PHYSICAL REVIEW VOLUME 84, NUMBER 3 NOVEMBER 1, 1951

Divergent Integrals in Renormalizable Field Theories

ABDUS SALAM*
Institute for Advanced Study, Princeton, New Jersey

(Received June 22, 1951)

The proofs of the renormalization of the theories of scalar and pseudoscalar mesons in scalar interaction with nucleons and with the electromagnetic field, given previously by the author, depended on a certain subtraction procedure. A general proof is here given that this subtraction procedure, when applied to a divergent integral in the S matrix of a renormalizable field theory, does, in fact, lead to an absolutely convergent, covariant, and unique remainder.

I

SINCE Dyson's[1] proof of the possibility of absorbing all divergences in spinor electrodynamics, by renormalizing mass and charge, it has been possible to extend his considerations and to renormalize the theories of scalar and pseudoscalar mesons in (scalar) interactions with the nucleons, and with the electromagnetic field.[2] The proofs in these cases depend upon the success of a generalized procedure[3] for subtracting divergences from the integrals occurring in these theories. This procedure was outlined in **1**, Sec. III, and it was asserted that after the subtractions are carried out in a prescribed manner, the remainder is an absolutely convergent integral in a strict mathematical sense. A proof of this statement is given in this note. The conditions under which such a procedure succeeds brings out clearly the difference between theories which can be renormalized and those for which the concept of renormalization of constants has proved inadequate.

Regarding the general structure of the graphs and the corresponding integrals occurring in the matrix elements of Dyson's S matrix, we note the following:

(i) In a connected Feynman-Dyson graph consisting of N vertices and F internal lines, $n=F-N+1$ lines can be chosen such that the momenta corresponding to the remaining $(N-1)$ lines can be expressed as linear sums of the momentum variables $t_1, t_2, \cdots t_n$ which can be assigned to these n lines. Thus, the graph gives rise to an n-fold integral $I(n)$ over these basic variables.[4]

(ii) For all "renormalizable" theories (i.e., the theories for which the number of primitive divergent graphs is governed by conditions of the type $E<a$, E being a linear sum of the number of external lines and a being a fixed constant), not only is the number of primitive divergent graphs limited, but further, perhaps more important, the degree of divergence of the integrals[5] corresponding to these graphs does not depend on the order of the graphs.

(iii) We are interested in the absolute and not the conditional convergence of our final expressions; that is to say, their convergence must be independent of the choice of basic variables. Irrespective of the choice of basic variables, all the possible divergent subintegrations can be detected graphically by seeing which parts of the graph satisfy the condition $E<a$, where E is the number of external lines of the part in question. By a "wrong" choice of basic variables, it is possible to make some of the integrals corresponding to these parts conditionally convergent; for example, a subintegration[6] which conditionally converges for t_1 and t_2 integrations may diverge if $t_1'=t_1+t_2$ is chosen as a basic variable. The proof the effectiveness of the subtraction procedure given in Sec. II depends on the following lemma

Lemma 1:—There exists at least one "correct" choice of basic variables, such that every possible genuinely divergent part of the graph has associated with it a divergent subintegration over one or more of the basic variables.

This lemma is proved in Sec. III. In any graph there may be more than one set of "correct" variables. It is also shown in Sec. III that the finite remainder after the proposed subtractions is the same for any "correct" set. It will be seen in Sec. III that the lemma is true under extremely restrictive conditions. These happen to be precisely the conditions which renormalizable field theories satisfy.

II

The detailed prescription for subtracting divergences from overlapping graphs has been given in **1**, Sec. III. We write the given integral $I(n)$ as

$$\begin{aligned}I(n)=&D(t_1)R(t_2t_3\cdots t_n)+D(t_2)R(t_1t_3\cdots t_n)+\cdots\\&+D(t_1t_2)R(t_3t_4\cdots t_n)+\cdots\\&+D(t_1t_2t_3)R(t_4t_5\cdots t_n)+\cdots+\cdots\\&+D(t_1t_2\cdots t_n)+I_c(t_1t_2\cdots t_n).\end{aligned}\quad(1)$$

* Now at Government College, Lahore, Pakistan.

[1] F. J. Dyson, Phys. Rev. **75**, 1736 (1949).

[2] Abdus Salam, Phys. Rev. **82**, 217 (1951), referred to as **1**; and "Renormalized S matrix for scalar electrodynamics," to be published.

[3] This procedure was suggested by F. J. Dyson, to whom the author is deeply indebted for kind help and discussion.

[4] The domain of integration is in fact a $4n$-dimensional space; but the treatment being wholly relativistic, the 4 components of any vector are never separately considered. We, therefore, speak of n-fold rather than $4n$-fold integrations.

[5] The degree of divergence of an integral is estimated by employing the basic considerations given in Sec. V of Dyson's paper and merely counting the powers of relevant t in the numerator and denominator of the integrand.

[6] If a set of variables $t_at_b\cdots t_m$ is held fixed, the integration over the remaining variables is called a subintegration.

Here $D(t_it_j\cdots)$ is the true divergence from the part of the graph corresponding to the $t_it_j\cdots$ subintegration and R's represent the reduced integrals. It is to be shown that I_c, the remainder left after all terms $D\times R$ have been subtracted from $I(n)$ is absolutely convergent.

In general, the integrand of $I(n)$ is a product of "simple" factors F, H, J, etc., each "simple" factor F being a function of one single variable t_i and external momenta (p), and of overlap factors G, K, etc., which are functions of more than one variable.

In any term like $D(t_1)R(t_2t_3\cdots t_n)$, $D(t_1)$ is a divergent constant; analytically, $D(t_1)$ is a divergent constant obtained by an operation $T(t_1)$ on the t_1 subintegration, the integrand of this subintegration consisting of all those factors F, G, etc., in the integrand of $I(n)$ which contain t_1. All momenta other than t_1 in these factors are treated as "external." There is one operator $T(t_1)$ for each type of divergence in the theory; this operator may correspond, for example, to giving all the "external" momenta special "free-particle" values or may also involve differentiations with respect to "external" momenta before such "free" values are substituted if the divergence over t_1 subintegration is more than logarithmic. In practice, the operation T may be even more complicated, but in all cases (so far developed) T acts by changing the values of "external" momenta. In actual fact, the result of operation $T(t_1)$ is not merely $D(t_1)$ but $D(t_1)\gamma_\mu$, $D(t_1)\delta_{\mu\nu}$, or $D(t_1)\Delta_F^{-1}$, etc., where γ_μ, $\delta_{\mu\nu}$, etc., will be called "vertex factors." These "vertex factors" are absorbed in the reduced integrals. $R(t_2t_3\cdots t_n)$, for example, is an integral with the integrand consisting of all factors G, H, etc., in the original integrand which do not contain t_1, multiplied by the "vertex factor" from the t_1 integration. $R(t_2t_3\ \cdots\ t_n)$ corresponds to a graph, called the reduced graph, which has exactly the same number of external lines as I, but is of lower order in e.

We notice at this stage that the operation

$$[1-T(t_1)]I(n)=[I(n)-D(t_1)R(t_2t_3\cdots t_n)]$$

can be thought of graphically as the insertion of the convergent part of the t_1 integration at the appropriate place in the reduced graph $R(t_2t_3\cdots t_n)$.

The general procedure for obtaining a true divergence, $D(t_1t_2)$, for example, has been formulated in **1**, Sec. III. Consider the integral

$$[1-T(t_1)-T(t_2)]I=[I-D(t_1)R(t_2t_3\cdots t_n)-D(t_2)R(t_1t_3\cdots t_n)]. \quad (2)$$

An operation $T(t_1t_2)$ on this new integral gives $D(t_1t_2)$. For this operation all momenta other than t_1, t_2 are "external" and explicitly the operation is only designed to make the double integration over t_1t_2 in

$$[1-T(t_1t_2)][1-T(t_1)-T(t_2)]I(n) \quad (3)$$

convergent. It was shown in **1**, Sec. II, that (3) in fact proves to be convergent over t_1 and t_2 integrations considered singly as well. The proof may be repeated. Holding t_2 fixed, we rewrite (3) as

$$=[1-T(t_1)]I-T(t_1t_2)[1-T(t_1)]I-[1-T(t_1t_2)]T(t_2)I. \quad (4)$$

The first two terms obviously converge over t_1 integration.

We can rewrite the third term as

$$[1-T(t_1t_2)]T(t_2)I=[1-T(t_1t_2)]D(t_2)R(t_1t_3\cdots t_n).$$

$T(t_1t_2)$ operates only on the t_1t_2 integration, giving all variables "external" to the t_1t_2 integration certain special values. Here, since $D(t_2)$ is a constant and does not contain any external momenta, $T(t_1t_2)$ operates, in effect, only on that part of the integrand which contains t_1. The convergence of $[1-T(t_1t_2)]D(t_2)\times R(t_1t_3\cdots t_n)$ over t_1 is clear dimensionally, since $D(t_2)$ being divergent, the degree of divergence of $R(t_1t_3\cdots t_n)$ in t_1 cannot exceed that of the t_1t_2 divergence, which $[1-T(t_1t_2)]$ is explicitly designed to remove. In the example considered in **1**, Sec. II, $T(t_1)$ and $T(t_2)$ represent extraction of (logarithmic) vertex part divergences, while $T(t_1t_2)$ is designed, in the first place, to extract (linear) self-energy divergences. $D(t_2)$ thus being a logarithmic divergence, the t_1 integration in $R(t_1t_3\cdots t_n)$ can at most be linearly divergent. The operation $[1-T(t_1t_2)]$ therefore suffices to produce convergence over t_1 in $[1-T(t_1t_2)]T(t_2)I$. This is an important point for the subsequent proof.

In the general case I_c is convergent for the over-all n-fold integration by its very construction, since it is obtained from Φ defined in Eq. (5) below by an operation $[1-T(t_1t_2\cdots t_n)]$, which operation subtracts from Φ the true divergence $D(t_1t_2\cdots t_n)$:

$$\Phi=I-D(t_1)R(t_2\cdots t_n)\cdots-D(t_1t_2)R(t_3\cdots t_n)\cdots \\ \cdots \\ -D(t_1t_2\cdots t_{n-1})R(t_n)\cdots. \quad (5)$$

We now want to show that I_c is also convergent over all subintegrations $t_1, \cdots, t_1t_2, \cdots, t_1t_2\cdots t_{n-1}, \cdots$ as well. Combining this with the discussion given in Sec. III, this will establish that I_c is absolutely convergent in a strict mathematical sense and all mathematical operations are valid for it.

By rearranging the terms in $I_c(n)$, we shall first exhibit that $I_c(n)$ is convergent over the $(t_2t_3\cdots t_n)$ subintegration, explicitly. The proof proceeds inductively. Suppose that $I_c(n-1)$ obtained from any $(n-1)$ fold integral $I(n-1)$ by subtracting true divergences $\times$reduced integrals converges over all its possible subintegrations as well as over the $(n-1)$ fold integration. We rewrite Eq. (5) as follows:

$$\Phi=[I-D(t_2)R(t_1t_3\cdots t_n)-D(t_3)R(t_1t_2\cdots t_n)$$
$$-\cdots-D(t_n)R(t_1\cdots t_{n-1})-D(t_2t_3)R(t_1t_4\cdots t_n)$$
$$-\cdots-D(t_2t_3\cdots t_n)R(t_1)]$$
$$-[D(t_1)R(t_2t_3\cdots t_n)-D(t_1t_2)R(t_3t_4\cdots t_n)\cdots$$
$$-D(t_1t_n)R(t_2\cdots t_{n-1})\cdots-D(t_1t_2t_3)R(t_4t_5\cdots t_n)\cdots$$
$$\cdots\cdots\cdots$$
$$-D(t_1t_3\cdots t_n)R(t_2)-D(t_1t_2\cdots t_n)R(t_3)\cdots$$
$$-D(t_1t_2\cdots t_{n-1})R(t_n)]. \quad (6)$$

Then we have

$$I_c(n)=[1-T(t_1\cdots t_n)]\Phi. \quad (7)$$

If t_1 is held fixed, it acts as an "external" momentum vector for the terms in the first bracket. The integrand in $R(t_1)$ contains (apart from the "vertex factor") those factors in the integrand of $I(n)$ which do not contain any other basic variable except t_1, and these factors are therefore common to all the terms in the first bracket in Eq. (6). Apart from these, the first bracket is precisely $I_c(n-1)$ and, consequently, convergent over $t_2t_3\cdots t_n$ integration, by assumption.

To exhibit the explicit convergence of the second bracket, we derive an alternative expression for the true divergences.

Consider $D(t_1t_2)$. We write

$$D(t_1t_2)R(t_3t_4\cdots t_n)=T(t_1t_2)[I-D(t_2)R(t_1t_3\cdots t_n)$$
$$-D(t_1)R(t_2t_3\cdots t_n)]$$
$$=T(t_1t_2)[I-D(t_2)R(t_1t_3\cdots t_n)]$$
$$-D(t_1)D_R(t_2)\times R(t_3t_4\cdots t_n). \quad (8)$$

Here $D_R(t_2)R(t_3t_4\cdots t_n)$ is the result of the operation $T(t_1t_2)$ (which, $D(t_1)$ being a constant, is effectively now an operation on t_2 integration alone) on $R(t_2t_3\cdots t_n)$. The reduced integral $R(t_3t_4\cdots t_n)$ from this operation, obtained by omitting all factors containing t_2 in the integrand of $R(t_2t_3\cdots t_n)$ is the same as that obtained by omitting all factors containing t_1t_2 from the integrand of I. As noted before $[I-D(t_2)R(t_1t_3\cdots t_n)]$ can be thought of as the graph obtained from $R(t_1t_3\cdots t_n)$ by inserting in the appropriate place the convergent remainder $G_c(t_2)$ of the t_2 integration in I. The operation $T(t_1t_2)$ now implies that from the resulting expression another divergent constant is evaluated. For our purposes, however, the essential point is that $T(t_1t_2)[I-D(t_2)R(t_1t_3\cdots t_n)]$ is convergent over t_2. To emphasize this we write $T(t_1t_2)[I-D(t_2)R(t_1t_3\cdots t_n)]$ as $(G_c(t_2)\rightarrow\int dt_1dt_2)_0R(t_3t_4\cdots t_n)$. Here, $(G_c(t_2)\rightarrow\int dt_1dt_2)_0$ is the divergent constant mentioned above and is such that in its integral representation the t_2 integration is convergent. Its multiplier $R(t_3t_4\cdots t_n)$ is an integral with the integrand consisting of those factors in the integrand of I which contain neither t_1 nor t_2 and is thus the correct reduced integral. Hence, finally we have

$$D(t_1t_2)R(t_3\cdots t_n)=[(G_c(t_2)\rightarrow\int dt_1dt_2)_0$$
$$-D(t_1)D_R(t_2)]\times R(t_3\cdots t_n). \quad (9)$$

It may be noted here that if we take two variables t_1 and t_i such that the parts corresponding to t_1 and t_i integrations do not overlap, then we have $D_R(t_i)=D(t_i)$. (The true divergence for the t_i subintegration of the reduced integral $R(t_2\cdots t_n)$ is the same as that of the original integral $I(t_1t_2\cdots t_n)$.) It was shown in Sec. 3, **1**, that for such a case we have $D(t_1t_i)=-D(t_1)D(t_i)$. Hence, it follows that $(G_c(t_i)\rightarrow\int dt_1dt_i)_0=0$, which is compatible with its graphical interpretation.

We shall prove a similar result for $D(t_1t_2t_3)$. By definition we have

$$D(t_1t_2t_3)R(t_4\cdots t_n)=T(t_1t_2t_3)[I-D(t_2)R(t_1t_3\cdots t_n)$$
$$-D(t_3)R(t_1t_2t_4\cdots t_n)-D(t_2t_3)R(t_1t_4\cdots t_n)]$$
$$+T(t_1t_2t_3)\begin{bmatrix}-D(t_1)R(t_2t_3\cdots t_n)\\ -D(t_1t_2)R(t_3t_4\cdots t_n)\\ -D(t_1t_3)R(t_2t_4\cdots t_n)\end{bmatrix}. \quad (10)$$

The expression in the first square bracket is convergent on account of our inductive assumption for the t_2t_3 integration. Precisely as above, we can write the result of operating $T(t_1t_2t_3)$ on the first bracket as

$$(G_c(t_2t_3)\rightarrow\int dt_1dt_2dt_3)_0R(t_4\cdots t_n).$$

By a repeated use of Eq. (9), the remaining terms in Eq. (10) can be written as

$$=-T(t_1t_2t_3)[D(t_1)\{R(t_2t_3\cdots t_n)-D_R(t_2)R(t_3t_4\cdots t_n)$$
$$-D_R(t_3)R(t_2t_4\cdots t_n)\}$$
$$+(G_c(t_3)\rightarrow\int dt_1dt_3)_0R(t_2t_4\cdots t_n)$$
$$+(G_c(t_2)\rightarrow\int dt_1dt_2)_0R(t_3t_4\cdots t_n)]$$
$$=-[D(t_1)D_R(t_2t_3)+(G_c(t_2)\rightarrow\int dt_1dt_2)_0D_R(t_3)$$
$$+(G_c(t_3)\rightarrow\int dt_1dt_3)_0D_R(t_2)]\times R(t_4\cdots t_n). \quad (11)$$

So that, finally we have

$$D(t_1t_2t_3)R(t_4\cdots t_n)=[(G_c(t_2t_3)\rightarrow\int dt_1dt_2dt_3)_0$$
$$-(G_c(t_3)\rightarrow\int dt_1dt_3)_0D_R(t_2)$$
$$-(G_c(t_2)\rightarrow\int dt_1dt_2)_0D_R(t_3)$$
$$-D(t_1)D_R(t_2t_3)]R(t_4\cdots t_n). \quad (12)$$

This can be generalized immediately by an inductive proof as follows:

$$D(t_1t_2\cdots t_r)R(t_{r+1}\cdots t_n)$$
$$=[(G_c(t_2\cdots t_r)\rightarrow\int dt_1dt_2\cdots dt_r)_0$$
$$-(G_c(t_3\cdots t_r)\rightarrow\int dt_1dt_3\cdots dt_r)_0D_R(t_2)$$
$$\cdots\cdots\cdots$$
$$-(G_c(t_2\cdots t_{r-1})\rightarrow\int dt_1dt_2\cdots dt_{r-1})_0D_R(t_r)$$
$$-(G_c(t_4\cdots t_r)\rightarrow\int dt_1dt_4\cdots dt_r)_0D_R(t_2t_3)$$
$$\cdots\cdots\cdots$$
$$-(G_c(t_5\cdots t_r)\rightarrow\int dt_1dt_5\cdots dt_r)_0D_R(t_2t_3t_4)$$
$$\cdots\cdots\cdots$$
$$-(G_c(t_r)\rightarrow\int dt_1dt_r)_0D_R(t_2t_3\cdots t_{r-1})$$
$$\cdots\cdots\cdots$$
$$-D(t_1)D_R(t_2t_3\cdots t_r)]\times R(t_{r+1}\cdots t_n). \quad (13)$$

All that remains now is to substitute from Eq. (13) in the expression obtained after operating $[1-T(t_1t_2\cdots t_n)]$ on the second bracket of Eq. (6). We obtain

$$
\begin{aligned}
&-D(t_1)[R(t_2t_3\cdots t_n)-D_R(t_2)R(t_3\cdots t_n)-\cdots\\
&\qquad -D_R(t_2t_3)R(t_4\cdots t_n)-\cdots\\
&\qquad \cdots\cdots\cdots\\
&\qquad -D_R(t_2t_3\cdots t_n)]\\
&-[(G_c(t_2)\to\textstyle\int dt_1dt_2)_0\{R(t_3\cdots t_n)-D_R(t_3)R(t_4\cdots t_n)\\
&\qquad -D_R(t_3t_4)R(t_5\cdots t_n)-\cdots-D_R(t_3\cdots t_n)\}]\\
&-[(G_c(t_3)\to\textstyle\int dt_1dt_3)_0\{R(t_2t_4\cdots t_n)\\
&\qquad -D_R(t_2)R(t_4\cdots t_n)\cdots\\
&\qquad -D_R(t_2t_4)R(t_5\cdots t_n)\cdots-D_R(t_2t_4\cdots t_n)\}]\\
&\qquad \cdots\cdots\cdots\\
&-[(G_c(t_2t_3)\to\textstyle\int dt_1dt_2dt_3)_0\{R(t_4\cdots t_n)\\
&\qquad -D_R(t_4)R(t_5\cdots t_n)\cdots\\
&\qquad -D_R(t_4t_5)R(t_6t_7\cdots t_n)-D_R(t_4t_5\cdots t_n)\}]\\
&\qquad \cdots\cdots\cdots\\
&-(G_c(t_2\cdots t_n)\to\textstyle\int dt_1dt_2\cdots dt_n)_0. \qquad (14)
\end{aligned}
$$

By our inductive hypothesis each expression in square brackets in the above converges for the $t_2t_3\cdots t_n$ integration if t_1 is held fixed. This proves the result. Since the inductive hypothesis assumes the convergence of $I_c(n-1)$ for all its possible subintegrations, we have succeeded in establishing the convergence of $I_c(n)$ over all subintegrations t_2, t_2t_3, $\cdots$, $t_2t_3\cdots t_{n-1}\cdots$ as well.

By different rearrangements we could prove the convergence of $I_c(n)$ over any other subintegration over $(n-1)$ variables. This shows that $I_c(n)$ is absolutely convergent, provided we can give a proof of the lemma stated in Sec. I. This is done in the next section.

III

In this section we give a proof of the lemma stated towards the end of Sec. I, on which the discussion in Sec. II depends. The proof is long, and only an outline will be given. We first investigate the condition for primitive divergents for the most general mixture of fields. Consider a graph with E_b as the total number of external boson and E_f as the total number of external fermion lines. Let N_i stand for the number of such vertices in the graph as have i lines incident; thus, i may be any integer equal or greater than two. The number of boson lines at each vertex of the type i is b_i and the number of fermion lines f_i.

If F stands for internal lines, then we have the relations,

$$E_b+2F_b=\sum b_iN_i,\quad E_f+2F_f=\sum f_iN_i.$$

We restrict ourselves to theories[7] in which the integrals for matrix-elements can be written in momentum space by setting in the integrand $\Delta_F(t)$ for each boson line, $S_F(t)$ for each fermion line, and a vertex factor of the general form t^{α_i} for each vertex of the type i. The condition for convergence of a graph, following Dyson, is derived as

$$2F_b+F_f-[4(F-N+1)+\sum\alpha_iN_i]>1.$$

Since $N=\sum N_i$, this reduces to

$$\tfrac{3}{2}E_f+E_b+\sum N_i(4-\alpha_i-b_i-\tfrac{3}{2}f_i)-4>1.$$

The upper limit to the number of primitive divergent graphs is therefore given by the following condition:[8]

$$\tfrac{3}{2}E_f+E_b<5+\sum(\alpha_i+b_i+\tfrac{3}{2}f_i-4)N_i.$$

This result shows that if for a mixture of fields the number of primitive divergent graphs does not depend on the order of the graphs considered, then this number is governed by a condition of the type $E<a$ with $a=5$. Any graph in these theories, with four or more external fermion lines or five or more external boson lines must be (at least "superficially"[9]) convergent. This implies that if we open one single line of any graph with at least two external fermion lines, and if the resulting graph is a "connected" graph (i.e., the resulting graph does not split into pieces each joined to the other by a single line), it must converge superficially. The same result holds *a fortiori* if two or more lines which leave the graph connected are opened simultaneously. It is also seen to be valid for all graphs with boson lines external except for boson self-energy graphs. In this case, opening one boson line may produce a connected graph with 4 external boson lines which may diverge logarithmically. However, precisely this was the case treated in 1(B), where it was shown that such divergences affect only the mass-renormalization constant. These divergences were called "final"[10] and it was proved quite generally that one could effectively proceed as if these divergences did not exist. We can therefore state the following as a general rule: *If in any graph one or more lines are opened such that the graph still remains "connected," the resulting graph is either superficially convergent or such that its divergence is "final" and we can effectively proceed as if the divergence doesn't exist.* Using the above result we proceed to the direct proof of Lemma 1. By "exhibiting" a part of a graph we mean opening enough lines so that the momenta

[7] This excludes β-formalism of meson theory, for which the propagation function is

$$T_F(t)=\{i\beta t-K+(1/K)[(\beta t)^2-t^2]\}/(t^2+K^2).$$

[8] For a theory to have a finite number of primitive divergents, it is clear that $\Sigma(\alpha+b+\frac{3}{2}f-4)N$ must equal zero. This therefore excludes all vector couplings for meson-nucleon interactions ($b=1$, $f=2$, $\alpha=1$), meson-pair theories ($b=2$, $f=2$) and Fermi types of interaction between four spinor particles ($f=4$).

[9] An integral is "superficially" convergent, if by counting the powers of t in the integrand, the integral itself is found to be convergent, while any one of its subintegrations diverges.

[10] One further instance of a "final" divergence [not noted in 1(B)] is provided by vertex parts with 2 meson and 1 external photon line. If the external photon line belongs to a 3-vertex, opening one of the internal meson lines at this vertex may produce a connected M-part. This M-divergence can be treated as "final." The reduced integral in this case is a vertex part which can be shown to be a function of the external photon momentum $(p-p')$ only. The reduced integral would therefore vanish from the familiar arguments of gauge-invariance.

associated with the external lines of the part become fixed. It is clear that any part (convergent or divergent) may be exhibited by opening all its external lines. If we can show that the number of lines, some or all of which need to be opened, to exhibit every divergent part contained in a graph, is never greater than n (the number of independent momenta at our disposal), we will have proved the lemma. For we can then associate basic momenta with these lines, and thus have a "correct" choice of basic variables.

All internal lines belonging to any graph can be divided into two mutually exclusive sets. Lines in set (a) have the property that if any one of them is opened the graph splits into two or more parts each joined to the other by a single line (called a "bridge"). All lines not belonging to set (a) fall in set (b) which thus includes lines which if opened singly leave the graph connected.

The lines in set (a) can further be divided into distinct classes; If opening a line l_a^1 splits the graph into separate pieces joined by single lines (bridges) l_a^2, l_a^3, $\cdots$, it is obvious that the result of opening l_a^2 would be to split the graph into pieces joined by bridges l_a^1, l_a^3, $\cdots$. If a basic momentum variable t is assigned to l_a^1, the lines l_a^2, l_a^3, etc., acquire momenta $t+p$, where p are linear sums of momenta of external lines. All these lines will belong by definition to class R_a. We can assign all lines in set (a) uniquely to various classes R_1, R_2, $\cdots R_s$. No line (a) can belong to more than one class at a time, and each class must contain at least two lines. If those vertices of the graph at which external lines are incident are 3-vertices, each internal line at such a vertex certainly belongs to some one of these classes. For distinction we shall label such classes as R^{ext}. If R^{ext} consists of just the two internal lines incident at the external vertex defining R^{ext}, then the effective result of opening one of these external lines is one single connected graph with external lines one more than the original graph. We further distinguish such R^{ext} by writing these as $\mathbf{R}^{\text{ext}}$.

The connection of these sets and classes with divergences in a graph is obvious. On account of the rule stated above, opening a line (b) leaves the graph connected and therefore at least superficially convergent, while a line (a) may split the graph into pieces at least one of which has external lines satisfying $E<a$. To "exhibit" these divergences, it is therefore essential that a basic momentum variable be associated with this line (a). If (a) belongs to a class R_a, an assignment of the basic variable to any other line in R_a is, from the definition of a class, clearly an equivalent and a "correct" procedure. As a first step towards the proof of Lemma 1 we show that s, the number of effective classes, which can split the graph into pieces all or some of which satisfy $E<5$ is always $\leqslant n$.

For graphs consisting entirely of 4-vertices the result is trivial. For this case, we have $n=F-N+1$, while $2F+E=4N$. Each of the s classes must contain at least two lines; since the minimum number of lines in the classes, $2s$, must not exceed the total number of internal lines F, we have $2s\leqslant F=2n+\frac{1}{2}E-2$. For the case of divergent graphs with $E\leqslant 4$, this shows that $s\leqslant n$.

To prove the result for graphs with 3-vertices we establish Lemma 2.

Lemma 2:—The maximum number of classes s for a proper graph with n basic vectors and no external lines is $n-2$.

For the proof it is unnecessary to distinguish between boson and fermion lines. We can always obtain any graph with n basic vectors from a graph with $(n-1)$ basic vectors by adding a new line with its terminating vertices on two existing lines of the graph with $(n-1)$ basic vectors. In general, the addition of a new line does not add to the number of the existing classes except for two cases: (i) when the terminating vertices of the new line both belong to the same line of the original graph and (ii) when these vertices belong to two lines already belonging to a class. In both these cases the number of classes is increased at most by one. The proof of the lemma follows by induction, since the result is obviously true for $n=2$. We also notice that at the new terminating vertices there are at most two lines belonging to set (a), while the newly added line itself must belong to set (b). Generalizing from this, we have the result that at no vertex can there be more than two (a) lines incident. We shall have occasion to use this result.

We now consider the effect of adding external lines to such graphs. Each external line may produce at most one new class $\mathbf{R}^{\text{ext}}$, so that the general result is that for a graph with n independent vectors and E external lines

$$\max s\leqslant n+E-2.$$

We can now show that for $E=3$, at least one of classes $\mathbf{R}^{\text{ext}}$ either produces a convergent connected graph when one of its 2 lines is opened, or a "final" divergence. The former is true for $\mathbf{R}^{\text{ext}}$ formed by the external photon or meson line in vertex parts in all theories except scalar electrodynamics where this $\mathbf{R}^{\text{ext}}$ introduces, as noted before, merely a "final" divergence. Excluding this from the classes of interest we have $s\leqslant n$ for $E=3$. For the case $E=4$, opening one of the lines of an $\mathbf{R}^{\text{ext}}$ must produce a connected graph with five external lines, i.e., a subintegration which is superficially convergent. This is not true, however, if two of the R^{ext} coincide (i.e., when two external lines are adjacent), in which case the graph produced has just four external lines. Thus, in this case too the effective number of classes which exhibit divergent parts is governed by $s\leqslant n$.

With these preliminary results we now prove that we can choose a set of n lines which if opened singly or in pairs, etc., would exhibit all the divergences of the graph. Consider the case $s=n$. We assign s basic momenta to the s lines, selected one each from the s classes R. The assignment of all momenta is complete and it remains to prove that we only need open some or all of these n lines to exhibit every possible divergence. We notice now that for this case the lines (b) acquire an additional property. By counting the lines and vertices we can show that at each vertex there are precisely two lines (a) and not more than one line (b). (We are here dealing with graphs with 3-vertices only), and also there are in all just $(n-1)$ lines (b). If we open one of these lines (b), the graph still remains "connected." Lines (a) defined for the original graph retain their property of splitting graphs even for the new graph; therefore, there are already two (a) lines at each vertex of the new graph. We have seen that at any vertex there cannot be more than two (a) lines. This shows that the (b) lines of the original graph remain the (b) lines of the new graph. The "connectedness" of the graph no longer depends on the lines (b); in other words, lines (b), which were defined in the first place on account of their property of leaving the graph connected if opened one at a time, acquire the further property that they still leave the graph connected if more than one of them are opened simultaneously.[11] The result of opening any set of these lines will therefore never be a divergent part. All divergent parts of the graphs are thus exhibited by opening one or more of lines (a); analytically, this assignment of basic momenta will insure that the entire number of divergences in the graph never exceeds the possible subintegrations at our disposal, and therefore the lemma is proved.

[11] This result is of course not true for $s<n$.

The uniqueness of $I_c(t_1t_2\cdots t_n)$ for a different "correct" choice of basic momenta follows immediately from the concept of classes. For the case dealt with above, the choice is correct only if the basic lines are selected one each from a class R. Since all other lines in a class acquire momenta $p+t$, if t is the basic line which belongs to R, all "correct" choices are completely equivalent, and obtainable one from the other by trivial transformations involving only linear sums of external momenta (p). This proves the result.

We will not deal in detail here with the proof of Lemma 1 for the case of $s<n$ where some at least of the $(n-1)$ fold integrations are "superficially" convergent. The proof requires formulation of rules for assignment of the remaining $(n-s)$ moments which is not difficult.

The proofs in this section, though tedious, are essential because it is important to realize that a subtraction procedure is worthless unless it can be shown that it leads to unique absolutely convergent results.

The author is indebted to Dr. P. T. Matthews and Dr. Res Jost for help in Sec. III and to Professor J. R. Oppenheimer for the very kind hospitality of the Institute for Advanced Study.

PHYSICAL REVIEW VOLUME 84, NUMBER 3 NOVEMBER 1, 1951

The Absorption Spectrum of GaCl

FRANKLYN K. LEVIN* AND J. G. WINANS
University of Wisconsin, Madison, Wisconsin
(Received April 23, 1951)

The absorption spectrum of GaCl has been measured and analyzed to provide values for molecular constants. The energy of dissociation for the ground state has been determined from a predissociation limit as less than 5.00 volts and probably greater than 4.98 volts.

INTRODUCTION

WHILE studies of the vibrational spectra of molecules formed by the elements from a single column of the periodic table are not uncommon, investigations of the rotational spectra from various elements in one column have been rare. Such a study has been started for the chlorides formed by the elements of column III. An analysis of the rotational structure of the InCl molecule was made by Froslie and Winans;[1] the rotational spectrum of AlCl was analyzed by Holst.[2]

Two previous investigations of the spectrum of GaCl have been made. Petrikalm and Hochberg[3] obtained the vibrational spectrum in absorption in the region from 3220A to 3470A. The dispersion was sufficient to check the isotope shift due to Cl^{35} and Cl^{37}, but not enough to make a vibrational analysis of the bands. Miescher and Wehrli[4] working with somewhat larger spectrographs confirmed the assignment of the spectrum to GaCl and showed that the heads around 3300A belonged to two overlapping systems. Although Miescher and Wehrli made a thorough and accurate vibrational analysis of the three systems obtained in absorption, the dispersion of their instruments was not sufficient to analyze the rotational structure or determine the position of the null lines.

In the present investigation, the spectrum was

* Now with Carter Oil Company, Tulsa, Oklahoma.
[1] H. M. Froslie and J. G. Winans, Phys. Rev. **72**, 481 (1947).
[2] W. Holst, Z. Physik **93**, 55 (1935).
[3] A. Petrikalm and J. Hochberg, Z. Physik **86**, 214 (1933).
[4] E. Miescher and M. Wehrli, Helv. Phys. Acta **7**, 331 (1934).