, condition (3) should hold rigorously. For many practical problems, however, Eq. (3) may be of little interest. Suppose we know what photons are present at a time represented by a space-like surface I in space-time, and that we want to make predictions on what will happen at later times. We can express $A_{\mathrm{ret}}{}^{\mu}(\mathbf{x},t)$ as an integral over space-time[6,7] as in Eq. (I.2) of the appendix, and we might divide the integral into two parts $A_{\mathrm{ret}<I}{}^{\mu}(\mathbf{x},t)$ and $A_{\mathrm{ret}>I}{}^{\mu}(\mathbf{x},t)$ corresponding to the parts of space-time before and after

[1] W. Pauli, Kapitel 2, B.8, p. 269 sqq. of Geiger and Scheel's *Handbuch der Physik*, second edition, Vol. **24/1** (1933).
[2] F. J. Belinfante and J. S. Lomont, Phys. Rev. **84**, 541 (1951). We shall refer to equations from this paper by (G-I:).
[3] J. Schwinger, Phys. Rev. **74**, 1439 (1948).
[4] F. J. Belinfante and J. S. Lomont, Phys. Rev. **78**, 346(A) (1950).
[5] F. J. Belinfante, Phys. Rev. **75**, 337(A) (1949).
[6] J. Schwinger, Phys. Rev. **75**, 651 (1949).
[7] F. J. Belinfante, Phys. Rev. **76**, 66 (1949).

the surface I,

$$ {}^{\mathrm{H}}A_{\mathrm{ret}>\mathrm{I}}{}^{\mu}(\mathbf{x}, t) = -4\pi \int_I^t d^4x\, D(x-x')\, {}^{\mathrm{H}}j^{\mu}(\mathbf{x}', t'), \quad (5) $$

where t on the integral denotes the surface $t=$constant, $d^4x = d^3\mathbf{x}\, dx^0$ with $x^0=-x_0=ct$, and $D(x)$ is Schwinger's D-function.[3,6] ${}^{\mathrm{H}}A_{\mathrm{ret}<\mathrm{I}}{}^{\mu}$ is a similar expression with integral from $-\infty$ to I. Then, by (1),

$$ A^{\mu}(\mathbf{x}, t) = A_{\mathrm{I}}{}^{\mu}(\mathbf{x}, t) + A_{\mathrm{ret}>\mathrm{I}}{}^{\mu}(\mathbf{x}, t), \quad (6) $$

where

$$ A_{\mathrm{I}}{}^{\mu}(\mathbf{x}, t) = A_{\circ}{}^{\mu}(\mathbf{x}, t) + A_{\mathrm{ret}<\mathrm{I}}{}^{\mu}(\mathbf{x}, t). \quad (7) $$

Now, in the problem considered, we may not be interested in this division (7) of $A_{\mathrm{I}}{}^{\mu}$ or of its transverse part $\mathfrak{A}_{\mathrm{I}}$ into two parts. We would simply postulate as initial condition the numbers of photons (and of other particles) present at I. For instance, in some very special problem, we may start out with a photon vacuum at I, so that we want to put

$$ {}^{\mathrm{H}}\mathcal{A}_{\mathrm{I}}{}^{(+)}(\mathbf{x}, t)\, \Phi = 0, \quad \text{(photon vacuum at } I). \quad (8) $$

Although we would believe that in principle Eq. (3) is still correct, we would make no attempt to use it or even to reconcile it with the condition (8), which is of more practical importance for such particular problem. The practical importance of Eq. (3), then,—beside its mere philosophical value as a general principle of physics,—lies in the fact that in many instances we idealize practical problems with initial photon vacuum at I, by taking the limit $I\rightarrow-\infty$. In such cases, the rigorous condition (3) can replace the ad hoc condition (8). But in a problem like the Compton effect, the ad hoc condition of presence of a photon at I would lead in the limit $I\rightarrow-\infty$ to a practical initial condition contrary to the fundamental but in this case not very interesting equation (3). In such case, we would therefore simply ignore Eq. (3) for the solution of such problem.

II. USE OF INTERACTION REPRESENTATION

The main task of this paper will further be to reformulate formula (3) in a different mathematical form, and to prove its covariance. For this purpose we shall relate our Heisenberg representation of q-numbers to an interaction representation ($^{\mathrm{I}}$) coinciding with the Heisenberg representation ($^{\mathrm{H}}$) on the initial surface I:

$$ {}^{\mathrm{H}}q(\mathbf{x}, t) = U_{\mathrm{I}}(t)^{-1} \cdot {}^{\mathrm{I}}q(\mathbf{x}, t) \cdot U_{\mathrm{I}}(t), \quad (9) $$

$$ U_{\mathrm{I}}(I) = 1. \quad (10) $$

We shall also consider the interaction representation ($^{\circ}$), which coincides with Heisenberg representation at $t=-\infty$:

$$ {}^{\mathrm{H}}q(\mathbf{x}, t) = U_{\circ}(t)^{-1} \cdot {}^{\circ}q(\mathbf{x}, t) \cdot U_{\circ}(t), \quad (9a) $$

$$ U_{\circ}(-\infty) = 1. \quad (10a) $$

This is simply a special case of (9)–(10) for $I\rightarrow-\infty$. In both cases, U satisfies the usual generalized Schrödinger equation (similar to Eq. (G-I:9) of reference 2), which for infinitesimal parallel displacements of the surface $t=$constant simplifies to

$$ i\hbar[dU_{\mathrm{I}}(t)/dt] = U_{\mathrm{I}}(t)\, {}^{\mathrm{H}}\mathcal{W}(t) = {}^{\mathrm{I}}\mathcal{W}(t)\, U_{\mathrm{I}}(t), \quad (11) $$

and its hermitian conjugate

$$ -i\hbar[dU_{\mathrm{I}}{}^{-1}/dt] = U_{\mathrm{I}}(t)^{-1}\, {}^{\mathrm{I}}\mathcal{W}(t), \quad (12) $$

where $\mathcal{W}(t)$ is the integral of the interaction operator $W(x)$ over the space-like surface t.

In Manifestly Covariant ("M.C.") Quantum Electrodynamics ("q.e.") it can then be shown easily (see Appendix I) that

$$ {}^{\mathrm{H}}A_{\mathrm{I}}{}^{\mu}(x)[\equiv {}^{\mathrm{H}}A^{\mu}(x) - {}^{\mathrm{H}}A_{\mathrm{ret}>\mathrm{I}}{}^{\mu}(x)] = {}^{\mathrm{I}}A^{\mu}(x). \quad (13) $$

In Gauge-Independent ("G.I.") q.e., no four-vector A^{μ} is defined, but one does define a solenoidal vector $\mathfrak{A}(x)$, which is equivalent to Schwinger's $\mathcal{A}^{\mu}$ when $\mathcal{A}^0$ vanishes. It is easily shown (see Appendix II) that in G.I.q.e.

$$ {}^{\mathrm{H}}\mathfrak{A}_{\mathrm{I}}(x)[\equiv {}^{\mathrm{H}}\mathfrak{A}(x) - {}^{\mathrm{H}}\mathfrak{A}_{\mathrm{ret}>\mathrm{I}}(x)] = {}^{\mathrm{I}}\mathfrak{A}(x), \quad (14) $$

where $\mathfrak{A}_{\mathrm{ret}}(x)$ is the transverse retarded vector potential, and where $>\mathrm{I}$ refers to sources "after" the surface I.

Taking in Eqs. (13)–(14) the limit $I\rightarrow\infty$, so that $\mathcal{A}_{\mathrm{I}}\rightarrow\mathcal{A}_{\circ}$ and ${}^{\mathrm{I}}\mathcal{A}\rightarrow{}^{\circ}\mathcal{A}$, we can rewrite our fundamental auxiliary condition (3) as

$$ {}^{\circ}\mathcal{A}^{(+)\mu}(\mathbf{x}, t)\, \Phi = 0. \quad (15) $$

In this fundamental formula, a "q-number in interaction representation" seems to operate on the state vector in Heisenberg representation. In fact, our result for $I\rightarrow-\infty$

$$ {}^{\circ}\mathcal{A}^{\mu} = {}^{\mathrm{H}}\mathcal{A}_{\circ}{}^{\mu} \equiv {}^{\mathrm{H}}\mathcal{A}^{\mu} - {}^{\mathrm{H}}\mathcal{A}_{\mathrm{ret}}{}^{\mu} \quad (14a) $$

shows that Eq. (9a) can be regarded as a canonical transformation by which—in Heisenberg representation—new variables ${}^{\circ}q$ are introduced, in which the retarded fields have been rigorously separated from the electromagnetic field. The limitations to the usefulness of such canonical transformation (which according to Eq. (G-I:54) of reference 2 eliminates the electromagnetic interactions altogether), and a more useful transformation leaving the interaction with external fields, were discussed for instance by Dyson.[8]

Note that, even in M.C.q.e., one cannot "generalize" Eq. (15) by replacing $\mathcal{A}$ in it by A. There is not only no physical reason for such a "generalization," but the $\mu=0$ component of such "generalized" Eq. (15) would even be an absurdity, as "there is no state, in which no scalar photons could be created."[9]

III. PROOF OF COVARIANCE

The Lorentz-invariance of (15) is not yet obvious. (By Lorentz-invariance we mean independence from the choice of the time-like four-vector n^{μ} used by

[8] F. J. Dyson, Phys. Rev. **75**, 486 (1949).
[9] F. J. Belinfante, Phys. Rev. **76**, 226 (1949).

Schwinger in his definition of $\mathfrak{A}$. We prefer to choose this n^μ always along the time-axis, so that out n^μ and $\mathfrak{A}^\mu$ do not transform as four-vectors.)

In M.C.q.e., the covariance of the condition (15) under Lorentz transformation of the time direction given by n can be proved as follows: By

$$^{\circ}A^\mu = {}^{\mathrm{H}}A_{\circ}{}^\mu \equiv {}^{\mathrm{H}}A^\mu - {}^{\mathrm{H}}A_{\mathrm{ret}}{}^\mu \tag{13a}$$

we find in Heisenberg representation

$$\partial_\mu\, {}^{\circ}A^\mu(x)\,\Phi = 0 \tag{16}$$

by subtracting the identity $\partial_\mu {}^{\mathrm{H}}A_{\mathrm{ret}}{}^\mu(x)=0$ from the Lorentz condition $\partial_\mu {}^{\mathrm{H}}A^\mu(x)\Phi=0$. But (16) with[3]

$$\Box\, {}^{\circ}A^\mu(x) = 0 \tag{17}$$

and the four-vector character of $^{\circ}A^\mu$ together guarantee the rigorous invariance of (15) by the same argument[9] as used to prove the invariance in zeroth-order approximation of the "photon vacuum" condition

$$^{\mathrm{H}}\{\mathfrak{A}^{(+)}(x)\}\,\Phi \approx 0$$

or

$$^{\mathrm{I}}\mathfrak{A}^{(+)}(\mathbf{x},t)\,\Psi_{\mathrm{I}}(t) \approx 0, \tag{18}$$

which has been used in various older publications.[6,9]

In G.I.q.e., the covariance of the condition (15) follows from the invariance of the tensor formula

$$^{\circ}\mathfrak{F}_{\lambda\mu}{}^{(+)}(x)\,\Phi = 0, \tag{19}$$

where $^{\circ}\mathfrak{F}_{\lambda\mu}$ is the electromagnetic field $^{\circ}\mathfrak{E}$, $^{\circ}\mathfrak{B}$ in interaction representation. Equation (19) follows from (15) by $^{\circ}\mathfrak{F}_{\lambda\mu}{}^{(+)}=\partial_\lambda\, {}^{\circ}\mathfrak{A}_\mu{}^{(+)}-\partial_\mu\, {}^{\circ}\mathfrak{A}_\lambda{}^{(+)}$, while Eq. (15) (with $^{\circ}\mathfrak{A}^0=0$) again follows from the space-like components of (19) ($^{\circ}\mathfrak{B}^{(+)}\Phi=0$) by the definition Eq. (G-I:3) of reference 2, of $\mathfrak{A}$ in terms of $\mathfrak{B}$.

IV. COMPARISON OF AUXILIARY CONDITION WITH PHOTON VACUUM CONDITIONS

In earlier publications, Eq. (18) was meant as either only a zeroth-order approximation for the state-vector, or as a boundary condition on a specific initial surface I, where it is by $\Psi_{\mathrm{I}}(t)=U_{\mathrm{I}}(t)\Phi$ with (10) and (14) equivalent to the special "practical" assumption (8). Equation (18) was not rigorously covariant, but only in zeroth order.[9] This is due to the fact that from (18) we cannot conclude the vanishing of $^{\mathrm{I}}\mathfrak{E}^{(+)}(\mathbf{x},t)\,\Psi_{\mathrm{I}}(t)$, since differentiation of (18) with respect to t introduces $d\Psi_{\mathrm{I}}/dt$. Thus, (19) is no longer valid, and in a different Lorentz frame $^{\mathrm{I}}\mathfrak{B}^{(+)}\Psi_{\mathrm{I}}$, and therefore $^{\mathrm{I}}\mathfrak{A}^{(+)}\Psi_{\mathrm{I}}$ will no longer vanish. On the other hand, the general law given by Eq. (15) (together with (16) in the M.F. theory) is rigorously covariant.

Let us remark here that in many calculations one uses neither (15) nor (18), but something in between:

$$^{\circ}\mathfrak{A}^{(+)}\,\Psi_1[\sigma] = 0, \tag{20}$$

where Ψ_1 is obtained from $\Psi_{\circ}=U_{\circ}\Phi$ by a transformation not quite equal to U_0^{-1}, but an approximation of it. With the first-order approximation sometimes used in the calculation of the self-energy of a free electron, use of (20) instead of (15) can be shown to lead to errors in the free-electron self-energy proportional to the fourth and higher powers of e only, so that, contrary to earlier belief,[10] this cannot account for the appearance of "kinetic self-energy" terms. (The latter are due to certain ambiguities in the evaluation of the divergent integrals of products of Schwinger's Δ and $\mathfrak{D}$ functions and their derivatives in the expression for the electron self-energy.)

APPENDIX

I. Manifestly Covariant Theory

In M.C.q.e. the interaction operator is

$$\mathcal{W}(t) = -\int d^3\mathbf{x}\, j_\lambda(\mathbf{x},t)\, A^\lambda(\mathbf{x},t). \tag{I.1}$$

In Heisenberg representation, the retarded field from a source $^{\mathrm{H}}j_\mu(\mathbf{x},t)$ (compare Eq. (9)) is given by[6,7]

$$^{\mathrm{H}}A_{\mathrm{ret}}{}^\mu(\mathbf{x},t) = -4\pi\int_{-\infty}^{t} dct' \int d^3\mathbf{x}'\, U_{\mathrm{I}}(t')^{-1} \times D(x-x')\, {}^{\mathrm{I}}j^\mu(\mathbf{x}',t')\, U_{\mathrm{I}}(t'). \tag{I.2}$$

By[3]

$$4\pi i\hbar c g^{\mu\lambda} D(x-x') = [{}^{\mathrm{I}}A^\mu(x)\,;\,{}^{\mathrm{I}}A^\lambda(x')] \tag{I.3}$$

and by (I.1) we find from (I.2), for the part of the integral between the surfaces I and t as in Eq. (5),

$$\begin{aligned} {}^{\mathrm{H}}A_{\mathrm{ret}>\mathrm{I}}{}^\mu(\mathbf{x},t) &= \int_I^t (dt'/i\hbar)\, U_{\mathrm{I}}(t')^{-1}[{}^{\mathrm{I}}A^\mu(\mathbf{x},t)\,;\,{}^{\mathrm{I}}\mathcal{W}(t')]\, U_{\mathrm{I}}(t') \\ &= \int_I^t dt'\,(d/dt')\{U_{\mathrm{I}}(t')^{-1}\, {}^{\mathrm{I}}A^\mu(\mathbf{x},t)\, U_{\mathrm{I}}(t')\} \\ &= U_{\mathrm{I}}(t)^{-1}\, {}^{\mathrm{I}}A^\mu(\mathbf{x},t)\, U_{\mathrm{I}}(t) - U_{\mathrm{I}}(I)^{-1}\, {}^{\mathrm{I}}A^\mu(\mathbf{x},t)\, U_{\mathrm{I}}(I) \\ &= {}^{\mathrm{H}}A^\mu(\mathbf{x},t) - {}^{\mathrm{I}}A^\mu(\mathbf{x},t), \end{aligned} \tag{I.4}$$

where we used (11)–(12), (9), and (10). From (I.4) with the definition (6) follows Eq. (13).

II. Gauge-Independent Theory

In G.I.q.e. the interaction operator is given by Eq. (G-I:10) of reference 2, or

$$\mathcal{W} = \int d^3\mathbf{x}\{(\mathbf{E}_{\shortparallel}{}^2/8\pi) - \mathfrak{A}\cdot\mathbf{j}\}. \tag{II.1}$$

Since in Heisenberg representation $^{\mathrm{H}}\mathfrak{A}$ satisfies the Maxwell equations (G-I:59–66), it must be equal, but for a solution $^{\mathrm{H}}\mathfrak{A}_{\circ}$ of the homogeneous equations, to

[10] Compare the wrong remark in F. J. Belinfante, Phys. Rev. **81**, 307(A) (1951), that the "kinetic self-energy" of an electron would be due to incorrect use of (20) instead of (15).

the retarded field, for which we find in Appendix III:

$$^{\mathrm{H}}\mathfrak{A}_l^{\mathrm{ret}}(\mathbf{x},t) = -4\pi\int_{-\infty}^{t} cdt'\int d^3\mathbf{x}'\,U_{\mathrm{I}}(t')^{-1}\sum_{m=1}^{3} D_{lm}^{\mathrm{tr}}(x-x') \times {}^{\mathrm{I}}j_m(\mathbf{x}',t')\,U_{\mathrm{I}}(t'). \quad \text{(II.2)}$$

Now, according to Eq. (3.14) of reference 3 and Eqs. (III.5–6) of Appendix III, we have[11]

$$[{}^{\mathrm{I}}\mathfrak{A}_l(x);{}^{\mathrm{I}}\mathfrak{A}_m(x')] = 4\pi i\hbar c\{\delta_{lm}\triangle-\nabla_l\nabla_m\}\mathfrak{D}(x-x') = 4\pi i\hbar c\,D_{lm}^{\mathrm{tr}}(x-x'). \quad \text{(II.3)}$$

Further, the coulomb field ${}^{\mathrm{I}}\mathbf{E}_{\parallel}(x')$ defined by Eqs. (G-I:6–7) of reference (2) commutes with ${}^{\mathrm{I}}\mathfrak{A}(x)$ even for time-like $x-x'$. Therefore, we find for the part of the integral (II.2) between I and t as in Eq. (5), by Eqs. (II.1), (II.3), (11)–(12), (9), and (10):

$$^{\mathrm{H}}\mathfrak{A}_l^{\mathrm{ret}>\mathrm{I}}(\mathbf{x},t) = \int_I^t (dt'/i\hbar)U_{\mathrm{I}}(t')^{-1}[{}^{\mathrm{I}}\mathfrak{A}_l(\mathbf{x},t);{}^{\mathrm{I}}\mathcal{W}(t')]U_{\mathrm{I}}(t') = \int_I^t dt'(d/dt')\{U_{\mathrm{I}}(t')^{-1}\,{}^{\mathrm{I}}\mathfrak{A}_l(\mathbf{x},t)\,U_{\mathrm{I}}(t')\} = {}^{\mathrm{H}}\mathfrak{A}_l(\mathbf{x},t)-{}^{\mathrm{I}}\mathfrak{A}_l(\mathbf{x},t). \quad \text{(II.4)}$$

Subtracting this from ${}^{\mathrm{H}}\mathfrak{A}_l(\mathbf{x},t)$ we get Eq. (14).

III. The Transverse D-Function

A solution of[12]

$$\Box\,{}^{\mathrm{H}}\mathfrak{A}_l(x) = -4\pi({}^{\mathrm{H}}j_\perp)_l \equiv -(1/4\pi^3)\int d^4k\int d^4x'\{\delta_{lm}-(k_lk_m/\mathbf{k}^2)\}\,{}^{\mathrm{H}}j^m(x') \times\exp\{ik_\mu(x-x')^\mu\} \quad \text{(III.1)}$$

(with summation over m from 1 to 3) is apparently given by

$$^{\mathrm{H}}\mathfrak{A}_l^{\mathrm{mix}}(x) = +(1/4\pi^3)\int d^4k\int d^4x'\,[k_\nu k^\nu]^{-1} \times\{\delta_{lm}-(k_lk_m/\mathbf{k}^2)\}\,{}^{\mathrm{H}}j^m(x')\exp\{ik_\mu(x-x')^\mu\}. \quad \text{(III.2)}$$

As the integrand has singularities for $k^0=\pm|\mathbf{k}|$, we take the principal value in these points. We complete the path of integration for $k^0(=-k_0)$ to a contour, closed along $\pm i\infty$ as $(t-t')\lesseqgtr 0$. Thence, with $\epsilon(t-t')=(t-t')/|t-t'|$, with $\triangle\equiv\boldsymbol{\nabla}^2$, and with Schwinger's D- and $\mathfrak{D}$-function,[3,6,11]

$$\begin{aligned}{}^{\mathrm{H}}\mathfrak{A}_l^{\mathrm{mix}}(\mathbf{x},t) &= (1/4\pi^2)\int d^4x'\,\epsilon(t-t')\int d^3\mathbf{k}|\mathbf{k}|^{-1} \times\textstyle\sum_m\{\delta_{lm}-(k_lk_m/\mathbf{k}^2)\}\,{}^{\mathrm{H}}j^m(x') \times\exp\{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')\}\sin\{kc(t-t')\}\\ &= -2\pi\int d^4x'\epsilon(t-t')\textstyle\sum_m {}^{\mathrm{H}}j^m(x') \times\{\delta_{lm}-(\nabla_l\nabla_m/\triangle)\}D(x-x')\\ &= -2\pi\int d^4x'\epsilon(t-t')\textstyle\sum_m {}^{\mathrm{H}}j^m(x') \times\{\delta_{lm}\triangle-\nabla_l\nabla_m\}\mathfrak{D}(x-x'). \end{aligned} \quad \text{(III.3)}$$

If the factor $\epsilon(t-t')$ is omitted, we apparently get a solution of the homogeneous equation, since $\Box\mathfrak{D}=0$. Adding this solution to ${}^{\mathrm{H}}\mathfrak{A}_l^{\mathrm{mix}}$, we find an integral over $t'<t$ only, therefore the retarded solution:

$$^{\mathrm{H}}\mathfrak{A}_l^{\mathrm{ret}}(\mathbf{x},t) = -4\pi\int_{-\infty}^{t} cdt'\int d^3\mathbf{x}'\sum_{m=1}^{3} \times D_{lm}^{\mathrm{tr}}(x-x')\,{}^{\mathrm{H}}j_m(\mathbf{x}',t'). \quad \text{(III.4)}$$

Here, we introduced the "transverse D-function," defined in analogy to the transverse delta-function[12] for $l, m=1, 2, 3$ by

$$D_{lm}^{\mathrm{tr}}(x) = \delta_{lm}D(x)-D_{lm}^{\mathrm{long}}(x), \quad \text{(III.5)}$$

$$D_{lm}^{\mathrm{long}}(x) = \nabla_l\nabla_m\mathfrak{D}(x);\quad D(x)=\textstyle\sum_l D_{ll}^{\mathrm{long}}(x). \quad \text{(III.6)}$$

Some of its properties are

$$\Box\,D_{lm}^{\mathrm{tr}}(x)=0,\quad D_{lm}^{\mathrm{tr}}(x)=D_{ml}^{\mathrm{tr}}(x)=-D_{lm}^{\mathrm{tr}}(-x), \quad \text{(III.7)}$$

and, by Eqs. (14a)–(14b) of reference 7 and Eqs. (12)–(13) and (17) of reference 12,

$$D_{lm}^{\mathrm{tr}}(t=0)=0,\quad \{\partial_0 D_{lm}^{\mathrm{tr}}(x)\}_{t=0} = -\delta_{lm}^{\mathrm{tr}}(\mathbf{x}). \quad \text{(III.8)}$$

Similar equations hold for $D_{lm}^{\mathrm{long}}(x)$.

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[11] The $\mathfrak{D}$-function has been defined by Schwinger (see references 3 and 6). It equals $\mathfrak{D}(\mathbf{r},t)=\{|r+ct|-|r-ct|\}/(8\pi r)$. (Similarly, Schwinger's $\mathfrak{D}^{(1)}$-function equals $[ct/4\pi^2r]\ln\{|r+ct|/|r-ct|\}+[1/4\pi^2]\ln|r^2-c^2t^2|+$constant.)

[12] Compare Sec. 2 of F. J. Belinfante, Physica **12**, 1 (1946).