For $|\arg z|>\pi/2$ we use the relation:

$$\operatorname{erfc}(z)=1-2\pi^{-\frac{1}{2}}\int_0^z \exp(-z^2)dz$$

$$=1+2\pi^{-\frac{1}{2}}\int_0^{-z}\exp(-z^2)dz=2-\operatorname{erfc}(-z), \quad (50)$$

so if z is outside the range of (49), $-z$ whose argument is $\pi-\arg z$ will be inside the range, and we have for $|z|\to\infty$ that:

$$\operatorname{erfc}(z)\to 2+\pi^{-\frac{1}{2}}z^{-1}\exp(-z^2), \text{ if } \pi/2<\arg z<3\pi/2. \quad (51)$$

Thus the asymptotic behavior of erfc(z) is defined in the whole complex plane.

Now the function $\chi(r, k, t)$ can be written as:

$$\chi(r, k, t)=\exp i(r^2/2t)\exp z^2\operatorname{erfc}(z), \quad (52)$$

where $z=(2t)^{-\frac{1}{2}}(r-kt)\exp(-i\pi/4)$. When $r\neq 0$ and $t\to 0$, we see that $|z|\to(2t)^{-\frac{1}{2}}r\to\infty$ and $\arg z\to-\pi/4$. Applying then (49) we see that for $t\to 0$,

$$\chi(r, k, t)\to\exp i(r^2/2t)\exp(i\pi/4)\pi^{-\frac{1}{2}}r^{-1}(2t)^{\frac{1}{2}}\to 0 \quad (53)$$

for any complex k.

When $t\to\infty$ we have that $|z|\to|k|(t/2)^{\frac{1}{2}}\to\infty$ and $\arg z\to\frac{3}{4}\pi+\arg k$. From (49, 51), and using the fact that the exponential and erfc(z) are single-valued functions, we have that for $t\to\infty$:

$$\chi(r, k, t)\to-(\pi k^2t/2)^{-\frac{1}{2}}\exp(i\pi/4)\exp i(r^2/2t)\to 0$$
$$\text{if } 3\pi/4<\arg k<7\pi/4,$$
$$\chi(r, k, t)\to 2\exp i(kr-\tfrac{1}{2}k^2t) \quad (54)$$
$$-(\tfrac{1}{2}\pi k^2t)^{-\frac{1}{2}}\exp(i\pi/4)\exp i(r^2/2t)$$
$$\to 2\exp i(kr-\tfrac{1}{2}k^2t) \text{ if } -\pi/4<\arg k<3\pi/4.$$

This result is valid also for $r=0$.

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Quantum Mechanics in Fock Space

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In the present paper we propose to develop a quantum-mechanical scheme in Fock space that would describe interactions that take place through the formation of a compound particle. The discussion will be restricted to a dynamical system representing a single-level scattering process. The state of this dynamical system can be found in two stages: initial particles and compound nucleus. The dynamical behavior of this state will be given in terms of a hamiltonian (that has no classical analog) which will be derived from the boundary conditions satisfied by the state. The representation of the dynamical variables, associated with the two stages of the state, will be discussed, and a complete set of constants of motion of our dynamical system will be given. The expansion of an arbitrary Fock state in terms of the simultaneous eigenstates of this complete set of constants of motion, leads to the generalized Hankel transforms, used in a preceding article for the determination of the time dependent states for scattering and disintegration. In a representation in which the constants of motion are diagonal, the operator representing the relative kinetic energy of the two initial particles will be nondiagonal, particularly in the neighborhood of the resonance energy. The unitary matrix connecting any initial Fock state with the corresponding state at time t will be obtained, and with its help the time-dependent behavior of the Heisenberg dynamical variables of our dynamical system can be derived.

I. INTRODUCTION AND SUMMARY

In preceding articles[1,2] a reformulation of the phenomenological picture of nuclear resonance reactions was achieved in terms of a description in Fock space,[3] in which interactions took place through appropriate boundary conditions. This reformulation was shown to be useful for the description of time dependent states and, in particular, the time dependent description of a nuclear scattering process was discussed in detail in (I).

In the present paper we propose to elaborate a quantum-mechanical scheme in Fock space, that would describe interactions that take place through the formation of a compound particle. For reasons of space, as well as of mathematical simplicity, this scheme will be developed in connection with the problem of scattering of spinless particles whose discussion was initiated in (I). It will be shown though, that the extension of the scheme to the other types of interaction discussed in (II) will not present any essential difficulty.

A scattering process taking place through the formation of a compound particle could be indicated schematically as:

$$A+a\rightleftarrows C, \quad (1)$$

where A represents the initial nucleus, a the incident

[1] M. Moshinsky, Phys. Rev. **84**, 525 (1951). This paper will be referred to as (I).
[2] M. Moshinsky, Phys. Rev. **81**, 347 (1951). This paper will be referred to as (II).
[3] V. Fock, Z. Physik **75**, 622 (1932).

particle, and C the compound nucleus, all of which will be assumed spinless. The description of the scattering process will not be given in the configuration space of the nucleons present, but in the configuration space of the particles present,[2] which will be considered as elementary particles. The scattering process (1) gives then rise to a dynamical system whose states can be found in two stages: The first one, in which two particles A, a are present, the second, where we have a single compound particle C.

The bra and ket notation of Dirac[4] will be used for the states describing the scattering process (1), and representations of these bras and kets can be obtained with the help of an observable that indicates the stages of the states. The connection between the different representations of these Fock states, i.e., transformation theory, will be developed, as well as the representation of linear operators in Fock space.

The dynamical behavior of the state, describing the scattering process (1), was determined in (I) by boundary conditions, obtained from considerations of conservation of probability, connecting the two stages of the state. In quantum mechanics, the dynamical behavior of a state should be determined by the hamiltonian, and in the present paper we shall introduce the hamiltonian for the dynamical system (1), as that linear operator in Fock space that transforms any time dependent Fock state $|Qt\rangle$ into the state $i(d|Qt\rangle/dt)$. With the help of the boundary conditions, an explicit form for this hamiltonian, in any representation, will be obtained.

Besides the hamiltonian there are other observables of the dynamical system (1) such as the relative angular momentum, relative momentum, etc., of the two particles in the first stage. We shall show how to represent these observables in terms of linear hermitian operators in Fock space. As usual, those operators that commute with the hamiltonian will correspond to the constants of motion of the dynamical system (1).

We will obtain the eigen Fock states corresponding to a complete set of constants of motion of the dynamical system (1). The expansion of an arbitrary Fock state in terms of these eigenstates, will lead us to the generalized Hankel transforms (19I), which were used in (I) for the determination of the time-dependent Fock states.

Finally, we shall proceed to determine the time dependent linear operator $\mathbf{U}(t)$ in Fock space, that transforms any initial Fock state $|Q\rangle$ of the dynamical system (1), into the corresponding state $|Qt\rangle$ at time t. The linear operator $\mathbf{U}(t)$ will be given by a unitary matrix which satisfies, as usual,[5] the equation:[6]

$$i(d\mathbf{U}/dt)=\mathbf{H}\mathbf{U}, \tag{2}$$

where $\mathbf{H}$ is the hamiltonian of the dynamical system (1). The unitary matrix $\mathbf{U}(t)$ determines, from a quantum-mechanical standpoint, the dynamical system (1) completely, i.e., it determines the time dependent behavior of all the Heisenberg dynamical variables[5] associated with this dynamical system.

II. TRANSFORMATION THEORY IN FOCK SPACE

We have shown in (I), that interactions between two spinless particles leading to single-level S-wave scattering, can be described in terms of a Fock state of the form:[7]

$$\Psi=\begin{bmatrix}\psi_1(\mathbf{r})\\ \psi_2\end{bmatrix}. \tag{3}$$

The two components of the Fock state describe the fact that it may be found, either in the form of the two particles A, a, represented in their relative configuration space by $\psi_1(\mathbf{r})$, or in the form of a single compound particle C represented by ψ_2.

It is convenient to represent states of the type (3) in the bra and ket notation of Dirac,[4] which greatly simplifies the following analysis. For this purpose we introduce an observable which is restricted to take the values 1 and 2 only, so as to indicate the stage of our state, i.e., if the observable has the value 1 we have the first stage where two particles are present, while for 2 we have the stage in which a single compound particle is present. States of the form (1) would then be denoted by a ket $|\ \rangle$, whose representation would be given by the vector:

$$|\ \rangle=\begin{bmatrix}\langle 1\mathbf{r}|\ \rangle\\ \langle 2|\ \rangle\end{bmatrix}. \tag{4}$$

The appearance of $\mathbf{r}$ in the component $\langle 1\mathbf{r}|\ \rangle$ of (4), indicates that the first stage of our state is represented in the relative configuration representation.

The bra state $\langle\ |$ corresponding to (4), will be represented[4] by the vector:

$$\langle\ |=[\langle\ |\mathbf{r}1\rangle\langle\ |2\rangle]. \tag{5}$$

The scalar product of two states $\langle P|$, $|Q\rangle$ will be defined by:

$$\langle P|Q\rangle=\langle P|\mathbf{r}1\rangle\langle 1\mathbf{r}|Q\rangle+\langle P|2\rangle\langle 2|Q\rangle, \tag{6}$$

where

$$\langle P|\mathbf{r}1\rangle\langle 1\mathbf{r}|Q\rangle\equiv\int\langle P|\mathbf{r}1\rangle d\mathbf{r}\langle 1\mathbf{r}|Q\rangle.$$

In the present paper the convention of summation of repeated indices, will be extended to variables that can

[4] P. A. M. Dirac, *Quantum Mechanics* (Clarendon Press, Oxford, 1947), third edition, Chapter III.

[5] W. Pauli, *Handbuch der Physik*, 2 Auf. Band 24, pp. 138–142; see reference 4, Chapter V, pp. 108–118.

[6] Same units will be used as in (I), i.e., $\hbar=c=\mu=1$.

[7] The Fock state is represented in the center-of-mass reference frame, and this restriction to states of zero total momentum, should be kept in mind. It will be removed at the conclusion of this article.

take a continuous range of values, thus:

$$\langle\,|\mathbf{r}\rangle\langle\mathbf{r}|\,\rangle \equiv \int \langle\,|\mathbf{r}\rangle d\mathbf{r}\langle\mathbf{r}|\,\rangle;$$
$$\langle\,|\mathbf{k}\rangle\langle\mathbf{k}|\,\rangle \equiv \int \langle\,|\mathbf{k}\rangle d\mathbf{k}\langle\mathbf{k}|\,\rangle. \tag{7}$$

The transformation of the first stage $\langle 1\mathbf{r}|\,\rangle$ of the ket (4), to the momentum representation $\langle 1\mathbf{k}|\,\rangle$, can be achieved with the help of the usual[8] transformation function:

$$\langle\mathbf{k}|\mathbf{r}\rangle = \langle\mathbf{r}|\mathbf{k}\rangle^* = (2\pi)^{-\frac{3}{2}} \exp(-i\mathbf{k}\cdot\mathbf{r}). \tag{8}$$

The second stage $\langle 2|\,\rangle$ of the ket (4) represents a single compound particle, and it should remain unaffected by a change of representation for the system of two particles in the first stage. The transformation matrices $\mathbf{T}$, $\mathbf{T}^{-1}$ which take the ket (4) from the relative configuration to the relative momentum representation, and *vice versa*, should have the form:

$$\mathbf{T} = \begin{bmatrix} \langle\mathbf{k}|\mathbf{r}\rangle & 0 \\ 0 & 1 \end{bmatrix}; \quad \mathbf{T}^{-1} = \begin{bmatrix} \langle\mathbf{r}|\mathbf{k}\rangle & 0 \\ 0 & 1 \end{bmatrix}. \tag{9}$$

Thus, the ket (4) in the momentum representation, is given by:

$$\begin{bmatrix} \langle 1\mathbf{k}|\,\rangle \\ \langle 2|\,\rangle \end{bmatrix} = \mathbf{T}|\,\rangle = \begin{bmatrix} \langle\mathbf{k}|\mathbf{r}\rangle\langle 1\mathbf{r}|\,\rangle \\ \langle 2|\,\rangle \end{bmatrix}. \tag{10}$$

Any linear operator $\mathbf{F}$ which acts on a Fock state (10) could be given in the momentum representation by the matrix:

$$\mathbf{F} = \begin{bmatrix} \langle 1\mathbf{k}'|F|\mathbf{k}''1\rangle & \langle 1\mathbf{k}'|F|2\rangle \\ \langle 2|F|\mathbf{k}''1\rangle & \langle 2|F|2\rangle \end{bmatrix}. \tag{11}$$

The action of the operator $\mathbf{F}$ on any ket state, should be interpreted in the sense of a matrix operation on a vector with respect to indices 1, 2 *and* $\mathbf{k}''$. The appearance of a repeated index $\mathbf{k}''$ will imply an integration, in accordance with the convention (7). A similar remark applies to operations on bra states, and for products of linear operators.

We shall now introduce the basic ket $|\mathbf{k}''1\rangle$ to represent a state in which two particles are present with relative momentum $\mathbf{k}''$, and the basic ket $|2\rangle$ to represent the state corresponding to a single compound particle. The vectors corresponding to these kets in the momentum representation are:

$$|\mathbf{k}''1\rangle = \begin{bmatrix} \delta(\mathbf{k}'-\mathbf{k}'') \\ 0 \end{bmatrix}, \quad |2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{12}$$

The basic bras $\langle 1\mathbf{k}'|$, $\langle 2|$ are given by the transposed conjugates of the vectors (12). The configuration representation of the kets (12) is obtained with the help of $\mathbf{T}^{-1}$, so we need only to replace $\delta(\mathbf{k}'-\mathbf{k}'')$ by $\langle\mathbf{r}'|\mathbf{k}''\rangle$ in (12) to have the corresponding vectors.

The unit operator in Fock space $\mathbf{I}$, can be defined in terms of the basic kets, and in the momentum representation it is given by:

$$\mathbf{I} \equiv \|\,|\mathbf{k}''1\rangle|2\rangle\,\| = \begin{bmatrix} \delta(\mathbf{k}'-\mathbf{k}'') & 0 \\ 0 & 1 \end{bmatrix}. \tag{13}$$

In what follows we shall indicate by $\langle P|$, $|Q\rangle$; etc., the constant states, and by $\langle Pt|$, $|Qt\rangle$, etc., the corresponding time dependent states, i.e., those time dependent Fock states which at $t=0$ reduce to $\langle P|$, $|Q\rangle$, etc.

III. THE HAMILTONIAN

We showed in (I) that the time-dependent behavior of the Fock state, representing a single-level scattering process, was determined by the Schrödinger equation (2I) and the boundary conditions (3, 4I). In the notation developed in the previous section, these equation and boundary conditions, applied to a time-dependent ket $|Qt\rangle$, become:

$$-i(\partial\langle 1\mathbf{r}|Qt\rangle/\partial t) = \tfrac{1}{2}\nabla^2\langle 1\mathbf{r}|Qt\rangle, \text{ for } \mathbf{r}\neq 0, \tag{14a}$$

$$-i(\partial\langle 2|Qt\rangle/\partial t) + E_0\langle 2|Qt\rangle = 2\pi C'(\partial r\langle 1\mathbf{r}|Qt\rangle/\partial r)_{r=0}, \tag{14b}$$

$$C'\langle 2|Qt\rangle = (r\langle 1\mathbf{r}|Qt\rangle)_{r=0} \tag{14c}$$

The ket $|Qt\rangle$ is determined completely by the relations (14) once its initial value $|Q\rangle$ is specified

We can now define the hamiltonian of our dynamical system (1), as the linear operator in Fock space $\mathbf{H}$ that transforms the ket $|Qt\rangle$ into $i(d|Qt\rangle/dt)$. To obtain the explicit form of this linear operator, let us consider the time derivative of the scalar product $\langle P|Qt\rangle$ of the form (6), in which $\langle P|$ is an arbitrary constant bra whose component $\langle P|\mathbf{r}1\rangle$ is bounded at $\mathbf{r}=0$, and vanishes appropriately at $\mathbf{r}\to\infty$. We have then:

$$i\frac{d}{dt}\langle P|Qt\rangle = \int \langle P|\mathbf{r}1\rangle i\frac{\partial}{\partial t}\langle 1\mathbf{r}|Qt\rangle d\mathbf{r} + \langle P|2\rangle i\frac{\partial}{\partial t}\langle 2|Qt\rangle. \tag{15a}$$

The component $\langle 1\mathbf{r}|Qt\rangle$ satisfies the Schrödinger equation (14a) for $\mathbf{r}\neq 0$, so that surrounding the origin by a sphere of radius r_0, we can transform the integral in (15a) into:

$$\lim_{r_0\to 0} \tfrac{1}{2}\int_0^{\pi}\int_0^{2\pi} [r\langle P|\mathbf{r}1\rangle(\partial r\langle 1\mathbf{r}|Qt\rangle/\partial r) - (\partial r\langle P|\mathbf{r}1\rangle/\partial r)r\langle 1\mathbf{r}|Qt\rangle]_{r=r_0} \sin\theta d\varphi d\theta$$
$$-\tfrac{1}{2}\int (\nabla^2\langle P|\mathbf{r}1\rangle)\langle 1\mathbf{r}|Qt\rangle d\mathbf{r}. \tag{15b}$$

[8] See reference 4, p. 97.

Since $\langle P|\,\mathbf{r}1\rangle$ is bounded at the origin, and $\langle 1\mathbf{r}|Qt\rangle$ has a singularity at $\mathbf{r}=0$ only for the S-wave component (as shown in (I)), the first term in (15b) reduces to:

$$-2\pi(\partial r\langle P|\,\mathbf{r}1\rangle/\partial r)_{r=0}(r\langle 1\mathbf{r}|Qt\rangle)_{r=0}. \qquad (15c)$$

Using now the boundary conditions (14b, c) we can write finally:

$$\begin{aligned} i(d\langle P|Qt\rangle/dt)=&\int \tfrac{1}{2}k^2\langle P|\,\mathbf{k}1\rangle\langle 1\mathbf{k}|Qt\rangle d\mathbf{k}\\ &-2\pi C'[(\partial r\langle P|\,\mathbf{r}1\rangle/\partial r)_{r=0}\langle 2|Qt\rangle\\ &+\langle P|\,2\rangle(\partial r\langle 1\mathbf{r}|Qt\rangle/\partial r)_{r=0}]\\ &+\langle P|\,2\rangle\tfrac{1}{2}k_0^2\langle 2|Qt\rangle\equiv\langle P|H|Qt\rangle, \qquad (16)\end{aligned}$$

where the integral stands for the second term in (15b) expressed in the momentum representation.

For $t=0$, (16) defines the matrix element of the hamiltonian $\mathbf{H}$, corresponding to two arbitrary Fock states $\langle P|$, $|Q\rangle$. As the representative of a bra is the complex conjugate of the representative for a ket,[4] we see from (16) that $\langle P|H|Q\rangle^*=\langle Q|H|P\rangle$, and the linear operator $\mathbf{H}$ is hermitian. An hermitian matrix for the hamiltonian $\mathbf{H}$ can then be obtained in any given representation.

The hamiltonian $\mathbf{H}$ defined by (16), has clearly no classical analog, as interactions between particles in terms of boundary conditions cannot be formulated in classical mechanics.

IV. PROJECTION OPERATORS AND OBSERVABLES

When our dynamical system (1) is in the first stage, i.e., when the two initial particles are present, there are dynamical variables such as the relative angular momentum, position, momentum, etc., of the two particles, whose expectation values can be obtained from measurements. We expect that these dynamical variables can be represented by linear operators in Fock space of the form (11), which are furthermore hermitian. Before giving their explicit form though, we need to discuss the two basic dynamical variables whose expectation values give the probability of finding a state in the first or in the second stage.

Let us introduce the two hermitian linear operators in Fock space $\mathbf{I}_1$, $\mathbf{I}_2$, defined in the momentum representation by:

$$\mathbf{I}_1=\begin{bmatrix}\delta(\mathbf{k}'-\mathbf{k}'') & 0\\ 0 & 0\end{bmatrix};\quad \mathbf{I}_2=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}. \qquad (17)$$

The operators $\mathbf{I}_1$, $\mathbf{I}_2$ are projection operators, as they reduce any ket of the form (10) to the first or the second stage. Furthermore, $\mathbf{I}_1+\mathbf{I}_2=\mathbf{I}$ where $\mathbf{I}$ is the unit operator (13).

The expectation values of $\mathbf{I}_1$, $\mathbf{I}_2$ for a state $|Q\rangle$ are $\langle Q|\,\mathbf{k}1\rangle\langle 1\mathbf{k}|Q\rangle$ and $\langle Q|\,2\rangle\langle 2|Q\rangle$, respectively. The operators $\mathbf{I}_1$, $\mathbf{I}_2$ are therefore, the ones associated with the first and second stage of the Fock state, as their expectation values give the probability[2] of finding the state in the form of two particles, or in the form of a single compound particle respectively.

With the help of $\mathbf{I}_1$ it now becomes clear what would be the representation of any dynamical variable associated with the relative motion of the two particles in the first stage. We just multiply the ordinary representation of the dynamical variable by $\mathbf{I}_1$ and have its representation in Fock space. Thus for example, for the ith component of the relative angular momentum $\mathbf{L}_i$, we have in the momentum representation a matrix of the form:

$$\begin{aligned}\mathbf{L}_i=&-i\epsilon_{ijl}k_j'\partial/\partial k_l'\mathbf{I}_1\\ \equiv&\begin{bmatrix}-i\epsilon_{ijl}k_j'\partial/\partial k_l'\delta(\mathbf{k}'-\mathbf{k}'') & 0\\ 0 & 0\end{bmatrix}, \qquad (18)\end{aligned}$$

in which ϵ_{ijl} is the usual antisymmetric symbol which is $\neq 0$ only if ijl are even or odd permutations of 1, 2, 3, when it equals $+1$ or -1 respectively.

The linear operators for the components of the relative position $\mathbf{X}_i$ and momentum $\mathbf{P}_j$ are given in the momentum representation by the matrices:

$$\mathbf{X}_i=i\partial/\partial k_i'\mathbf{I}_1,\quad \mathbf{P}_j=k_j'\mathbf{I}_1. \qquad (19)$$

The components $\mathbf{L}_i$ of the vector of relative angular momentum, are constants of motion of the dynamical system (1), as we shall show that they commute with the hamiltonian.

Designating by $\langle P|$, $|Q\rangle$ two arbitrary states, and defining $\langle P'|$, $|Q'\rangle$ by:

$$\langle P'|=\langle P|\mathbf{L}_i,\ |Q'\rangle=\mathbf{L}_i|Q\rangle \qquad (20)$$

we see from (16) that:

$$\begin{aligned}\langle P|HL_i-L_iH|Q\rangle&=\langle P|H|Q'\rangle-\langle P'|H|Q\rangle\\ &=-2\pi C'[\langle P|\,2\rangle(\partial r\langle 1\mathbf{r}|Q'\rangle/\partial r)_{r=0}\\ &\quad-(\partial r\langle P'|\,\mathbf{r}1\rangle/\partial r)_{r=0}\langle 2|Q\rangle], \qquad (21)\end{aligned}$$

where use was made of the hermiticity of $\mathbf{L}_i$ and of the fact that from (18) and (20) we have $\langle 2|Q'\rangle=\langle P'|\,2\rangle=0$. The term $\langle 1\mathbf{r}|Q'\rangle$ could be expanded in spherical waves in which the S-wave would be missing from the definition (20). As we showed in (I) that there is interaction between the initial particles only in the S-wave, we conclude that $\langle 1\mathbf{r}|Q'\rangle$ is regular at $r=0$ and therefore:

$$(\partial r\langle 1\mathbf{r}|Q'\rangle/\partial r)_{r=0}=0.$$

A similar analysis holds for $\langle P'|\,\mathbf{r}1\rangle$ so that we obtain:

$$\langle P|HL_i-L_iH|Q\rangle=0, \qquad (22)$$

and as $\langle P|$, $|Q\rangle$ are arbitary, we see that $\mathbf{L}_i$ commutes with $\mathbf{H}$.

The total angular momentum

$$\mathbf{L}^2=\sum_{i=1}^{3}\mathbf{L}_i^2$$

is clearly also a constant of motion, and $\mathbf{H}$, $\mathbf{L}^2$, $\mathbf{L}_3$ will form a set of commuting constants of motion.

V. EIGENSTATES AND EXPANSION THEOREMS

We proceed to obtain the simultaneous eigenstates of commuting set of constants of motion:

$$\mathbf{H},\ \mathbf{L}^2,\ \mathbf{L}_3. \tag{23}$$

We shall show that this set of constants of motion is a complete set of commuting observables,[4] and that therefore, an arbitrary Fock state can be expanded in terms of the simultaneous eigenstates of this set.

We first notice that if in (16), $\langle P|$ is put equal to the basic bra $\langle 1\mathbf{r}|$ defined by:

$$\langle 1\mathbf{r}| = [\delta(\mathbf{r}-\mathbf{r}') \quad 0]$$

we obtain

$$\langle 1\mathbf{r}|H|Ql\rangle = -\tfrac{1}{2}\nabla^2\langle 1\mathbf{r}|Ql\rangle - 2\pi C'[\partial r'\delta(\mathbf{r}-\mathbf{r}')/\partial r']_{r'=0}\langle 2|Ql\rangle. \tag{24}$$

The term inside the square bracket can be written as:

$$\delta(\mathbf{r}-\mathbf{r}') - \mathbf{r}'\cdot\nabla\delta(\mathbf{r}-\mathbf{r}'),$$

which when $\mathbf{r}'\to 0$ clearly reduces to $\delta(\mathbf{r})$. In Eq. (24) we can thus replace the square bracket by $\delta(\mathbf{r})$.

We now designate by $|Elm\rangle$ the ket which is an eigenstate of $\mathbf{H}$, $\mathbf{L}^2$, $\mathbf{L}_3$ corresponding to the eigenvalues E, $l(l+1)$ and m, respectively.[9] From (16, 18) the eigenket $|Elm\rangle$ in the configuration representation satisfies the equations:

$$-\tfrac{1}{2}\nabla^2\langle 1\mathbf{r}|Elm\rangle - 2\pi C'\delta(\mathbf{r})\langle 2|Elm\rangle = E\langle 1\mathbf{r}|Elm\rangle, \tag{25a}$$

$$-2\pi C'(\partial r\langle 1\mathbf{r}|Elm\rangle/\partial r)_{r=0} + \tfrac{1}{2}k_0^2\langle 2|Elm\rangle = E\langle 2|Elm\rangle, \tag{25b}$$

$$L^2\langle 1\mathbf{r}|Elm\rangle = l(l+1)\langle 1\mathbf{r}|Elm\rangle, \tag{26a}$$

$$L_3\langle 1\mathbf{r}|Elm\rangle = m\langle 1\mathbf{r}|Elm\rangle. \tag{26b}$$

In (26) L^2, L_3 stand for the usual differential operators for the total angular momentum and its projection.

From Eqs. (25a), (26) we see that for $\mathbf{r}\neq 0$, $\langle 1\mathbf{r}|Elm\rangle$ must have the form:

$$\langle 1\mathbf{r}|Elm\rangle = [a_{lm}j_l(kr) + b_{lm}n_l(kr)]P_l^m(\cos\theta)e^{im\varphi}, \tag{27}$$

where a_{lm}, b_{lm} are numerical factors that depend on $k=(2E)^{\frac{1}{2}}$, and as $\mathbf{r}=0$ is excluded, the irregular spherical Bessel function[10] $n_l(kr)$ also appears in (27).

If $b_{lm}\neq 0$ we have that $\langle 1\mathbf{r}|Elm\rangle$ has a singularity[10] of the order $r^{-(l+1)}$ when $r\to 0$. The singularity in $\langle 1\mathbf{r}|Elm\rangle$ for $l\neq 0$ would be too high to satisfy (25), as we know that[11]

$$\nabla^2 r^{-1} = -4\pi\delta(\mathbf{r}). \tag{28}$$

It is clear, therefore, that for $l\neq 0$ the only possible solution of the system of Eqs. (25), (26) is:

$$\langle 1\mathbf{r}|Elm\rangle = a_{lm}j_l(kr)P_l^m(\cos\theta)e^{im\varphi},\ \langle 2|Elm\rangle = 0. \tag{29}$$

We see that the two particles in the first stage of our dynamical system do not interact if their relative angular momentum is larger than zero.

The first component of the eigenket $|E00\rangle$ is given by (27) when l, $m=0$, and it has a singularity of the order r^{-1} when $r\to 0$. Substituting this value of $\langle 1\mathbf{r}|E00\rangle$ into Eq. (25a) and making use of (28) we obtain:

$$2\pi[r\langle 1\mathbf{r}|E00\rangle - C'\langle 2|E00\rangle]\delta(\mathbf{r}) = 0. \tag{30}$$

This equation implies that the square bracket must vanish for $\mathbf{r}=0$, so that we obtain the boundary condition:

$$(r\langle 1\mathbf{r}|E00\rangle)_{r=0} = C'\langle 2|E00\rangle. \tag{31}$$

From (31) and Eq. (25b) corresponding to the ket $|E00\rangle$, we obtain:

$$(r\langle 1\mathbf{r}|E00\rangle)_{r=0} = 4\pi C'^2(k_0^2-k^2)^{-1}(\partial r\langle 1\mathbf{r}|E00\rangle/\partial r)_{r=0} \tag{32}$$

which is precisely the boundary condition for stationary states given in (6I).

Equations (31), (32) determine the eigenket $|E00\rangle$ up to a factor, and its components are given by:

$$\langle 1\mathbf{r}|E00\rangle = a_{00}(k)[j_0(kr) - 4\pi C'^2k(k_0^2-k^2)^{-1}n_0(kr)], \tag{33a}$$

$$\langle 2|E00\rangle = a_{00}(k)4\pi C'(k_0^2-k^2)^{-1}. \tag{33b}$$

The set of commuting constants of motion of our dynamical system is a complete set,[4] as the eigenkets common to them are determined up to a factor. This factor should be determined by the normalization condition, so as to make the scalar product of two eigenstates $|E'l'm'\rangle$, $|E''l''m''\rangle$ equal to:

$$\langle E'l'm'|E''l''m''\rangle = \delta(E'-E'')\delta_{l'l''}\delta_{m'm''}. \tag{34}$$

We see from (29) and (33) that eigenkets corresponding to different l or m are orthogonal. The eigenkets corresponding to the same l, m in which $l>0$, form an orthonormal set, if the a_{lm} are taken essentially as the normalization coefficients for spherical harmonics. It remains to show that for a convenient value of $a_{00}(k)$ the eigenkets $|E00\rangle$ form an orthonormal set.

In the configuration representation, the scalar product of the two eigenkets $|E'00\rangle$, $|E''00\rangle$ is given by

$$\langle E'00|E''00\rangle = 4\pi\int_0^\infty \langle 1\mathbf{r}|E'00\rangle^*\langle 1\mathbf{r}|E''00\rangle r^2dr + \langle 2|E'00\rangle^*\langle 2|E''00\rangle. \tag{35}$$

Introducing the explicit form (33) of the component of

[9] We restrict ourselves to a dynamical system (1) for which the binding energy (see reference 1) $E_0>0$. In this case, we showed in (I) that the eigenvalues E are restricted to real positive values.

[10] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), pp. 404–405.

[11] See reference 4, p. 156.

these kets, and making use of the relation:[12]

$$(2\pi)^{-1}\int_0^\infty \exp(ikr)dr=\delta_+(k)=\tfrac{1}{2}[\delta(k)+i(\pi k)^{-1}]$$

we obtain straightforwardly that:

$$\langle E'00|E''00\rangle=a_{00}^*(k')a_{00}(k'')2\pi^2k'^{-1}(k_0^2-k'^2)^{-2}\times[(k_0^2-k'^2)^2+C^4k'^2]\delta(E'-E''). \quad (36)$$

The ket $|E00\rangle$ would be normalized if we give to the corresponding factor[13] $a_{00}(k)$ the value:

$$a_{00}(k)=(2\pi^2)^{-1}k^{\frac{1}{2}}(k_0^2-k^2)(k_0^2-k^2-ikC^2)^{-1}. \quad (37)$$

With the coefficients determined as above, the eigenkets $|Elm\rangle$ form a complete orthonormalized set,[4] so that an arbitrary ket $|Q\rangle$ can be expanded in terms of them in the usual way:

$$|Q\rangle=\int_0^\infty\sum_{l=0}^\infty\sum_{m=-l}^{l}|Elm\rangle\langle Elm|Q\rangle dE. \quad (38)$$

If the ket $|Q\rangle$ corresponds to zero angular momentum, i.e., $\mathbf{L}^2|Q\rangle=0$, then $\langle Elm|Q\rangle=0$ if $l\neq0$, and in the configuration representation the expansion theorem becomes

$$\begin{bmatrix}\langle 1\mathbf{r}|Q\rangle\\ \langle 2|Q\rangle\end{bmatrix}=\int_0^\infty\begin{bmatrix}\langle 1\mathbf{r}|E00\rangle\\ \langle 2|E00\rangle\end{bmatrix}\langle E00|Q\rangle dE, \quad (39a)$$

$$\langle E00|Q\rangle=4\pi\int_0^\infty\langle 1\mathbf{r}|E00\rangle^*\langle 1\mathbf{r}|Q\rangle r^2dr+\langle 2|E00\rangle^*\langle 2|Q\rangle. \quad (39b)$$

If we introduce the values of $\langle 1\mathbf{r}|E00\rangle$, $\langle 2|E00\rangle$ given by (33), (37), and put $E=\frac{1}{2}k^2$, we see that (39) becomes identical with the generalized Hankel transforms (19I). We have therefore justified, from a general quantum-mechanical standpoint, the mathematical developments introduced in (I), which were used for the time dependent description of the scattering and disintegration processes.

With the help of the eigenstates $|Elm\rangle$, given by (29), (33), any observable $\mathbf{F}$ could be expressed in the hamiltonian-angular momentum representation, i.e., we could find the matrix elements:

$$\langle E'l'm'|F|E''l''m''\rangle. \quad (40)$$

The operators $\mathbf{H}$, $\mathbf{L}^2$, $\mathbf{L}_3$ are, of course, diagonal in this representation, and it is interesting to obtain the representation of the relative kinetic energy operator

$$\tfrac{1}{2}\mathbf{P}^2=\tfrac{1}{2}\sum_{i=1}^3\mathbf{P}_i^2.$$

From (19) we see that

$$\langle E'l'm'|\tfrac{1}{2}P^2|E''l''m''\rangle=\int\tfrac{1}{2}k^2\langle E'l'm'|\mathbf{k}1\rangle\langle 1\mathbf{k}|E''l''m''\rangle d\mathbf{k}. \quad (41)$$

Making use of (16) in which we put $\langle P|=\langle E'l'm'|$, $|Q\rangle=|E''l''m''\rangle$, and of the diagonal form of $\mathbf{H}$, we obtain straightforwardly that

$$\langle E'l'm'|\tfrac{1}{2}P^2|E''l''m''\rangle=E'\delta(E'-E'')\delta_{l'l''}\delta_{m'm''}, \quad (42a)$$

if $l'\neq0$, or $l''\neq0$, or both l', $l''\neq0$; and that

$$\langle E'00|\tfrac{1}{2}P^2|E''00\rangle=E'\delta(E'-E'')+\frac{4C'^2(k_0^2-k'^2-k''^2)(k''k')^{\frac{1}{2}}}{(k_0^2-k''^2-ik''C^2)(k_0^2-k'^2+ik'C^2)}, \quad (42b)$$

where $E'=\frac{1}{2}k'^2$, $E''=\frac{1}{2}k''^2$.

In the hamiltonian-angular momentum representation, the relative kinetic energy of the two particles in the first stage, is a nondiagonal matrix. From (42) we see that the nondiagonal terms become particularly prominent in the neighborhood of the resonance energy $E_0=\frac{1}{2}k_0^2$.

VI. THE UNITARY MATRIX U(t)

We want now to obtain a linear time dependent operator in Fock space $\mathbf{U}(t)$, that transforms any initial Fock state into the corresponding state at time t. For any ket $|Q\rangle$ we have then:

$$|Qt\rangle=\mathbf{U}(t)|Q\rangle. \quad (43)$$

Introducing this relation into (16), we obtain immediately that:

$$i(d\langle P|U|Q\rangle/dt)=\langle P|HU|Q\rangle \quad (44)$$

for any two arbitrary Fock states $\langle P|$, $|Q\rangle$, so that $\mathbf{U}(t)$ satisfies (2).

The simplest way of obtaining $\mathbf{U}(t)$, is to solve (44) in the hamiltonian-angular momentum representation, in which $\mathbf{H}$ is diagonal, so that $\mathbf{U}(t)$ in this representation, which at $t=0$ reduces to the unit matrix, becomes

$$\langle E'l'm'|U|E''l''m''\rangle=\exp(-iE't)\delta(E'-E'')\delta_{l'l''}\delta_{m'm''}. \quad (45)$$

The matrix $\mathbf{U}(t)$ is unitary in this representation, a property which is clearly maintained in any other representation as well.[5]

Using (45), any time dependent state could be expressed in the form,

$$|Qt\rangle=\mathbf{U}(t)|Q\rangle=\int_0^\infty\sum_{l=0}^\infty\sum_{m=-l}^{l}|Elm\rangle\exp(-iEt)\langle Elm|Q\rangle dE. \quad (46)$$

[12] W. Heisenberg, Z. Physik **120**, 519 (1943).

[13] The normalization condition (36) determines $a_{00}(k)$ only up to an arbitrary phase factor, but this phase factor does not affect the form of the expansion theorem (38).

If we express the $|Qt\rangle$ and the scalar product $\langle Elm|Q\rangle$ in the configuration representation, we obtain precisely the procedure followed in Secs. II, III of (I) for the determination of any time dependent state. When we equate $|Q\rangle$ to the basic kets $|\mathbf{k}''1\rangle$, $|2\rangle$ of (12), we obtain, respectively, the time dependent states for scattering and disintegration $\Psi_1(\mathbf{k}'', t)$, $\Psi_2(t)$, given by (23, 27I). In the configuration representation we have, therefore,

$$\begin{aligned}&\langle 1\mathbf{r}|U|\mathbf{k}''1\rangle=\psi_{11}(\mathbf{r},\mathbf{k}'',t),\quad \langle 1\mathbf{r}|U|2\rangle=\psi_{12}(\mathbf{r},t),\\&\langle 2|U|\mathbf{k}''1\rangle=\psi_{21}(\mathbf{k}'',t),\quad \langle 2|U|2\rangle=\psi_{22}(t).\end{aligned}\tag{47}$$

If we want to express $\mathbf{U}(t)$in a pure momentum representation, i.e., to have it in the form of the operator (11), we apply the transformation matrix $\mathbf{T}$ to (47) and obtain

$$\begin{aligned}\langle 1\mathbf{k}'|U|\mathbf{k}''1\rangle&=\langle\mathbf{k}'|\mathbf{r}\rangle\langle 1\mathbf{r}|U|\mathbf{k}''1\rangle,\\\langle 1\mathbf{k}'|U|2\rangle&=\langle\mathbf{k}'|\mathbf{r}\rangle\langle 1\mathbf{r}|U|2\rangle,\end{aligned}\tag{48}$$

while $\langle 2|U|\mathbf{k}''1\rangle$, $\langle 2|U|2\rangle$ remain unaffected.

In the appendix we evaluate the Fourier transform of the function $r^{-1}\chi(r, k, t)$ which appears in ψ_{11}, ψ_{12}. Using this result, we can evaluate the Fourier transforms (48), and obtain the following explicit form of $\mathbf{U}(t)$ in the momentum representation

$$\langle 1\mathbf{k}'|U|\mathbf{k}''1\rangle=\delta(\mathbf{k}'-\mathbf{k}'')\exp(-i\tfrac{1}{2}k''^2t)+(C^2/2\pi^2)\sum_{i=1}^{6}\left\{k_i\left[\prod_{j=1}^{6}{}'(k_i-k_j)\right]^{-1}\chi_0(k_i,t)\right\},\tag{49a}$$

$$\langle 1\mathbf{k}''|U|2\rangle=-C'(2/\pi)^{\frac{1}{2}}\times\sum_{i=1}^{4}\left\{k_i\left[\prod_{j=1}^{4}{}'(k_i-k_j)\right]^{-1}\chi_0(k_i,t)\right\}=\psi_{21}(\mathbf{k}'',t)=\langle 2|U|\mathbf{k}''1\rangle,\tag{49b}$$

$$\langle 2|U|2\rangle=\psi_{22}(t)=(k_1-k_2)^{-1}[k_1\chi_0(k_1,t)-k_2\chi_0(k_2,t)].\tag{49c}$$

In (49), k_1, k_2 stand for the poles of the scattering matrix $S(k)$ of (15I), while $k_3=k''$, $k_4=-k''$, and $k_5=k'$, $k_6=-k'$. The terms

$$\prod_{j=1}^{6}{}'(k_i-k_j),\quad \prod_{j=1}^{4}{}'(k_i-k_j)$$

mean products in which the factor $j=i$ has been omitted. The function $\chi_0(k, t)=\chi(0, k, t)$, so that for any complex k it is given by:

$$\chi_0(k,t)=\exp(-i\tfrac{1}{2}k^2t)\,\mathrm{erfc}[-(1-i)k(t/4)^{\frac{1}{2}}],\tag{50}$$

where erfc(z) is the error integral function of (25I).

The component $\langle 1\mathbf{k}'|U|\mathbf{k}''1\rangle$ is symmetric with respect to $\mathbf{k}'$, $\mathbf{k}''$ as can be seen from (49a) by writing out explicitly the coefficients of the χ_0's. Furthermore, from (49b) we see that $\langle 1\mathbf{k}''|U|2\rangle=\langle 2|U|\mathbf{k}''1\rangle$, so that we conclude that the unitary matrix $\mathbf{U}(t)$ is symmetric[14] in the momentum representation, i.e., $\mathbf{U}'(t)=\mathbf{U}(t)$.

With the help of $\mathbf{U}(t)$ the Schrödinger dynamical variables given in the momentum representation in Sec. IV, could be transformed into the corresponding Heisenberg dynamical variables, in terms of the usual[5] relation:

$$\mathbf{F}(t)=\mathbf{U}^{*\prime}(t)\mathbf{F}\mathbf{U}(t),\tag{51}$$

where $\mathbf{U}^{*\prime}(t)$ stands for the conjugate transposed[14] of $\mathbf{U}(t)$, and the matrix product should be interpreted in the sense of (11). In principle, the time dependence of all Heisenberg dynamical variables associated with the dynamical system (1) can be determined.

An interesting application of (51) is that the commutator of the relative momentum and position of the two initial particles is not a constant of motion. In fact, from (19, 51) we have:

$$\mathbf{X}_i(t)\mathbf{P}_j(t)-\mathbf{P}_j(t)\mathbf{X}_i(t)=i\delta_{ij}\mathbf{I}_1(t)=i\delta_{ij}[\mathbf{I}-\mathbf{I}_2(t)],\tag{52}$$

where in the momentum representation $\mathbf{I}_2(t)$ is given by:

$$\mathbf{I}_2(t)=\begin{bmatrix}\langle 1\mathbf{k}'|U|2\rangle^*\langle 2|U|\mathbf{k}''1\rangle & \langle 1\mathbf{k}'|U|2\rangle^*\langle 2|U|2\rangle\\ \langle 2|U|2\rangle^*\langle 2|U|\mathbf{k}''1\rangle & \langle 2|U|2\rangle^*\langle 2|U|2\rangle\end{bmatrix}.\tag{53}$$

VII. CONCLUSION

We have assumed in all the preceding developments that the dynamical system (1) was described in the center-of-mass frame of reference. It is clear that in arbitrary frame of reference, our states $|Q\rangle$ would be characterized also by the total momentum $\boldsymbol{\kappa}$ of the system. The unitary matrix $\mathbf{U}(t)$ of the previous section, would just have to be multiplied by the factor:[15]

$$\delta(\boldsymbol{\kappa}'-\boldsymbol{\kappa}'')\exp(-i\boldsymbol{\kappa}''^2t/2m)\tag{54}$$

to apply to states in an arbitrary frame of reference.

The developments of the present paper were restricted to the dynamical system (1) representing a single-level S-wave scattering process. We have obtained in (I) boundary conditions for the description of the many-level elastic scattering process, as well as for two-particle nuclear reactions, which could be indicated schematically as:

$$A+a\rightleftarrows\begin{Bmatrix}C_1\\ \vdots\\ C_n\end{Bmatrix},\tag{55a}$$

and

$$A+a\rightleftarrows C\rightleftarrows B+b,\tag{55b}$$

respectively. The procedure followed in the present paper suggests how to formulate the states of the dynamical systems (55) in Fock space, and also how to obtain the corresponding hamiltonians from the bound-

[14] In the present notation, $\mathbf{F}'$ means that we take the transposed of the 2×2 matrix (11), and also interchange $\mathbf{k}'$, $\mathbf{k}''$.

[15] m stands for the sum of the masses of the two particles in the first stage, in units of the reduced mass.

ary conditions. We expect, therefore, no essential difficulty in the generalization of the present developments to the dynamical systems (55).

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APPENDIX

We shall evaluate here the Fourier transform of the function:

$$r^{-1}\chi(r,k,t)=r^{-1}\exp i(kr-\tfrac{1}{2}k^2t)\,\mathrm{erfc}(u), \qquad (56)$$

where

$$\mathrm{erfc}(u)=2\pi^{-\frac{1}{2}}\int_u^\infty \exp(-z^2)dz, \quad u=(1-i)(4t)^{-\frac{1}{2}}(r-kt),$$

and k is an arbitrary complex number.

As χ depends only on the magnitude of the position vector, we have:

$$I(k,k',t)=(2\pi)^{-\frac{3}{2}}\int \exp(-i\mathbf{k}'\cdot\mathbf{r})r^{-1}\chi(r,k,t)d\mathbf{r}$$

$$=(2/\pi)^{\frac{1}{2}}k'^{-1}\int_0^\infty \chi(r,k,t)\sin k'r\,dr$$

$$=(2/\pi)^{\frac{1}{2}}(2ik')^{-1}\exp(-i\tfrac{1}{2}k^2t)$$

$$\times[I_+(k,k',t)-I_-(k,k',t)], \qquad (57)$$

where

$$I_\pm(k,k',t)=\int_0^\infty \exp[i(k\pm k')r]\,\mathrm{erfc}(u)dr.$$

Integrating by parts we obtain:

$$I_\pm(k,k',t)=[-i(k\pm k')^{-1}\exp[i(k\pm k')r]\,\mathrm{erfc}(u)]_0^\infty$$

$$-i(k\pm k')^{-1}\int_0^\infty \exp[i(k\pm k')r]$$

$$\times\exp(-u^2)[2\pi^{-\frac{1}{2}}(1-i)(4t)^{-\frac{1}{2}}]dr. \qquad (58)$$

When $r\to\infty$, $\mathrm{erfc}(u)\to 0$ as shown in Appendix 2 of (I), so that the first parenthesis reduces to its value at $r=0$. Denoting by $u_\pm'$ the expression:

$$u_\pm'=(1-i)(r\pm k't)(4t)^{-\frac{1}{2}} \qquad (59)$$

we can reduce $I_\pm(k,k',t)$ to the form:

$$I_\pm(k,k',t)=i(k\pm k')^{-1}\,\mathrm{erfc}[-(1-i)k(t/4)^{\frac{1}{2}}]$$

$$-i(k\pm k')^{-1}\exp[i(k^2-k'^2)(t/2)]$$

$$\times\left\{2\pi^{-\frac{1}{2}}\int_0^\infty \exp(-u_\pm'^2)(1-i)(4t)^{-\frac{1}{2}}dr\right\}. \qquad (60)$$

The term in the curly bracket can be written as:

$$2\pi^{-\frac{1}{2}}\int_{V_\pm}^\infty \exp(-u'^2)du'=\mathrm{erfc}(V_\pm), \qquad (61)$$

where $V_\pm=\pm(1-i)k'(t/4)^{\frac{1}{2}}$.

Using (57, 60, 61), we can write finally:

$$I(k,k',t)=-(2/\pi)^{\frac{1}{2}}(k^2-k'^2)^{-1}\chi_0(k,t)$$

$$+(2/\pi)^{\frac{1}{2}}(2k')^{-1}(k-k')^{-1}\chi_0(k',t)$$

$$-(2/\pi)^{\frac{1}{2}}(2k')^{-1}(k+k')^{-1}\chi_0(-k',t), \qquad (62)$$

where $\chi_0(k,t)$ is given by (50).