Eqs. (184), (186), (187) or (175), (169), and (172)-(173), of Pauli's *Handbuch* article,[3] so that essentially we have given a new proof of the intrinsic Lorentz covariance of Pauli's theory (confirming that $\mathbf{E}_{\parallel}$ should not be quantized as a photon field). At the same time, we have shown how Pauli's theory can be extended to electron fields quantized according to the exclusion principle, and also how it can be formulated in interaction representation, making possible some unique covariant kind of distinction between positon and negaton states.

In some following papers we shall formulate a covariant auxiliary condition stating that all photons present can be regarded as at some time having been emitted by a source,[17] we shall discuss the energy density tensor in gauge independent quantum electrodynamics, and we shall show how our proof of the integrability of the generalized Schrödinger equation lends itself to interesting speculations on how a covariant quantum electrodynamics freed from self-interactions might be formulated.[18]

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[17] Compare also F. J. Belinfante, Phys. Rev. **81**, 307(A) (1951).
[18] F. J. Belinfante and J. S. Lomont, Phys. Rev. **83**, 225(A) (1951). See also F. J. Belinfante, Prog. Theor. Phys. **6**, 202 (1951), and Phys. Rev. **82**, 767(A) (1951).

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A Variational Principle for Gauge-Independent Electrodynamics

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A new lagrangian for the maxwell field is defined, expressed completely in terms of gauge-independent transverse field-strengths ($\mathfrak{E}$, $\mathfrak{B}$) and matter variables ($\mathbf{x}_i$, $\mathbf{v}_i$, and $\mathbf{p}_i$ in the classical theory; ψ and $\psi^\dagger$ in the wave-mechanical theory). The coulomb field is defined in terms of matter variables, and a (solenoidal) vector potential is defined in terms of the magnetic field strength $\mathfrak{B}$. The variational principle gives the usual equations of motion and field equations. Quantization leads automatically to the formulas of gauge-independent quantum electrodynamics.

1. CLASSICAL THEORY

THE usual lagrangian of electrodynamics has the disadvantage of being expressed in terms of the potentials, which are not uniquely determined and have no direct physical meaning. However, it is possible to derive the maxwell equations for the field and the relativistic equations of motion for charged particles in the field from a variational principle, in which only physically meaningful quantities are to be varied. For instance, in classical electrodynamics we may put, for charged particles (point charges) moving in the microscopic maxwell field with transverse (solenoidal) part $\mathfrak{E}$, $\mathfrak{B}$,

$$\mathfrak{L}=\mathfrak{L}_m+\int[\mathfrak{A}\cdot\mathfrak{E}^t/4\pi c-(\mathfrak{E}^2+\mathfrak{B}^2)/8\pi+\mathfrak{A}\cdot\mathbf{j}-\tfrac{1}{2}\rho V], \quad (1)$$

with[1]

$$\mathfrak{L}_m = \textstyle\sum_i \mathbf{p}_i\cdot(\mathbf{x}^t{}_i-\mathbf{v}_i) - m_ic^2(1-\mathbf{v}_i^2/c^2)^{\frac{1}{2}}, \quad (2)$$

where ^{t} denotes time differentiation ($\mathfrak{E}^t=\partial\mathfrak{E}/\partial t$, $\mathbf{x}^t{}_i=d\mathbf{x}_i/dt$); and then vary the functions $\mathbf{x}_i(t)$, $\mathbf{p}_i(t)$, $\mathbf{v}_i(t)$, and the transverse fields $\mathfrak{E}(\mathbf{x},t)$ and $\mathfrak{B}(\mathbf{x},t)$ independently in

$$\delta\int\mathfrak{L}dt=0, \quad (3)$$

keeping the variations zero at the limits of integration. In Eq. (1), ρ, $\mathbf{j}$, V, $\mathfrak{A}$, and $\int$ are abbreviations for

$$\rho(\mathbf{x},t)=\textstyle\sum_i e_i\delta(\mathbf{x}-\mathbf{x}_i(t)), \quad (4a)$$

$$\mathbf{j}(\mathbf{x},t)=\textstyle\sum_i(e_i/c)\mathbf{v}_i\delta(\mathbf{x}-\mathbf{x}_i(t)), \quad (4b)$$

$$V(\mathbf{x},t)=\int'\rho(\mathbf{x}',t)/r, \quad (5a)$$

$$\mathfrak{A}(\mathbf{x},t)=\int'[\mathrm{curl}'\mathfrak{B}(\mathbf{x}',t)]/4\pi r, \quad (5b)$$

$$\int=\int d^3\mathbf{x}, \quad \int'=\int d^3\mathbf{x}', \quad r=|\mathbf{x}-\mathbf{x}'|. \quad (6)$$

From the definition of the electromagnetic radiation field $\mathfrak{E}$, $\mathfrak{B}$ as being a transverse field it follows that

$$\mathrm{div}\mathfrak{B}=0, \quad \mathrm{div}\mathfrak{E}=0. \quad (7a\text{-}b)$$

From (7a) and the definition (5b) we deduce the identities

$$\mathrm{curl}\mathfrak{A}=\mathfrak{B}, \quad \mathrm{div}\mathfrak{A}=0. \quad (8a\text{-}b)$$

We shall further define another vector field

$$\mathbf{E}=-\nabla V-\mathfrak{A}^t/c. \quad (9)$$

On account of (8b), the second term is transverse, while the first term is obviously longitudinal (irrotational), so that we may write

$$\mathbf{E}_{\parallel}=-\nabla V, \quad \mathbf{E}_{\perp}=-\mathfrak{A}^t/c, \quad \mathbf{E}=\mathbf{E}_{\parallel}+\mathbf{E}_{\perp}. \quad (10a\text{-}b\text{-}c)$$

[1] For a discussion of this form for $\mathfrak{L}_m$ see F. J. Belinfante, Phys. Rev. **74**, 779 (1948).

The field $\mathbf{E}_{\parallel}$ defined by (10a) with (5a) is what one calls the coulomb field. From (9)–(10) and (5a) we find the identity

$$\text{div}\mathbf{E} = \text{div}\mathbf{E}_{\parallel} = 4\pi\rho. \tag{11}$$

Thus, the *identities* among the maxwell equations (Eqs. (7a) and (11)) are a consequence of the definitions of the field. (Compare also the alternative definition of $\mathbf{E}$ by Eq. (14).) The *field equations of motion* for the electromagnetic radiation field will now follow by variation of the transverse fields $\mathfrak{E}(\mathbf{x}, t)$ and $\mathfrak{B}(\mathbf{x}, t)$ in (3) with (1). Properly speaking, one should keep these fields transverse while they are varied. This can easily be done by equating these fields temporarily to the curl of other vector fields and then varying the latter. The results thus obtained turn out to be identical with the results Eqs. (12) and (15) obtained by direct arbitrary variation of $\mathfrak{E}$ and $\mathfrak{B}$ themselves. This is due to the fact that the members of the latter equations appear to have no longitudinal parts anyhow, on account of Eqs. (7)–(8).

We thus obtain, by variation of $\mathfrak{E}$,

$$\mathfrak{E} = -\mathfrak{A}^{t}/c. \tag{12}$$

By the definition (5b), this relates $\mathfrak{E}$ to $\mathfrak{B}^{t}$. By (8) and (10) this relation can also be written as

$$\text{curl}\mathbf{E} = \text{curl}\mathfrak{E} = -\partial\mathfrak{B}/c\partial t. \tag{13}$$

Moreover, (12) yields a new interpretation of (9)–(10) as

$$\mathbf{E} = -\nabla V + \mathfrak{E}, \quad \mathbf{E}_{\perp} = \mathfrak{E}, \tag{14}$$

so that $\mathbf{E}$ is the sum of the coulomb field $\mathbf{E}_{\parallel}$ and the electric radiation field $\mathfrak{E}$. Therefore we call $\mathbf{E}$ the *total* electric field strength. (We also could have started with (14) as a definition, and then have derived (9)–(10) by means of (12).)

Variation of $\mathfrak{B}$ in (3) with (1) alters also $\mathfrak{A}$ in (1) according to the definition (5b), so that this variation yields

$$-\frac{\mathfrak{B}}{4\pi} + \text{curl}\int d^3\mathbf{x}' \frac{\mathbf{j}(\mathbf{x}') + \mathfrak{E}^{t}(\mathbf{x}')/4\pi c}{4\pi r} = 0; \tag{15}$$

thence

$$\text{curl}\mathfrak{B} = 4\pi\mathbf{j}_{\perp} + \mathfrak{E}^{t}/c, \tag{16}$$

where $\mathbf{j}_{\perp}(\mathbf{x}) = \text{curl curl}\int'\mathbf{j}(\mathbf{x}')/4\pi r$ is the solenoidal part[2] of $\mathbf{j}$.

We now turn to the equations of motion for the matter variables. Variation of $\mathbf{x}_i$ yields

$$-\mathbf{p}^{t}_i + (e_i/c)\nabla_i\mathfrak{A}(\mathbf{x}_i)\cdot\mathbf{v}_i - e_i\nabla_i V(\mathbf{x}_i) = 0, \tag{17}$$

where we used the dependence of $\mathbf{j}$ and ρ and thence of V by the definitions (4) and (5a) on the function $\mathbf{x}_i(t)$ being varied.

Variation of $\mathbf{p}_i$ yields

$$\mathbf{x}^{t}_i = \mathbf{v}_i. \tag{18}$$

Independent variation of $\mathbf{v}_i$ yields[3]

$$-\mathbf{p}_i + m_i\mathbf{v}_i(1-\mathbf{v}_i^2/c^2)^{-\frac{1}{2}} + (e_i/c)\mathfrak{A}(\mathbf{x}_i) = 0. \tag{19}$$

Substitution of $\mathbf{p}_i$ from (19) in Eq. (17) gives the equation of motion

$$(d/dt)\{m_i\mathbf{v}_i(1-\mathbf{v}_i^2/c^2)^{-\frac{1}{2}}\} = e_i\mathbf{E}(\mathbf{x}_i) + (e_i/c)\mathbf{v}_i\times\mathfrak{B}(\mathbf{x}_i), \tag{20}$$

where we used the equality

$$d\mathfrak{A}(\mathbf{x}_i(t), t)/dt = \partial\mathfrak{A}(\mathbf{x}_i, t)/\partial t + \mathbf{x}^{t}_i\cdot\nabla_i\mathfrak{A}(\mathbf{x}_i, t), \tag{21}$$

the equation (18), the identity (8a) and the definition (9).

From Eqs. (4) we find

$$\rho^{t}/c + \text{div}\mathbf{j} = \textstyle\sum_i(e_i/c)\{-\mathbf{x}^{t}_i + \mathbf{v}_i\}\cdot\nabla\,\delta(\mathbf{x}-\mathbf{x}_i(t)) = 0, \tag{22}$$

where we finally used Eq. (18). From (5a) and (22) follows

$$V^{t}/c = -\int'\text{div}'\mathbf{j}(\mathbf{x}')/r = -\text{div}\int'\mathbf{j}(\mathbf{x}')/r, \tag{23}$$

so that, by (10a) and footnote 2,

$$\mathbf{E}^{t}_{\parallel}/c = -\nabla V^{t}/c = \nabla\,\text{div}\int'\mathbf{j}(\mathbf{x}')/r = -4\pi\mathbf{j}_{\parallel}. \tag{24}$$

Adding (24) finally to (16), we find by (10c) with (14)

$$\text{curl}\mathfrak{B} = 4\pi\mathbf{j} + \partial\mathbf{E}/c\partial t. \tag{25}$$

By (7a), (11), (13), (25), and (20) we have derived Maxwell's equations and the relativistic equation of motion of point charges in the electromagnetic field, both featuring $\mathbf{E}$ according to (14) composed of the coulomb field and the electric radiation field, only the latter being considered as an independent variable in the variational principle (3). We obtained these results without introducing potentials as independent variables, and the only potentials introduced by Eqs. (5) as mathematical abbreviations have by their definition a uniquely determined gauge depending only on the Lorentz frame, in which we work.

2. QUANTUM THEORY

In relativistic quantum electrodynamics, we replace the matter terms (2) in the lagrangian (1) by the analogous terms of the Dirac theory,

$$\mathfrak{L}_m = \int\psi^{\dagger}(i\hbar\partial/\partial t - c\boldsymbol{\alpha}\cdot\mathbf{p}_{\text{op}} - mc^2\beta)\psi, \tag{26}$$

where $\mathbf{p}_{\text{op}} = -i\hbar\nabla$, while the definitions (4) are now replaced by

$$\rho = e:\psi^{\dagger}\psi:, \quad \mathbf{j} = e:\psi^{\dagger}\boldsymbol{\alpha}\psi:. \tag{27}$$

Here, : : is Wick's notation for omission of the vacuum current and density.[4]

[2] This follows from $\mathbf{j} = (\text{curl curl} - \nabla\,\text{div})\int'\mathbf{j}(\mathbf{x}')/4\pi r = \mathbf{j}_{\perp} + \mathbf{j}_{\parallel}$ with $\text{div}\mathbf{j}_{\perp} = \text{curl}\mathbf{j}_{\parallel} = 0$.

[3] This equation defines the derived variables $\mathbf{v}_i$ in terms of the canonical variables $\mathbf{p}_i$, $\mathbf{x}_i$, and $\mathfrak{B}(\mathbf{x})$. Compare page 781 of reference 1.

[4] G. S. Wick, Phys. Rev. **80**, 268 (1950).

Variation of $\psi^\dagger$ and of ψ now yields

$$i\hbar\partial\psi/\partial t=\{mc^2\beta+c\boldsymbol{\alpha}\cdot\mathbf{p}_{\mathrm{op}}-e\mathfrak{A}\cdot\boldsymbol{\alpha}\}\psi+\tfrac{1}{2}e(V\psi+\psi V) \quad (28)$$

and hermitian conjugate equation. This is Dirac's equation, including self-interaction as V is given by (5a) and ψ is quantized according to the exclusion principle. Equations (17) and (18) are now replaced by the results of application of the generalized theorem of Ehrenfest, which gives the time-derivatives of $\int\psi^\dagger\mathbf{p}_{\mathrm{op}}\psi$ and of $\int\psi^\dagger\mathbf{x}\psi$, as calculated from (28) and conjugate equation.

From (27) and (28) and the commutativity of ρ and V one also easily finds again the continuity equation $\rho^t/c+\mathrm{div}\mathbf{j}=0$ (Eq. (22)), which we used in deriving Eq. (25) from Eq. (16). There are no other alterations in the derivation of the maxwell equations from (3) with (1) and (26) in quantum theory.

We remark that $\mathcal{L}$ is linear in time-derivatives. The pairs of canonical conjugates are $\mathbf{x}_i$ with $\mathbf{p}_i$ in the classical theory ($\mathbf{v}_i$ being derived variables, see footnote 3), or ψ with $i\hbar\psi^\dagger$ in wave mechanics; and $\mathfrak{E}(\mathbf{x})$ with $\mathfrak{A}(\mathbf{x})/4\pi c$ in both cases. The hamiltonian is obtained from $-\mathcal{L}$ by omission of the terms $-pq^t$ from it, and gives for the total energy the usual expression,

$$\mathcal{H}=\int\{\psi^\dagger(mc^2\beta-i\hbar c\boldsymbol{\alpha}\cdot\nabla)\psi+\tfrac{1}{2}\rho V -\mathbf{j}\cdot\mathfrak{A}+(\mathfrak{E}^2+\mathfrak{B}^2)/8\pi\}. \quad (29)$$

Expanding the transverse fields $\mathfrak{E}$ and $\mathfrak{A}/4\pi c$ in fourier components, and quantizing these in the usual way as canonically conjugate variables, one finds, after recombination of the fourier components to transverse fields, that the commutator of $\mathfrak{E}$ and $\mathfrak{A}/4\pi c$ is $i\hbar\times$ the so-called *transverse delta-function*,[5] or

$$[\mathfrak{E}_k(\mathbf{x},t);\mathfrak{A}_l(\mathbf{x}',t)]=4\pi i\hbar c\delta_{kl}{}^{\mathrm{tr}}(\mathbf{x}-\mathbf{x}') \equiv i\hbar c\{\nabla_k\nabla_l-\delta_{kl}\nabla\cdot\nabla\}(1/r). \quad (30)$$

The commutation relations of the coulomb field $\mathbf{E}_{\parallel}$, on the other hand, simply follow from those of the matter field, by (10a) with (5a). These are just the commutation relations of the gauge-independent quantum electrodynamics recently proposed by the author in collaboration with J. S. Lomont.[6]

[5] F. J. Belinfante, Physica **12**, 1 (1946).

[6] F. J. Belinfante and J. S. Lomont, Phys. Rev. **83**, 225(A) (1951) and Phys. Rev. **84**, 541 (1951).

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Size and Thermal Conductivity Effects in Paramagnetic Relaxation*

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The Casimir-du Pré thermodynamic theory of paramagnetic relaxation is generalized by taking account of the thermal conductivity of the paramagnetic salt. The case of a spherical specimen in a constant temperature bath is considered. One finds that infinitely many times are required to characterize the relaxation when a magnetic field is suddenly applied. The alternating current susceptibility is calculated and is shown to contain terms depending on the size and thermal conductivity of the specimen. A limited comparison of the theory with experimental data is made.

IN 1938 Casimir and du Pré[1] developed a simple thermodynamic interpretation of the frequency dependence of the complex paramagnetic susceptibility. They considered the system of all ionic spins to be an entity separate from the crystalline lattice and assumed that the spin system was always in thermodynamic equilibrium with an oscillatory magnetic field. Energy is transferred from the field to the spin system which rises in temperature and transfers heat to the lattice. If isothermal conditions are maintained, there is a further transfer of heat from the lattice to the constant temperature bath. The mathematical development of these ideas leads to Debye-type curves for the susceptibility.

The Casimir-du Pré theory is in fairly good agreement with experiment.[2] However recent measurements by Kramers, Bijl, and Gorter[3] and by Benzie and Cooke[4] have revealed discrepancies. These authors have suggested that a suitable distribution of spin-lattice relaxation times could account for the experimental results. An alternative suggestion—that the thermal conductivity of the specimen alters the theoretical susceptibility curves—is considered here.

We shall consider a spherical specimen of a paramagnetic salt immersed in a constant temperature bath since no new results are obtained for an adiabatically isolated specimen. The differential equation for the

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[1] H. B. G. Casimir and F. K. du Pré, Physica **5**, 507 (1938).

[2] For an extensive survey of experimental results see C. J. Gorter, *Paramagnetic Relaxation* (Elsevier Publishing Company, Inc., Amsterdam, 1947).

[3] Kramers, Bijl, and Gorter, Physica **16**, 65 (1950).

[4] R. J. Benzie and A. H. Cooke, Proc. Phys. Soc. (London) **A63**, 20 (1950).