Electron Density Distribution in a High Frequency Discharge in the Presence of Plasma Resonance*

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(Received July 20, 1951)

In a high frequency discharge plasma resonance maximizes the electric field, thus producing a high ionization rate in the regions near resonance. The effect of this on the distribution of electrons and of ionization in a parallel plane discharge is calculated and compared with the observation that the light from such a discharge often is a minimum at the center.

I. INTRODUCTION

IT was pointed out by Schumann[1] that when a high frequency field is applied to an ionized gas of high electron concentration, the field tends to concentrate in those parts of the discharge where the plasma is near resonance.[2] The admittance of a high frequency discharge under such conditions was discussed in a recent paper.[3] The appearance of a microwave discharge at high current is shown in Fig. 1 in which the discharge is taking place between parallel plates at the right and left of the bright glow, and it is noted that the light intensity is greatest near the electrodes. It may be inferred that the field is greatest here, and this corresponds, according to Schumann, to regions where the plasma is in resonance and is shielding the central part of the discharge. It is the purpose of this paper to calculate the electron density throughout the discharge under such conditions, and to show that the ionization has sharp maxima in the resonance regions.

II. THE IONIZATION FUNCTION

For a given applied field of radian frequency, ω, plasma resonance occurs at an electron density,[2]

$$n = m\omega^2\epsilon_0/e^2, \tag{1}$$

where e and m are the charge and mass of an electron, and ϵ_0 is the electric permittivity of free space. It is convenient to measure electron concentration in terms of this by the variable,

$$r = ne^2/m\omega^2\epsilon_0, \tag{2}$$

which is unity when the plasma is in resonance. A damping constant for the plasma is given by

$$\beta = \nu_c/\omega, \tag{3}$$

where ν_c is the average frequency of collision of an electron with the molecules of the gas. The formula for the total current density, including both electron current and displacement current is given by Eqs. (5) and (9) of reference 3. In terms of the parameters defined above it is

$$J = [ne^2/m(\nu_c + j\omega) + j\omega\epsilon_0]E$$
$$= [r(\beta - j)/(\beta^2+1) + j]\omega\epsilon_0 E, \tag{4}$$

from which the magnitude of the resistivity is

$$|E/J| = (1/\omega\epsilon_0)(\beta^2+1)^{\frac{1}{2}}[(r-1)^2+\beta^2]^{-\frac{1}{2}}. \tag{5}$$

This quantity has a maximum at resonance where $r = 1$.

The ionization frequency per electron ν_i depends on the applied field, and over a limited range about a central value E_0 we shall assume, following Herlin and Brown,[4] that it varies as a power 2α of the field:

$$\nu_i/\nu_0 = |E/E_0|^{2\alpha}. \tag{6}$$

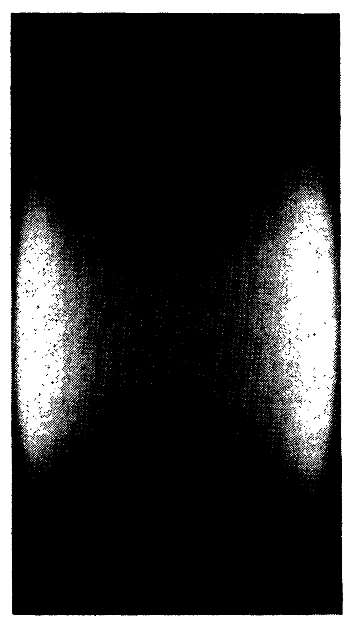


FIG. 1. Appearance of the high frequency discharge.

* This work has been supported in part by the Signal Corps, the Air Materiel Command, and ONR.

[1] W. O. Schumann, Z. Physik 7, 121 (1942).
[2] L. Tonks and I. Langmuir, Phys. Rev. 33, 195 and 990 (1929).
[3] E. Everhart and S. C. Brown, Phys. Rev. 76, 839 (1949).
[4] M. A. Herlin and S. C. Brown, Phys. Rev. 74, 291 (1948) and Phys. Rev. 74, 910 (1948). Our α is $\beta/2$ in their notation.

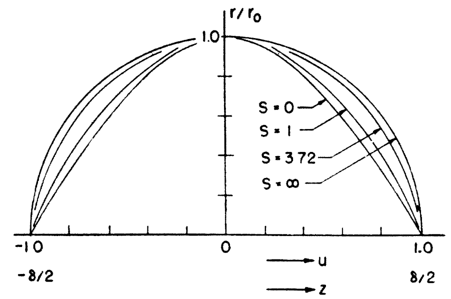


FIG. 2. Variation of electron density r across a discharge for which $\alpha=2$ in the limit of high pressures for various ratios $s=r_0(\beta^2+1)^{-\frac{1}{2}}$.

With the value of E from Eq. (5) this is

$$\nu_i=\nu_0|J/\omega\epsilon_0E_0|^{2\alpha}(\beta^2+1)^\alpha[(r-1)^2+\beta^2]^{-\alpha}. \quad (7)$$

The ionization rate is thus a function of the electron density with a maximum at $r=1$.

III. THE DIFFUSION EQUATION

The distribution of electrons in the discharge is determined by the production rate $n\nu_i$ in the gas and diffusion to the walls, and it will be assumed that the electron density is such that ambipolar diffusion exists. Limiting ourselves to the case of extended plane parallel plates perpendicular to the z-axis, the diffusion equation is[3]

$$d^2n/dz^2+n\nu_i/D_a=0 \quad (8)$$

where D_a is the ambipolar diffusion coefficient. It is convenient to normalize distance by the variable $u=2z/\delta$, where z is measured from the mid-plane and δ is the plate separation.

After substituting from Eq. (7) we then express Eq. (8) in terms of r and u:

$$d^2r/du^2+k^2r/(r^2-2r+1+\beta^2)^\alpha=0, \quad (9)$$

where

$$k^2=[\delta^2\nu_0(1+\beta^2)^\alpha/4D_a]|J/\omega\epsilon_0E_0|^{2\alpha}, \quad (10)$$

is a constant. It is determined as a characteristic value for Eq. (9) with the boundary conditions:

$$\begin{aligned} &r=r_0, \quad dr/du=0, \quad \text{at} \quad u=0\\ &r=0, \quad \text{at} \quad u=\pm 1. \end{aligned} \quad (11)$$

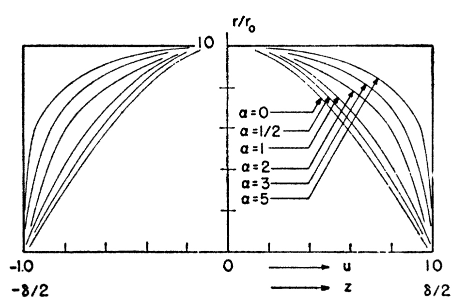


FIG. 3. Variation of electron density r across a discharge. Limiting form at very high electron concentrations $s\gg 1$ for various values of α.

Equation (9) integrates once to give

$$\left(\frac{dr}{du}\right)^2=k^2\int_r^{r_0}\frac{2rdr}{(r^2-2r+1+\beta^2)^\alpha}=k^2L^2. \quad (12)$$

The function L introduced here is proportional to the electron density gradient and, therefore, also to the electron diffusion current. A second integration gives the electron density as an implicit function of position

$$ku=\int_r^{r_0}dr/L. \quad (13)$$

The characteristic value of k is then given by setting the lower limit $r=0$ with $u=1$.

IV. INTEGRATION

The integration in (12) is readily performed for any integral or half-integral value of α but the integration (13) causes trouble. We shall carry it out with four different approximations applicable to different discharges or to different parts of the same discharge.

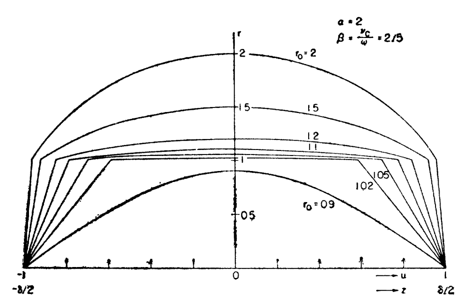


FIG. 4. Variation of electron density across a discharge at low pressures.

From these the general form of the solution may be readily visualized. The four approximations will be:

$$\text{A:} \quad 2r\ll r^2+1+\beta^2.$$

By neglecting the only negative term in the denominator of (12) all resonance effects on the electron distribution are eliminated. This approximation is always good at high pressures $\beta^2\gg 1$ and also at any pressure for electron concentrations not near resonance. The next two approximations are sub-groups of A.

$$\text{B:} \quad r^2-2r\ll 1+\beta^2.$$

This amounts to neglecting the ac electron current relative to the displacement current or to assuming a uniform field. It is good only for electron concentrations well below resonance or for sufficiently high pressures.

$$\text{C:} \quad \beta^2+1\ll r^2.$$

This approximation neglects the displacement current. It is good only at electron concentrations above resonance and therefore never good near the walls.

$$\text{D:} \quad \beta\ll 1.$$

This neglects the resistive current and is the low pressure limit. The effects of resonance dominate this approximation.

A: Neglecting the term $2r$ in the denominator of (12) the integration yields, for $\alpha\neq 1$,

$$L^2=[(r^2+1+\beta^2)^{1-\alpha}-(r_0^2+1+\beta^2)^{1-\alpha}]/(\alpha-1). \quad (14)$$

Substitution in (13) then leads to incomplete elliptic integrals[5] for all integral values of α. Substituting

$$\kappa=r_0(r_0^2+1+\beta^2)^{-\frac{1}{2}} \quad \text{and} \quad \cos\phi=r/r_0 \quad (15)$$

we find for $\alpha=2$

$$ku=(r_0^2+1+\beta^2)E(\kappa,\phi) \quad (16)$$

and for $\alpha=3$

$$ku=(r_0^2+1+\beta^2)^{\frac{1}{2}}[2E(\kappa/\sqrt{2},\phi)-F(\kappa/\sqrt{2},\phi)]. \quad (17)$$

Curves of r/r_0 *vs* u for $\alpha=2$ and various values of $r_0(\beta^2+1)^{-\frac{1}{2}}=s$ are shown in Fig. 2. The two limiting curves in this case ($\alpha=2$) are a cosine and a circle, the effect of increasing electron concentration being always to increase the concentration near the walls relative to that in the center.

B: Neglecting r^2-2r in the denominator of (12) leads to elementary integrations,

$$L^2=(r_0^2-r^2)/(1+\beta^2)^\alpha \quad (18)$$

$$r=r_0\cos[ku/(1+\beta^2)^{\alpha/2}]=r_0\cos(\pi u/2) \quad (19)$$

$$k=(1+\beta^2)^{\alpha/2}\pi/2. \quad (20)$$

This is the solution of the simple diffusion equation and substitution of this last expression into (10) and (7) leads to the usual equilibrium condition

$$\nu_i=\pi^2 D_a/\delta^2. \quad (21)$$

The field and ionization rate per electron are, in this case, uniform throughout the discharge.

C: The neglect of all but r^2 in the denominator of (12) leads to the integral

$$ku=(\alpha-1)^{\frac{1}{2}}r_0^{\alpha-1}\int_r^{r_0} r^{\alpha-1}(r_0^{2\alpha-2}-r^{2\alpha-2})^{-\frac{1}{2}}dr. \quad (22)$$

This integration has been performed for a number of values of α and the results are shown in Fig. 3. For $\alpha=0$ the curve is a cosine, for $\alpha=2$ it is a circle, and as $\alpha\rightarrow\infty$ it approaches a rectangular box.

D: The integrand of (12) has singularities at $r=1\pm j\beta$ but the approximations made in A, B, and C put these singularities back on the imaginary axis so that the path of integration did not go between them. At low pressures these singularities lie close to, and on either side of, the real axis and the integration (12) must pass between them if $r_0>1$. The major part of the integral then comes, for small β, from the saddle-point region near $r=1$. dr/du changes rapidly in this region and there is a bend in the plot of r *vs* u. In order to obtain the value L_0 of L at the boundary we may set $r=1$ in the numerator of (12) and integrate from zero to infinity without appreciable error.

$$L_0^2=\int_0^{r_0}\frac{2rdr}{[(r-1)^2+\beta^2]^\alpha}=\int_{-\infty}^{+\infty}\frac{2dr}{[(r-1)^2+\beta^2]^\alpha}=\frac{2\sqrt{\pi}}{\beta^{2\alpha-1}}\frac{\Gamma(\alpha-1/2)}{\Gamma(\alpha)}. \quad (23)$$

In the central part of the discharge not near resonance approximation C holds, so that at the resonance point u_1

$$ku_1=(\alpha-1)^{\frac{1}{2}}r_0^{\alpha-1}\int_1^{r_0} r^{\alpha-1}(r_0^{2\alpha-2}-r^{2\alpha-2})^{-\frac{1}{2}}dr. \quad (24)$$

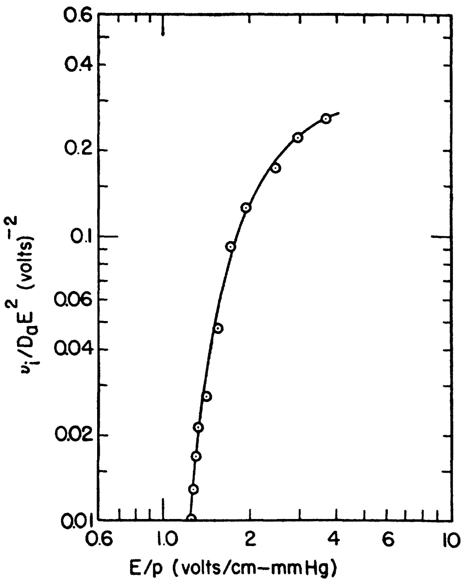


FIG. 5. Variation of ν_i/D_aE^2 with E/p in helium for large β.

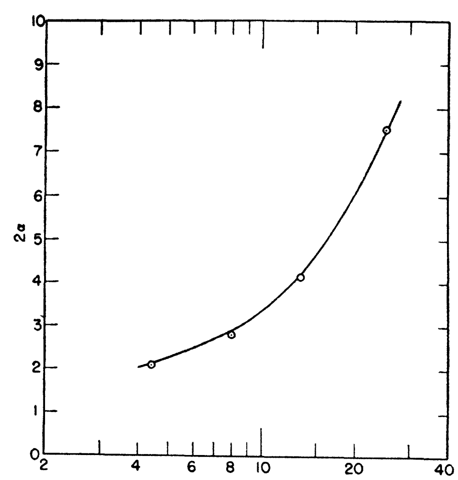


FIG. 6. Variation of α with $p\delta$ in helium for large β.

[5] E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, New York, 1943), pp. 52–72.

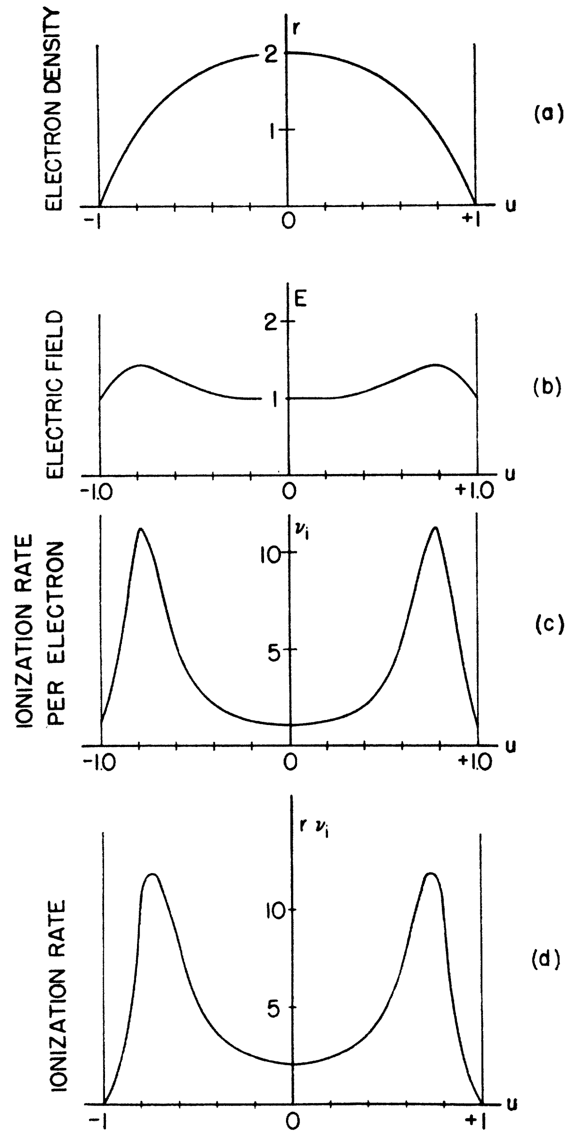


FIG. 7. Variation of electron density, electric field, and ionization rate across the discharge.

Near the walls approximation B holds because $r \ll 1$.

$$L^2 = L_0^2 - \int_0^r 2r dr = L_0^2 - r^2, \tag{25}$$

$$k(1-u_1) = \int_0^1 (L_0^2 - r^2)^{-\frac{1}{2}} dr = \sin^{-1}(1/L_0) \approx 1/L_0, \tag{26}$$

because L_0 is large when β is small. Equations (24) and (26) determine k and u_1.

These relations are particularly simple for $\alpha = 2$. Then $L_0^2 = \pi/\beta^3$, $k = r_0(r_0^2-1)^{\frac{1}{2}} + 1/L_0$,

$$\frac{1}{u_1} = 1 + \frac{1}{L_0 r_0 (r^2-1)^{\frac{1}{2}}}.$$

Curves for this case are shown in Fig. 4. These curves have sharp bends at the resonance points because of the approximations made. The lowest curve does not reach resonance and is a cosine.

V. DETERMINATION OF α

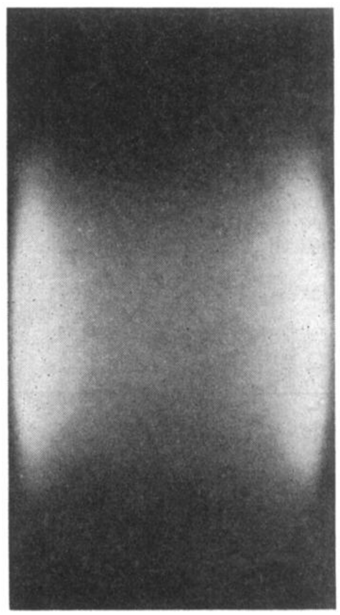
One method of determining α experimentally will be described to show that at a given frequency there is a relationship between α and β. The values of α can be obtained from measurements of the operating field E at various pressures p when the electron concentration is kept below resonance but not so small that ambipolar diffusion does not prevail. Under these conditions E is constant at a given pressure independent of both discharge current and position. Therefore ν_i/D_a will also be constant—equal, in fact, to π^2/δ^2 as is given by (21). Data are thus obtained for making a plot of ν_i/D_aE^2 *vs* E/p which are the proper variables for the problem as Herlin and Brown[4] have shown. The slope of this curve on logarithmic paper is $2\alpha-2$, from Eq. (6), and thus α is determined at various electric fields and pressures.

The procedure outlined above is essentially similar to that of Herlin and Brown who treated the breakdown condition instead of the steady-state discharge studied here. They point out further that there is a curve of ν_i/D_aE^2 *vs* E/p for each value of $p\lambda$, where λ is the free space wavelength of the exciting field, and $\beta = 0.013p\lambda$ (cm-mm Hg) for helium.

Data for the limiting case of large β have been obtained for helium and plotted in Fig. 5. The corresponding values of α *vs* $p\delta$ obtained from the slope of this curve are plotted in Fig. 6.

VI. DISTRIBUTION OF IONIZATION

The electric field and ionization distribution will be computed in detail for a particular discharge in helium. For a value of $p\delta$ of 23 we find from Fig. 6 that $\alpha = 7/2$. Also if $p\lambda = 77$ we find that $\beta = 1$. We further choose the electron density at the center of the discharge to be such that $n\lambda^2 = 2.2\times 10^{13}$/cm so that $r_0 = 2$. The curve of electron density *vs* position for these values is shown in Fig. 7(a). The variation of electric field across the discharge is computed from Eq. (5) and shown in Fig. 7(b). Equation (6) can then be used to compute Fig. 7(c) which shows the variation of the ionization rate per electron at various positions across the discharge. The ionization rate will be the product of the ionization rate per electron by the electron concentration and is thus the product of the curve of Fig. 7(c) with that of Fig. 7(a). This is shown in Fig. 7(d). Assuming that the excitation at each point is proportional to the ionization at that point this should also correspond to the light intensity emitted by various portions of the discharge.



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