Gauge-Independent Quantum Electrodynamics*

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We discuss a *covariant*, but not manifestly covariant, form of quantum electrodynamics. No gauge-dependent potentials are introduced as independent (canonical) variables. Only the transverse electromagnetic field is quantized as a photon field. We formulate the theory first in interaction representation, although only flat space-like surfaces σ are considered. The interaction operator given by Eq. (10) is then used for describing Lorentz transformations (rotations of σ) as well as time dependence (parallel progress of σ). The integrability of the generalized Schrödinger equation is then proved. As we transform to Heisenberg representation the electron wave function ψ loses its spinor character and the transverse photon field $\mathfrak{E}$, $\mathfrak{B}$ its tensor character, but by adding the coulomb field $\mathbf{E}_{\parallel}$ to $\mathfrak{E}$ we restore the tensor character of the electromagnetic field. The gauge-independent quantum electrodynamics of Pauli's *Handbuch* article is a special form of the result thus obtained for the particular case that the number of electrons is known and finite. Our theory has a more general form allowing use of positon (hole) theory.

INTRODUCTION

QUANTUM electrodynamics in its conventional form is gauge-invariant, that is, physical results calculated by it are invariant under a change of the gauge of the longitudinal and scalar potentials used as some of the canonical variables in that theory. It is, however, slightly unsatisfactory that these physically meaningless gauge-dependent variables enter the theory at all. The interesting fact that gauge-invariance is in some way linked up with the fact of conservation of charge is not very essential[1] and certainly does not outweigh the bothersome facts that (1) quantization of these gauge-dependent variables necessitates discussion of such unphysical features as longitudinal and even "scalar" photons, the latter of negative energy;[2] (2) consequently, the redundant degrees of freedom of the electromagnetic field thus introduced have to be undone again by "auxiliary conditions," which render the state vector unnormalizable and are a source of much nuisance and ambiguity if not mathematical inconsistency in various problems.[2]

In his article in the *Handbuch der Physik*, Pauli[3] has developed long ago a different type of quantum electrodynamics, in which the gauge-dependent variables did not occur, so that auxiliary conditions could be avoided. This theory was relativistically covariant in its essence although not in its external form. It was given in Heisenberg representation, which made the proof of its invariance somewhat cumbersome. The fact that Pauli did not use the method of second quantization, and thus made his theory formally not applicable to positon theory, is of little importance, as his theory is easily generalized in this regard.[4] The theory developed below might have been derived from Pauli's theory in this generalized form by transforming it to interaction representation by methods similar to the ones used for nonscalar interactions.[5,6] For the validity of such derivation, use would have to be made of the Lorentz-invariance of the generalized Pauli theory. It seemed, however, more appropriate, in particular with a view to certain future applications, to prove the invariance of our theory directly in interaction representation, and then reversely derive the invariance of Pauli's theory from that of our theory.

1. THE FIELD VARIABLES IN INTERACTION REPRESENTATION

The variables describing the electron field in interaction representation are the undors (Dirac spinors) $^\circ\psi$ and $^\circ\psi^\dagger$ satisfying the free electron Dirac equation and the usual covariant four-dimensional anticommutation relations.

The photon field is described by a tensor field $^\circ\mathfrak{F}_{\mu\nu}(^\circ\mathfrak{E}, {}^\circ\mathfrak{B})$ satisfying in interaction representation the vacuum maxwell equations (charges omitted), so that by $\mathrm{div}\,^\circ\mathfrak{E}=\mathrm{div}\,^\circ\mathfrak{B}=0$ the field is purely transverse. The four-dimensional commutation relations for $^\circ\mathfrak{F}_{\mu\nu}$ are the usual ones:[7]

$$[^\circ\mathfrak{F}_{\lambda\mu}(x);\,^\circ\mathfrak{F}_{\rho\sigma}(x')]=4\pi i\hbar c\{g_{\mu\rho}\partial_\lambda\partial_\sigma+g_{\lambda\sigma}\partial_\mu\partial_\rho-g_{\mu\sigma}\partial_\lambda\partial_\rho-g_{\lambda\rho}\partial_\mu\partial_\sigma\}D(x-x'),\quad(1)$$

with $D(x)$ given by Eqs. (12)–(15) of reference 5. For

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1 See the remark below Eq. (23) of F. J. Belinfante, Physica **7**, 449 (1940).

2 F. J. Belinfante, Phys. Rev. **76**, 226 (1949).

3 W. Pauli, Kapitel 2, B.8, p. 269 of Geiger and Scheel's *Handbuch der Physik* (Springer, Berlin, 1933), second edition, Vol. **24/1**.

4 F. J. Belinfante, Phys. Rev. **75**, 1633(A) (1949).

5 F. J. Belinfante, Phys. Rev. **76**, 66 (1949).

6 F. J. Belinfante and J. S. Lomont, Phys. Rev. **77**, 757(A) (1950).

7 *Notation:* $\mathbf{x}$ is the spatial part (with components x_k) of x (with components x^μ); italic indices run from 1 to 3 and Greek indices run from 0 to 3; $x^0=-x_0=ct$; $\mathbf{r}'=\mathbf{x}'-\mathbf{x}=-\mathbf{r}$, $r=|\mathbf{r}|$; $\partial_\mu=\partial/\partial x^\mu$; $\nabla_k=\partial/\partial x_k$, $\nabla_k'=\partial/\partial x_k'$; $\triangle=\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$. Often we write F for $F(\mathbf{x})$ and F' for $F(\mathbf{x}')$, $\int=\int d^3\mathbf{x}$ and $\int'=\int d^3\mathbf{x}'$. $[A;B]=AB-BA$. $\delta_{lm}=$Kronecker symbol. Superscript $^\circ$ denotes interaction representation, $^\mathrm{H}$ denotes Heisenberg representation. $a=(i\hbar c)^{-1}$.

$t=t'$ this yields the usual commutation relations for $\mathfrak{E}$ and $\mathfrak{B}$:

$$[\mathfrak{E}_k(\mathbf{x});\mathfrak{E}_l(\mathbf{x}')]=[\mathfrak{B}_{kl}(\mathbf{x});\mathfrak{B}_{mn}(\mathbf{x}')]=0,$$
$$[\mathfrak{B}_{kl}(\mathbf{x});\mathfrak{E}_m(\mathbf{x}')]=4\pi i\hbar c\{\delta_{km}\nabla_l-\delta_{lm}\nabla_k\}\delta^{(3)}(\mathbf{r}'). \quad (2)$$

We shall introduce the following functional of $\mathfrak{B}$:

$$\mathfrak{A}(\mathbf{x},t)\equiv\int d^3\mathbf{x}'\,\mathrm{curl}'\mathfrak{B}(\mathbf{x}',t)/4\pi r, \quad (3)$$

so that

$$\mathrm{div}\mathfrak{A}=0,\quad \mathrm{curl}\mathfrak{A}=\mathfrak{B}. \quad (4)$$

Then

$$[\mathfrak{A}_l(\mathbf{x},t);\mathfrak{E}_m(\mathbf{x}',t)]=i\hbar c\{\delta_{lm}\triangle'-\nabla_l'\nabla_m'\}(1/r)$$
$$=-4\pi i\hbar c\{\delta_{lm}\delta^{(3)}(\mathbf{r}')-\delta_{lm}{}^{\mathrm{long}}(\mathbf{r}')\}$$
$$=-4\pi i\hbar c\delta_{lm}{}^{\mathrm{tr}}(\mathbf{r}'). \quad (5)$$

Here, $\delta_{lm}{}^{\mathrm{long}}(\mathbf{x})$ and $\delta_{lm}{}^{\mathrm{tr}}(\mathbf{x})$ are the longitudinal and transverse delta-functions defined earlier by one of the authors.[8] In fact, the commutation relation (5) is identical with the commutation relation between the transverse parts of the fields **E** and **A** in conventional quantum electrodynamics. In the present theory, we

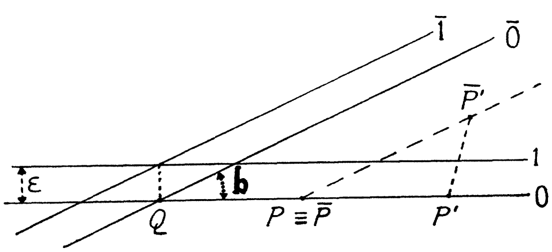

FIG. 1. Possible displacements $\epsilon(\xi,\eta,\zeta)$ of the flat surface 0. The displacement $0\rightarrow\bar{0}$ represents a rotation; $0\rightarrow 1$ represents a parallel displacement.

are not going to define any longitudinal counterpart to $\mathfrak{A}$. In fact, our solenoidal $\mathfrak{A}$ is a most natural description of the potential field in interaction representation.[9]

We shall also introduce a longitudinal field

$$\mathbf{E}_{\shortparallel}(\mathbf{x},t)\equiv-\nabla V(\mathbf{x},t), \quad (6)$$

$$V(\mathbf{x},t)\equiv\int d^3\mathbf{x}'\rho(\mathbf{x}',t)/r. \quad (7)$$

[8] F. J. Belinfante, Physica **12**, 1 (1946). Note that these longitudinal and transverse delta-functions are different from zero for large as well as for small values of their space-like arguments.

[9] If in interaction representation the potentials A^μ are resolved into Fourier components $A^\mu(k)\exp(ik_\nu x^\nu)$, the component of $A^\mu(k)$ outside the plane tangent along k^μ to the light cone has no physical meaning due to the Lorentz condition $k_\mu A^\mu(k)=0$. Furthermore, in this plane, components of $A^\mu(k)$ parallel to k^μ are physically meaningless as they do not contribute to the field strengths. All four-vectors connecting any pair of points on two lines parallel to k^μ describe therefore from a physical point of view the same field. In particular we may choose $A^\mu(k)$ in the intersection of the plane tangent along k^μ to the light cone, with the plane t=constant. Doing so, we find a potential without time component and with solenoidal spatial component $\mathfrak{A}$ as used in our theory. The fact that in a different Lorentz frame $\mathfrak{A}$ then represents a different four-vector does not matter, as we have seen that such different "arrows in space-time" connecting points of the same pair of parallel lines // k^μ are physically equivalent (different by a gauge transformation only).

Here, for electrons,

$$\rho\equiv e{:}\psi^\dagger\psi{:},\quad \mathbf{j}\equiv e{:}\psi^\dagger\boldsymbol{\alpha}\psi{:}, \quad (8)$$

where : : is Wick's notation for subtraction of the vacuum charge density.[10] In (6), we used a letter **E** instead of $\mathfrak{E}$, because V and $\mathbf{E}_{\shortparallel}$ are derived from the field of charged matter, and not from the electromagnetic field.

2. THE GENERALIZED SCHRÖDINGER EQUATION

Let $\Psi_0[\sigma]$ be the state vector on a space-like surface σ; let $d\sigma_0=d\xi d\eta d\zeta$ be the projection of a surface element on the plane τ=constant of some Lorentz frame $\xi\eta\zeta\tau$. Let $\sigma+\delta\sigma$ denote a new surface obtained from σ by infinitesimal displacement $\delta(c\tau)=\epsilon(\xi,\eta,\zeta)$ of its points. Then, with $a=(i\hbar c)^{-1}$, the generalized Schrödinger equation is

$$\Psi_0[\sigma+\delta\sigma)-\Psi_0[\sigma]$$
$$=a\int d\sigma_0\,\epsilon(\xi,\eta,\zeta)\,{}^\circ W(\xi,\eta,\zeta;\sigma)\,\Psi_0[\sigma]. \quad (9)$$

Here, ${}^\circ W$ is a function of the point ξ,η,ζ as well as a functional of the field variables in all points of the surface σ. We shall define the value of this functional of σ for flat surfaces σ only. Therefore we restrict the displacement $\epsilon(\xi,\eta,\zeta)$ in (9) to either a rotation like $0\rightarrow\bar{0}$ in Fig. 1, or a parallel displacement like $0\rightarrow 1$.

It is then convenient to express ${}^\circ W(\boldsymbol{\xi},\sigma)$ in terms of the field components calculated in a Lorentz frame ($xyzt$), in which σ is a plane t=constant. In such special frame of reference we put[11]

$$W=(\mathbf{E}_{\shortparallel}{}^2/8\pi)+(\mathfrak{E}\cdot\mathbf{E}_{\shortparallel}/4\pi)-\mathfrak{A}\cdot\mathbf{j}. \quad (10)$$

Note that (10) by Eqs. (3), (6)–(7) is expressed completely in terms of gauge-independent variables $\mathfrak{E}$, $\mathfrak{B}$, ρ, and **j** without introduction of redundant variables. Although, by div$\mathfrak{E}=0$, the middle term in the right hand member of (10) will drop out in Eq. (9) for parallel displacements $\delta c\tau=\epsilon$=constant, we yet need this term for rotations of σ, which are then accompanied by a Lorentz transformation of the frame of reference used in defining W by (10).

3. PROOF OF LORENTZ INVARIANCE OF THE THEORY

The Lorentz invariance of the field equations and commutation relations for the tensor $\{{}^\circ\mathfrak{E},{}^\circ\mathfrak{B}\}$ and the undors ${}^\circ\psi$ and ${}^\circ\psi^\dagger$ is obvious. What we have to show is merely the covariance of the Schrödinger equation, which means that (9) with (10) should (in Fig. 1) give the same change $\Psi_0[1]-\Psi_0[0]$ of the state vector with time when calculated directly ($0\rightarrow 1$) as when calculated in a different Lorentz frame ($0\rightarrow\bar{0}\rightarrow\bar{1}\rightarrow 1$). In other words, we must prove the integrability of (9).

We follow almost exactly the method used in Chapter 2 of reference (5) for the Proca field. As explained there, it is sufficient to show the equality, up to terms bilinear

[10] G. S. Wick, Phys. Rev. **80**, 268 (1950).

[11] Compare Eq. (49) of F. J. Belinfante, Physica **12**, 17 (1946).

in the infinitesimal displacements $\epsilon=$ constant for $0\to1$ and $\bar{0}\to\bar{1}$, and [12] $\varphi(\mathbf{x}')=\mathbf{b}\cdot(\mathbf{x}'-\mathbf{x}_Q)$ for $0\to\bar{0}$ and $1\to\bar{1}$, of

$$\Psi_\circ[\bar{1}]=[1+a\textstyle\int'\mathbf{b}\cdot(\mathbf{x}'-\mathbf{x}_Q)\{{}^\circ W'+\epsilon\partial_0\,{}^\circ W'\}]\Psi_\circ[1]$$

and

$$\Psi_\circ[\bar{1}]=[1+a\epsilon\textstyle\int\{{}^\circ W+\mathbf{b}\cdot(\mathbf{x}-\mathbf{x}_Q)\partial_0\,{}^\circ W+\delta\,{}^\circ W\}]\Psi_\circ[\bar{0}],$$

where

$$\Psi_\circ[1]=[1+a\epsilon\textstyle\int{}^\circ W]\Psi_\circ[0],$$
$$\Psi_\circ[\bar{0}]=[1+a\textstyle\int'\mathbf{b}\cdot(\mathbf{x}'-\mathbf{x}_Q)\,{}^\circ W']\Psi_\circ[0].$$

For this purpose we need only show that[7]

$$a\textstyle\int\int'\mathbf{b}\cdot(\mathbf{x}'-\mathbf{x}_Q)[{}^\circ W';{}^\circ W]=\int\delta^\circ W. \tag{11}$$

Here, $\delta\,{}^\circ W$ is the variation ${}^\circ\overline{W}(P)-{}^\circ W(P)$ of (10) under the infinitesimal Lorentz transformation $0\to\bar{0}$ in the point $P(\mathbf{x})$.

As, for $t'=t$, $\rho(\mathbf{x}')$ commutes with $\rho(\mathbf{x})$ and with $\mathbf{j}(\mathbf{x})$, also $\mathbf{E}_{11}(\mathbf{x}')$ and $\mathbf{E}_{11}(\mathbf{x})$ defined by (6)–(7) will commute with each other and with $\mathbf{j}(\mathbf{x})$ and $\mathbf{j}(\mathbf{x}')$ in the commutator $[W';W]$ with W from (10). Naturally they also commute with the photon field $\mathfrak{E}$, $\mathfrak{B}$, $\mathfrak{A}$. The only contribution to the left-hand member of (11) therefore arises from Eq. (5). This yields

$$\textstyle\int\int'\mathbf{b}\cdot(\mathbf{x}'-\mathbf{x}_Q)\sum_{l,m}({}^\circ j_l'\,{}^\circ E_{11m}-{}^\circ E_{11l}'\,{}^\circ j_m)\delta_{lm}{}^{\text{tr}}(\mathbf{r}).$$

Integration over $\mathbf{x}$ gives by Eq. (22) of reference (8)

$$\textstyle\int'\mathbf{b}\cdot(\mathbf{x}'-\mathbf{x}_Q)\{{}^\circ\mathbf{j}'\cdot({}^\circ\mathbf{E}_{11}')_\perp-{}^\circ\mathbf{E}_{11}'\cdot{}^\circ\mathbf{j}_\perp'\}.$$

The first term apparently vanishes. In the second term substitute (6) and integrate by parts. Since $\text{div}\mathbf{j}_\perp=0$, this yields

$$-\textstyle\int\mathbf{b}\cdot{}^\circ\mathbf{j}_\perp\,{}^\circ V \tag{12}$$

for the left-hand member of (11).

For calculating the right-hand member of (11) we first note that ${}^\circ W(P)$ is invariant under (finite) translation of the origin, so that it is allowed to consider the co-ordinates $\mathbf{x}$, t of the point P invariant. (P is the point where we calculate $\delta\,{}^\circ W$ in the integrand of (11).) Then, for the point $P(x)$ and for a different point $P'(x')$ the infinitesimal Lorentz transformation from the x-frame to the $\bar{x}$-frame gives

$$\left.\begin{aligned}\delta\mathbf{x}'&\equiv\bar{\mathbf{x}}(P')-\mathbf{x}(P')=-\mathbf{b}(ct'-ct);\quad \delta\mathbf{x}=0;\\ \delta ct'&\equiv c\bar{t}(P')-ct(P')=-\mathbf{b}\cdot(\mathbf{x}'-\mathbf{x});\quad \delta ct=0.\end{aligned}\right\} \tag{13}$$

Under this transformation, we have (with $\delta f\equiv\bar{f}-f$):

$$\begin{aligned}&\delta\,{}^\circ\mathfrak{E}=\mathbf{b}\times{}^\circ\mathfrak{B}, &&\delta\,{}^\circ\mathfrak{B}=-\mathbf{b}\times{}^\circ\mathfrak{E};\\ &\delta\,{}^\circ\rho=-\mathbf{b}\cdot{}^\circ\mathbf{j}, &&\delta\,{}^\circ\mathbf{j}=-\mathbf{b}\,{}^\circ\rho;\\ &\delta\,{}^\circ\psi^\dagger=-\tfrac{1}{2}\,{}^\circ\psi^\dagger\alpha\cdot\mathbf{b}, &&\delta\,{}^\circ\psi=-\tfrac{1}{2}\mathbf{b}\cdot\alpha\,{}^\circ\psi;\end{aligned} \tag{14}$$

$$\delta\nabla'=\mathbf{b}\partial/c\partial t'(=\mathbf{b}\partial/c\partial t,\text{ when }t'=t). \tag{15}$$

Let a new point $\bar{P}'$ be defined by $\bar{x}(\bar{P}')=x(P')\equiv x'$, (see Fig. 1), so that, for $F(\mathbf{x},t)\equiv\int d^3\mathbf{x}'f(\mathbf{x}',t)/r$, we find

$$\begin{aligned}\delta F(x)&=\int_{t'=t}d^3\mathbf{x}'\frac{\bar{f}(\bar{P}')-f(P')}{r}\\ &=\int d^3\mathbf{x}'\frac{\delta f(x')+f(\bar{P}')-f(P')}{r},\end{aligned} \tag{16}$$

where $r\equiv|\mathbf{x}'-\mathbf{x}|=|\bar{\mathbf{x}}(\bar{P}')-\bar{\mathbf{x}}(\bar{P})|$, while $\bar{t}(\bar{P}')=\bar{t}(\bar{P})$ as $t'=t$. Now, by (13) with $t'=t$, the vector $P'\bar{P}'$ has components

$$\begin{aligned}\bar{\mathbf{x}}(\bar{P}')-\bar{\mathbf{x}}(P')&=\mathbf{x}(P')-\bar{\mathbf{x}}(P')=-\delta\mathbf{x}'=0,\\ c\bar{t}(\bar{P}')-c\bar{t}(P')&=ct(P')-c\bar{t}(P')=-\delta ct'=\mathbf{b}\cdot\mathbf{r}'.\end{aligned} \tag{17}$$

Thence, (16) gives

$$\delta(\textstyle\int'f'/r)=\int'\{\delta f'+(\mathbf{b}\cdot\mathbf{r}')(\partial f'/c\partial t)\}/r. \tag{18}$$

This may be used for calculating $\delta\,{}^\circ\mathfrak{A}$ and $\delta\,{}^\circ V$. By (3), (18), (15), and the maxwell equation $\partial\,{}^\circ\mathfrak{B}/c\partial t=-\nabla\times{}^\circ\mathfrak{E}$, we find

$$\begin{aligned}\delta\,{}^\circ\mathfrak{A}(\mathbf{x})&=\textstyle\int'(4\pi r)^{-1}\cdot[(\delta\nabla')\times{}^\circ\mathfrak{B}'+\nabla'\times\delta\,{}^\circ\mathfrak{B}'\\ &\qquad+(\mathbf{b}\cdot\mathbf{r}')\nabla'\times\partial\,{}^\circ\mathfrak{B}'/c\partial t]\\ &=-\textstyle\int'(4\pi r)^{-1}\cdot[\mathbf{b}\times(\nabla'\times{}^\circ\mathfrak{E}')\\ &\qquad+\nabla'\times(\mathbf{b}\times{}^\circ\mathfrak{E}')+(\mathbf{b}\cdot\mathbf{r}')\nabla'\times(\nabla'\times{}^\circ\mathfrak{E}')]\\ &=\textstyle\int'(4\pi r)^{-1}\cdot[2(\mathbf{b}\cdot\nabla'){}^\circ\mathfrak{E}'-\nabla'({}^\circ\mathfrak{E}'\cdot\mathbf{b})-\mathbf{b}(\nabla'\cdot{}^\circ\mathfrak{E}')\\ &\qquad+(\mathbf{b}\cdot\mathbf{r}')\{\triangle'\,{}^\circ\mathfrak{E}'-\nabla'(\nabla'\cdot{}^\circ\mathfrak{E}')\}].\end{aligned} \tag{19}$$

Integrating by parts wherever ∇' acts on ${}^\circ\mathfrak{E}'$, we obtain, by

$$\mathbf{r}'\triangle'(1/4\pi r)=-\mathbf{r}'\delta^{(3)}(\mathbf{r}')=0, \tag{20}$$

$$\delta\,{}^\circ\mathfrak{A}(\mathbf{x})=-\textstyle\int'(\mathbf{b}\cdot\mathbf{r}')({}^\circ\mathfrak{E}'\cdot\nabla')\nabla'(1/4\pi r). \tag{21}$$

Similarly, by (7), (18) and $\partial\,{}^\circ\rho/c\partial t=-\text{div}{}^\circ\mathbf{j}$:

$$\begin{aligned}\delta\,{}^\circ V(\mathbf{x})&=\textstyle\int'\{-(\mathbf{b}\cdot{}^\circ\mathbf{j}')-(\mathbf{b}\cdot\mathbf{r}')(\nabla'\cdot{}^\circ\mathbf{j}')\}/r\\ &=\textstyle\int'(\mathbf{b}\cdot\mathbf{r}')({}^\circ\mathbf{j}'\cdot\nabla')(1/r),\end{aligned} \tag{22}$$

and by (6), (15), and (22)

$$\begin{aligned}\delta\,{}^\circ\mathbf{E}_{11}(\mathbf{x})&=-\nabla(\delta\,{}^\circ V)-\textstyle\int'(\mathbf{b}/r)\partial\,{}^\circ\rho'/c\partial t\\ &=\textstyle\int'[-(\mathbf{b}\cdot\mathbf{r}')({}^\circ\mathbf{j}'\cdot\nabla')\nabla(1/r)\\ &\qquad+\mathbf{b}({}^\circ\mathbf{j}'\cdot\nabla')(1/r)+\mathbf{b}(\nabla'\cdot{}^\circ\mathbf{j}')/r]\\ &=\textstyle\int'(\mathbf{b}\cdot\mathbf{r}')({}^\circ\mathbf{j}'\cdot\nabla')\nabla'(1/r).\end{aligned} \tag{23}$$

From (21) and (23) we find

$$\textstyle\int\{({}^\circ\mathfrak{E}\cdot\delta\,{}^\circ\mathbf{E}_{11}/4\pi)-{}^\circ\mathbf{j}\cdot\delta\,{}^\circ\mathfrak{A}\}=0. \tag{24}$$

Further, from (6)–(7), (14), and (3),

$$\begin{aligned}&\textstyle\int\{({}^\circ\mathbf{E}_{11}\cdot\delta\,{}^\circ\mathfrak{E}/4\pi)-{}^\circ\mathfrak{A}\cdot\delta\,{}^\circ\mathbf{j}\}\\ &\quad=-\textstyle\int(\mathbf{b}\times{}^\circ\mathfrak{B})\cdot\nabla\int'{}^\circ\rho'/4\pi r\\ &\qquad+\textstyle\int{}^\circ\rho\mathbf{b}\cdot\int'(\nabla'\times{}^\circ\mathfrak{B}')/4\pi r\\ &\quad=\textstyle\int'{}^\circ\rho'\mathbf{b}\cdot\int\nabla\times({}^\circ\mathfrak{B}/4\pi r)=0.\end{aligned} \tag{25}$$

Adding (24) and (25) we find

$$\textstyle\int\delta\{({}^\circ\mathfrak{E}\cdot{}^\circ\mathbf{E}_{11}/4\pi)-{}^\circ\mathfrak{A}\cdot{}^\circ\mathbf{j}\}=0, \tag{26}$$

[12] Notation as in reference (5), except **b** here for $\bar{\mathbf{b}}$ there. Also we use here the *xyzt*-frame tangent to (and coincident with) the surface 0 for the $\xi\eta\zeta\tau$-system of reference 5.

so, by (10), (6), and (23),

$$\int \delta\, {}^\circ W = -\int\int'(\mathbf{b}\cdot\mathbf{r}')({}^\circ\mathbf{j}'\cdot\nabla')\nabla'(1/4\pi r)\cdot\nabla\ {}^\circ V. \quad (27)$$

By Eqs. (13), (22), and (1) of reference 8 this gives

$$\begin{aligned}\int \delta\, {}^\circ W &= \int\int'(\mathbf{b}\cdot\mathbf{r}')\textstyle\sum_{m,n} {}^\circ j'_m(\nabla_n\, {}^\circ V)\delta_{mn}{}^{\mathrm{long}}(\mathbf{r}')\\ &= \int'(\mathbf{b}\cdot\mathbf{x}'){}^\circ\mathbf{j}'\cdot(\nabla'\, {}^\circ V')_{\parallel} - \int(\mathbf{b}\cdot\mathbf{x}){}^\circ\mathbf{j}_{\parallel}\cdot\nabla\, {}^\circ V\\ &= \int(\mathbf{b}\cdot\mathbf{x}){}^\circ\mathbf{j}_{\perp}\cdot\nabla\ {}^\circ V = -\int {}^\circ V\ {}^\circ\mathbf{j}_{\perp}\cdot\mathbf{b}. \quad (28)\end{aligned}$$

Comparing (28) with (12) we see that we have proved (11) and by this the integrability of the generalized Schrödinger equation (9) with (10) and the Lorentz-invariance of our theory.

4. HEISENBERG FORM OF FIELD EQUATIONS IN INTERACTION REPRESENTATION

Obviously the field equations of motion

$$\left.\begin{aligned}\partial\, {}^\circ\mathfrak{E}/c\partial t = \mathrm{curl}\, {}^\circ\mathfrak{B},\quad \partial\, {}^\circ\mathfrak{B}/c\partial t = -\mathrm{curl}\, {}^\circ\mathfrak{E},\\ i\hbar\partial\, {}^\circ\psi/\partial t = (mc^2\beta - i\hbar c\boldsymbol{\alpha}\cdot\nabla)\, {}^\circ\psi,\end{aligned}\right\} \quad (29)$$

and all derived equations such as

$$\left.\begin{aligned}\partial\, {}^\circ\mathfrak{A}/c\partial t = -{}^\circ\mathfrak{E},\quad \partial\, {}^\circ V/c\partial t = -\mathrm{div}\int'\, {}^\circ\mathbf{j}'/r,\\ \partial\, {}^\circ\mathbf{E}_{\parallel}/c\partial t = -4\pi\, {}^\circ\mathbf{j}_{\parallel},\end{aligned}\right\} \quad (29a)$$

can be summarized by

$$i\hbar\partial\, {}^\circ q(t)/\partial t = {}^\circ q(t)\ {}^\circ\mathcal{H}_0(t) - {}^\circ\mathcal{H}_0(t)\ {}^\circ q(t), \quad (30)$$

where

$$\mathcal{H}_0 \equiv \int\{\mathfrak{E}^2+\mathfrak{B}^2\}/8\pi + \int\psi^\dagger(mc^2\beta - i\hbar c\boldsymbol{\alpha}\cdot\nabla)\psi - \mathcal{H}_{\mathrm{vac}}. \quad (31)$$

The subtraction of the c-number $\mathcal{H}_{\mathrm{vac}}$ is most easily performed by splitting up the field variables into positive and negative frequency parts and then rearranging factors, which could be indicated by Wick's[10] symbol $:\cdots:$. Omitting vanishing integrals such as $\int\{{}^\circ\mathfrak{E}^{(+)}\cdot{}^\circ\mathfrak{E}^{(+)} + {}^\circ\mathfrak{B}^{(+)}\cdot{}^\circ\mathfrak{B}^{(+)}\}$ or $\int {}^\circ\overline{\psi^{(+)}}(\cdots){}^\circ\psi^{(-)}$, we may also write

$$\begin{aligned}{}^\circ\mathcal{H}_0 = \int\{{}^\circ\mathfrak{E}^{(-)}\cdot{}^\circ\mathfrak{E}^{(+)} + {}^\circ\mathfrak{B}^{(-)}\cdot{}^\circ\mathfrak{B}^{(+)}\}/4\pi\\ + \int {}^\circ\overline{\psi^{(+)}}(mc^2 + \hbar c\boldsymbol{\gamma}\cdot\nabla)\ {}^\circ\psi^{(+)}\\ - \int {}^\circ\psi^{(-)}(mc^2 - \hbar c\boldsymbol{\gamma}^{\mathrm{T}}\cdot\nabla)\ {}^\circ\overline{\psi^{(-)}}. \quad (32)\end{aligned}$$

Here we put $\bar\psi = \psi^\dagger\beta$, and $\boldsymbol{\gamma} = -i\beta\boldsymbol{\alpha}$.

5. TRANSFORMATION TO HEISENBERG REPRESENTATION

In Eq. (9) replace $\Psi_\circ[\sigma]$ by an operator $U[\sigma]$, and solve this equation with [13]

$$U(-\infty) = 1 \quad (33)$$

as boundary condition for U on σ at $t = -\infty$. If we write t instead of σ for surfaces $t = \mathrm{constant}$, Dyson's solution for U is then the infinite series[14]

$$\begin{aligned}U(t) = 1 + (i\hbar)^{-1}\int_{-\infty}^{t} dt_1\, {}^\circ\mathcal{W}_1\\ + (i\hbar)^{-2}\int_{-\infty}^{t} dt_1\int_{-\infty}^{t_1} dt_2\, {}^\circ\mathcal{W}_1\, {}^\circ\mathcal{W}_2 + \cdots. \quad (34)\end{aligned}$$

where $\mathcal{W}_n = \mathcal{W}(t_n) = \int d^3\mathbf{x}W(\mathbf{x}, t_n)$. The inverse of U is seen to be

$$\begin{aligned}U(t)^{-1} = 1 - (i\hbar)^{-1}\int_{-\infty}^{t} dt_1\, {}^\circ\mathcal{W}_1\\ + (i\hbar)^{-2}\int_{-\infty}^{t} dt_1\int_{-\infty}^{t_1} dt_2\, {}^\circ\mathcal{W}_2\, {}^\circ W_1 - \cdots = U(t)^\dagger, \quad (35)\end{aligned}$$

so that U is unitary. Apparently

$$i\hbar dU(t)^{-1}/dt = -U(t)^{-1}\, {}^\circ\mathcal{W}(t) = -{}^{\mathrm{H}}\mathcal{W}(t)U(t)^{-1}, \quad (36)$$

where the superscript prefix $^{\mathrm{H}}$ is defined by

$${}^{\mathrm{H}}Q(t) = U(t)^{-1}\, {}^\circ Q(t)\ U(t). \quad (37)$$

We also put

$$U(t)^{-1}\Psi_\circ(t) = \Phi. \quad (38)$$

Then, by (36), and by (9) or $i\hbar d\Psi_\circ(t)/dt = {}^\circ\mathcal{W}(t)\Psi_\circ(t)$, one easily finds $d\Phi/dt = 0$, so that Φ is a time-independent state vector. From (37) and (38) and the unitariness (35) of U, we find for matrix elements of $Q(t)$ between states a and b:

$$(\Psi_{\circ a}(t)^\dagger,\ {}^\circ Q(t)\Psi_{\circ b}(t)) = (\Phi_a{}^\dagger,\ {}^{\mathrm{H}}Q(t)\Phi_b). \quad (39)$$

We thus have performed a transformation from interaction representation to Heisenberg representation.

6. LORENTZ COVARIANCE OF FIELD VARIABLES IN HEISENBERG REPRESENTATION

Since ${}^\circ W$ was no scalar, we should expect the variables in Heisenberg representation to satisfy Lorentz transformations different from those for the variables in interaction representation,[6,15] but we might try to find new variables, which form tensors in Heisenberg representation. From (37), and from Eq. (9) with U for $\Psi_\circ$, with $\mathbf{x}'$ for $\boldsymbol{\xi}$ and with $\mathbf{b}\cdot\mathbf{r}'$ for ϵ, we find

$$\begin{aligned}\delta^{\mathrm{H}}q(\mathbf{x}) &= U^{-1}(\delta^\circ q)U + U^{-1}\, {}^\circ q(\delta U) - U^{-1}(\delta U)U^{-1}\, {}^\circ qU\\ &= U^{-1}(\delta^\circ q)U + a\int'(\mathbf{b}\cdot\mathbf{r}')U^{-1}[{}^\circ q;\, {}^\circ W']U; \quad (40)\end{aligned}$$

so, if $\delta^\circ q = {}^\circ f(q)$, then

$$\delta^{\mathrm{H}}q = {}^{\mathrm{H}}f(q) + a\int'(\mathbf{b}\cdot\mathbf{r}')^{\mathrm{H}}[q;\, W']. \quad (41)$$

It is easily seen that $\delta\rho$, δV, $\delta\mathbf{E}_{\parallel}$ and $\delta\mathbf{j}$ are not affected but for labels $^{\mathrm{H}}$ replacing $^\circ$ in Eqs. (14), (22), (23). (In the case of $\delta^{\mathrm{H}}\mathbf{j}$, the delta-functions from $[\mathbf{j};\, W']$ are cancelled by the factor $\mathbf{r}'$ in (41).)

However, we find, by (41), (14), (10), (5), (20), (23),

[13] Assuming this in every Lorentz frame we tacitly assume that in some way $W(-\infty) = 0$ in Eq. (9) when used for a change of slope of σ at $t = -\infty$.

[14] F. J. Dyson, Phys. Rev. 75, 486 (1949).

[15] See chapter 3 of reference 5.

and (2):

$$\begin{aligned}\delta\,{}^{\mathrm{H}}\mathfrak{E}(\mathbf{x})&=(\mathbf{b}\times{}^{\mathrm{H}}\mathfrak{B})+\int'(\mathbf{b}\cdot\mathbf{r}')\{{}^{\mathrm{H}}\mathbf{j}'\triangle-({}^{\mathrm{H}}\mathbf{j}'\cdot\nabla)\nabla\}(1/r)\\&=(\mathbf{b}\times{}^{\mathrm{H}}\mathfrak{B})-\delta\,{}^{\mathrm{H}}\mathbf{E}_{\parallel}(\mathbf{x});\qquad(42)\end{aligned}$$

$$\begin{aligned}\partial\,{}^{\mathrm{H}}\mathfrak{B}(\mathbf{x})&=-(\mathbf{b}\times{}^{\mathrm{H}}\mathfrak{E})+\int'(\mathbf{b}\cdot\mathbf{r}')({}^{\mathrm{H}}\mathbf{E}_{\parallel}'\times\nabla)\delta(\mathbf{r}')\\&=-\mathbf{b}\times({}^{\mathrm{H}}\mathfrak{E}+{}^{\mathrm{H}}\mathbf{E}_{\parallel}).\qquad(43)\end{aligned}$$

These equations suggest the introduction of a new variable

$$\mathbf{E}=\mathfrak{E}+\mathbf{E}_{\parallel},\qquad(44)$$

so that from (42)–(43) we find the infinitesimal tensor transformation

$$\delta^{\mathrm{H}}\mathbf{E}=\mathbf{b}\times{}^{\mathrm{H}}\mathfrak{B},\quad\delta^{\mathrm{H}}\mathfrak{B}=-\mathbf{b}\times{}^{\mathrm{H}}\mathbf{E}.\qquad(45)$$

Since $\mathfrak{E}$ is transverse (in either representation), the notation (44) is consistent as far as $\mathbf{E}_{\parallel}$ can now be regarded as the longitudinal part of $\mathbf{E}$, while apparently

$$\mathbf{E}_{\perp}=\mathfrak{E}.\qquad(46)$$

Thus the electromagnetic tensor ${}^{\mathrm{H}}\mathbf{E}$, ${}^{\mathrm{H}}\mathfrak{B}$ in Heisenberg representation has a longitudinal electric part which by (6)–(7) is just the coulomb field derived from the field of charged matter, while only the transverse part of the electromagnetic field is quantized as a photon field.

In a similar way it is easily shown by (41), (21), (10), (5), (20), and (44), that

$$\delta^{\mathrm{H}}\mathfrak{A}(\mathbf{x})=-\int'(\mathbf{b}\cdot\mathbf{r}')({}^{\mathrm{H}}\mathbf{E}'\cdot\nabla')\nabla'(1/4\pi r),\qquad(47)$$

which differs from (21) by the occurrence of $\mathbf{E}$ instead of $\mathfrak{E}$.

Finally we find from (41), (14), (10), from

$$\left.\begin{aligned}&[\psi;\rho']=e\psi'\delta(\mathbf{r}'),\quad[\psi;\mathbf{j}']=e\boldsymbol{\alpha}\psi'\delta(\mathbf{r}'),\\&[\psi;V']=e\psi/r,\quad[\psi;\mathbf{E}_{\parallel}']=-e\psi\nabla'(1/r),\end{aligned}\right\}\ (48)$$

and from (44), that

$$\begin{aligned}\delta^{\mathrm{H}}\psi&=-\tfrac{1}{2}\mathbf{b}\cdot\boldsymbol{\alpha}{}^{\mathrm{H}}\psi-(e/i\hbar c)\int'(\mathbf{b}\cdot\mathbf{r}')\{{}^{\mathrm{H}}\psi{}^{\mathrm{H}}\mathbf{E}'\\&\qquad+{}^{\mathrm{H}}\mathbf{E}'{}^{\mathrm{H}}\psi\}\cdot\nabla'(1/8\pi r)\\&=-\tfrac{1}{2}\mathbf{b}\cdot\boldsymbol{\alpha}{}^{\mathrm{H}}\psi+(e/4\pi i\hbar c)\int'\mathbf{b}\cdot{}^{\mathrm{H}}\mathbf{E}'\,{}^{\mathrm{H}}\psi/r\\&\qquad+(e/i\hbar c)\int'(\mathbf{b}\cdot\mathbf{r}'){}^{\mathrm{H}}\rho'\,{}^{\mathrm{H}}\psi/r.\qquad(49)\end{aligned}$$

In the last term we used $\mathrm{div}\mathbf{E}=\mathrm{div}\mathbf{E}_{\parallel}=4\pi\rho$ (by (44), (6), (7)), and we used (20). In the one but last term, the special ordening of factors was superfluous according to (48).

We conclude that ${}^{\mathrm{H}}\psi$ is no undor under Lorentz transformations, although ${}^{\mathrm{H}}\rho$ and ${}^{\mathrm{H}}\mathbf{j}$ defined by (8) form a four-vector.[16]

In order to eliminate $\mathbf{E}_{\parallel}$ from (49) we use

$$\begin{aligned}&\tfrac{1}{2}\int'(\mathbf{b}\cdot\mathbf{r}'){}^{\mathrm{H}}\rho'/r=-\int'(\mathbf{b}\cdot\mathbf{r}'/8\pi r)\triangle'\,{}^{\mathrm{H}}V'\\&=\text{(integrating by parts and using (20))}\\&\qquad=-\int'\,{}^{\mathrm{H}}V'(\mathbf{b}\cdot\nabla')(1/4\pi r)=-\int'\,{}^{\mathrm{H}}\mathbf{E}_{\parallel}'\cdot\mathbf{b}/4\pi r\end{aligned}$$

[16] A transformation like (49) but with $\mathrm{div}^{\mathrm{H}}\mathbf{E}/4\pi$ replacing ${}^{\mathrm{H}}\rho$ is obtained in manifestly covariant (conventional) quantum electrodynamics for the quantity $\psi'=\exp(ieL/\hbar c)\psi$, where $-\nabla L=\mathbf{A}_{\parallel}$. This ψ' there replaced ψ as $\mathbf{E}$ is replaced by $\mathbf{E}-(1/\mathrm{div})(4\pi\rho)$ when the coulomb interaction is separated and the longitudinal photon field is eliminated by introduction of a new set of canonical variables. See reference 8.

so that

$$\begin{aligned}\delta^{\mathrm{H}}\psi=\{&-\tfrac{1}{2}\mathbf{b}\cdot\boldsymbol{\alpha}+(e/i\hbar c)\mathbf{b}\cdot\int'\,{}^{\mathrm{H}}\mathfrak{E}'/4\pi r\\&+(e/2i\hbar c)\int'(\mathbf{b}\cdot\mathbf{r}'){}^{\mathrm{H}}\rho'/r\}{}^{\mathrm{H}}\psi.\qquad(50)\end{aligned}$$

We shall need all these equations, when we want to investigate the covariance of the energy density tensor.

7. HEISENBERG FIELD EQUATIONS OF MOTION

The equations of motion for ${}^{\circ}q(t)$ or for any function of the ${}^{\circ}q(t)$ can be written in the form (30). In particular the ${}^{\mathrm{H}}q(t)$ are such functions, so that

$$i\hbar\partial^{\mathrm{H}}q(t)/\partial t={}^{\mathrm{H}}q(t){}^{\circ}\mathcal{H}_0(t)-{}^{\circ}\mathcal{H}_0(t){}^{\mathrm{H}}q(t).\qquad(30\mathrm{a})$$

However, we can also write, by (37), (36), and (9),

$$\begin{aligned}i\hbar\partial^{\mathrm{H}}q(t)/\partial t=U(t)^{-1}\{&i\hbar\partial^{\circ}q(t)/\partial t\\&+{}^{\circ}q(t){}^{\circ}\mathcal{W}(t)-{}^{\circ}\mathcal{W}(t){}^{\circ}q(t)\}U(t).\qquad(51)\end{aligned}$$

Putting

$$\mathcal{H}_0+\mathcal{W}=\mathcal{H}\qquad(52)$$

we find from (51) and (30)

$$i\hbar\partial^{\mathrm{H}}q(t)/\partial t={}^{\mathrm{H}}q(t){}^{\mathrm{H}}\mathcal{H}(t)-{}^{\mathrm{H}}\mathcal{H}(t){}^{\mathrm{H}}q(t).\qquad(53)$$

In fact,

$${}^{\mathrm{H}}\mathcal{H}(t)={}^{\circ}\mathcal{H}_0(t)\qquad(54)$$

follows directly from (52), (37), (34)–(35), and (30).

From (10) and (44) we derive

$$W=(\mathbf{E}^2-\mathfrak{E}^2)/8\pi-\mathfrak{A}\cdot\mathbf{j},\qquad(55)$$

so that, by (31) and (52),

$$\begin{aligned}\mathcal{H}=\int\{\mathbf{E}^2+\mathfrak{B}^2\}/8\pi-\int\mathfrak{A}\cdot\mathbf{j}\\+\int\psi^{\dagger}(mc^2\beta-i\hbar c\boldsymbol{\alpha}\cdot\nabla)\psi-\mathcal{H}_{\mathrm{vac.}}\qquad(56)\end{aligned}$$

From (53), (31), (52), (10), (48), (5), (44), (6)–(7), (2) we find the field equations of motion in Heisenberg representation, while the identities are the same as before:

$$\begin{aligned}i\hbar\partial^{\mathrm{H}}\psi/\partial t=(mc^2\beta-i\hbar c\boldsymbol{\alpha}\cdot\nabla-e\boldsymbol{\alpha}\cdot{}^{\mathrm{H}}\mathfrak{A}){}^{\mathrm{H}}\psi\\+\tfrac{1}{2}e({}^{\mathrm{H}}V{}^{\mathrm{H}}\psi+{}^{\mathrm{H}}\psi{}^{\mathrm{H}}V),\qquad(57)\end{aligned}$$

$$\partial^{\mathrm{H}}\rho/c\partial t=-\mathrm{div}^{\mathrm{H}}\mathbf{j},\qquad(58)$$

$$\partial^{\mathrm{H}}\mathfrak{A}/c\partial t=-{}^{\mathrm{H}}\mathfrak{E},\ \text{so that}\ {}^{\mathrm{H}}\mathbf{E}=-\nabla\,{}^{\mathrm{H}}V-\partial^{\mathrm{H}}\mathfrak{A}/c\partial t,\qquad(59)$$

$$\mathrm{div}^{\mathrm{H}}\mathfrak{A}=0,\quad\mathrm{curl}^{\mathrm{H}}\mathfrak{A}={}^{\mathrm{H}}\mathfrak{B},\qquad(60)$$

$$\partial^{\mathrm{H}}V/c\partial t=-\mathrm{div}\int d^3\mathbf{x}'\,{}^{\mathrm{H}}\mathbf{j}(\mathbf{x}')/r,\qquad(61)$$

$$\partial^{\mathrm{H}}\mathbf{E}_{\parallel}/c\partial t=-4\pi{}^{\mathrm{H}}\mathbf{j}_{\parallel},\qquad(62)$$

$$\partial^{\mathrm{H}}\mathfrak{E}/c\partial t=\mathrm{curl}^{\mathrm{H}}\mathfrak{B}-4\pi{}^{\mathrm{H}}\mathbf{j}_{\perp},\qquad(63)$$

$$\partial^{\mathrm{H}}\mathbf{E}/c\partial t=\mathrm{curl}^{\mathrm{H}}\mathfrak{B}-4\pi{}^{\mathrm{H}}\mathbf{j},\qquad(64)$$

$$\begin{aligned}\mathrm{div}^{\mathrm{H}}\mathfrak{E}=0,\quad\mathrm{div}^{\mathrm{H}}\mathbf{E}_{\parallel}=\mathrm{div}^{\mathrm{H}}\mathbf{E}=4\pi{}^{\mathrm{H}}\rho,\\\mathrm{div}^{\mathrm{H}}\mathfrak{B}=0,\qquad(65)\end{aligned}$$

$$\partial^{\mathrm{H}}\mathfrak{B}/c\partial t=-\mathrm{curl}^{\mathrm{H}}\mathfrak{E}=-\mathrm{curl}^{\mathrm{H}}\mathbf{E}.\qquad(66)$$

Our Eqs. (44), (6)–(7), (56), (57), and (64)–(66) are equivalent to or are field-theoretical generalizations of

Eqs. (184), (186), (187) or (175), (169), and (172)-(173), of Pauli's *Handbuch* article,[3] so that essentially we have given a new proof of the intrinsic Lorentz covariance of Pauli's theory (confirming that $\mathbf{E}_{\parallel}$ should not be quantized as a photon field). At the same time, we have shown how Pauli's theory can be extended to electron fields quantized according to the exclusion principle, and also how it can be formulated in interaction representation, making possible some unique covariant kind of distinction between positon and negaton states.

In some following papers we shall formulate a covariant auxiliary condition stating that all photons present can be regarded as at some time having been emitted by a source,[17] we shall discuss the energy density tensor in gauge independent quantum electrodynamics, and we shall show how our proof of the integrability of the generalized Schrödinger equation lends itself to interesting speculations on how a covariant quantum electrodynamics freed from self-interactions might be formulated.[18]

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[17] Compare also F. J. Belinfante, Phys. Rev. **81**, 307(A) (1951).
[18] F. J. Belinfante and J. S. Lomont, Phys. Rev. **83**, 225(A) (1951). See also F. J. Belinfante, Prog. Theor. Phys. **6**, 202 (1951), and Phys. Rev. **82**, 767(A) (1951).

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A Variational Principle for Gauge-Independent Electrodynamics

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A new lagrangian for the maxwell field is defined, expressed completely in terms of gauge-independent transverse field-strengths ($\mathfrak{E}$, $\mathfrak{B}$) and matter variables ($\mathbf{x}_i$, $\mathbf{v}_i$, and $\mathbf{p}_i$ in the classical theory; ψ and $\psi^\dagger$ in the wave-mechanical theory). The coulomb field is defined in terms of matter variables, and a (solenoidal) vector potential is defined in terms of the magnetic field strength $\mathfrak{B}$. The variational principle gives the usual equations of motion and field equations. Quantization leads automatically to the formulas of gauge-independent quantum electrodynamics.

1. CLASSICAL THEORY

The usual lagrangian of electrodynamics has the disadvantage of being expressed in terms of the potentials, which are not uniquely determined and have no direct physical meaning. However, it is possible to derive the maxwell equations for the field and the relativistic equations of motion for charged particles in the field from a variational principle, in which only physically meaningful quantities are to be varied. For instance, in classical electrodynamics we may put, for charged particles (point charges) moving in the microscopic maxwell field with transverse (solenoidal) part $\mathfrak{E}$, $\mathfrak{B}$,

$$\mathfrak{L}=\mathfrak{L}_m+\int[\mathfrak{A}\cdot\mathfrak{E}^t/4\pi c-(\mathfrak{E}^2+\mathfrak{B}^2)/8\pi+\mathfrak{A}\cdot\mathbf{j}-\tfrac{1}{2}\rho V], \quad (1)$$

with[1]

$$\mathfrak{L}_m = \textstyle\sum_i \mathbf{p}_i\cdot(\mathbf{x}^t{}_i-\mathbf{v}_i) - m_ic^2(1-\mathbf{v}_i^2/c^2)^{\frac{1}{2}}, \quad (2)$$

where ^{t} denotes time differentiation ($\mathfrak{E}^t=\partial\mathfrak{E}/\partial t$, $\mathbf{x}^t{}_i=d\mathbf{x}_i/dt$); and then vary the functions $\mathbf{x}_i(t)$, $\mathbf{p}_i(t)$, $\mathbf{v}_i(t)$, and the transverse fields $\mathfrak{E}(\mathbf{x},t)$ and $\mathfrak{B}(\mathbf{x},t)$ independently in

$$\delta\textstyle\int\mathfrak{L}dt=0, \quad (3)$$

keeping the variations zero at the limits of integration. In Eq. (1), ρ, $\mathbf{j}$, V, $\mathfrak{A}$, and $\int$ are abbreviations for

$$\rho(\mathbf{x},t)=\textstyle\sum_i e_i\delta(\mathbf{x}-\mathbf{x}_i(t)), \quad (4a)$$

$$\mathbf{j}(\mathbf{x},t)=\textstyle\sum_i(e_i/c)\mathbf{v}_i\delta(\mathbf{x}-\mathbf{x}_i(t)), \quad (4b)$$

$$V(\mathbf{x},t)=\textstyle\int'\rho(\mathbf{x}',t)/r, \quad (5a)$$

$$\mathfrak{A}(\mathbf{x},t)=\textstyle\int'[\mathrm{curl}'\mathfrak{B}(\mathbf{x}',t)]/4\pi r, \quad (5b)$$

$$\textstyle\int=\int d^3\mathbf{x}, \quad \int'=\int d^3\mathbf{x}', \quad r=|\mathbf{x}-\mathbf{x}'|. \quad (6)$$

From the definition of the electromagnetic radiation field $\mathfrak{E}$, $\mathfrak{B}$ as being a transverse field it follows that

$$\mathrm{div}\mathfrak{B}=0, \quad \mathrm{div}\mathfrak{E}=0. \quad (7a\text{-}b)$$

From (7a) and the definition (5b) we deduce the identities

$$\mathrm{curl}\mathfrak{A}=\mathfrak{B}, \quad \mathrm{div}\mathfrak{A}=0. \quad (8a\text{-}b)$$

We shall further define another vector field

$$\mathbf{E}=-\nabla V-\mathfrak{A}^t/c. \quad (9)$$

On account of (8b), the second term is transverse, while the first term is obviously longitudinal (irrotational), so that we may write

$$\mathbf{E}_{\parallel}=-\nabla V, \quad \mathbf{E}_{\perp}=-\mathfrak{A}^t/c, \quad \mathbf{E}=\mathbf{E}_{\parallel}+\mathbf{E}_{\perp}. \quad (10a\text{-}b\text{-}c)$$

[1] For a discussion of this form for $\mathfrak{L}_m$ see F. J. Belinfante, Phys. Rev. **74**, 779 (1948).