Some Notes on Multiple-Boson Processes

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Methods of calculation with nonlinear functions of quantized boson fields are developed during the discussion of two problems involving multiple boson processes. In the first of these a simple treatment is given of the multiple radiation of photons by classical current distributions, a special case of which, in effect, is the infrared catastrophe.

In the second illustration, generalizations of the scalar and pseudoscalar meson theories are considered in which the interaction hamiltonian depends exponentially on the meson field. In the pseudoscalar case such hamiltonians are closely related to the familiar form of pseudovector coupling. Assuming the over-all coupling of the nucleon and meson fields to be weak, calculations are made of the nuclear forces, and of the multiple production of mesons in meson-nucleon and in nucleon-nucleon collisions. In the latter events statistical independence of meson emissions is found to prevail.

I. INTRODUCTION

DEFICIENCIES in the mathematical techniques for handling quantized field theories obscure many questions of critical importance, such as the extent to which difficulties of the theory arise from a questionable expansion in powers of a coupling constant, and the importance of higher order corrections. An attempt has, therefore, been made, based on the developments due to Schwinger and others,[1,2] to find improved methods of computing the various matrix elements and expectation values of interest. Ideally these methods should be capable of handling rather general functions of quantized field variables, wherever possible without resorting to power series expansions. Simple rules accomplishing these ends have been found for dealing with functions of fields whose commutators with themselves are c-numbers, i.e., boson fields. While the problem of treating spinor fields remains, these methods make possible the simplification of some parts of the theory, and the generalization of others.

To illustrate both aspects of the work, the mathematical methods are developed during a discussion of two problems involving multiple-boson processes. In part II we discuss the radiation of quanta by a classical current distribution and in particular the well-known "infrared catastrophe" (the emission of an infinite number of soft photons when a charged particle is suddenly accelerated). The familiar results of Bloch and Nordsieck[3] are obtained in a rather direct way. In part III we discuss generalizations of the usual neutral scalar and pseudoscalar meson theories in which the interaction hamiltonian is allowed to depend exponentially on the meson field. A particular case of such an exponential hamiltonian involving the pseudoscalar field has been shown by Dyson[4] to result from a contact transformation performed on the familiar hamiltonian for pseudovector coupling of the pseudoscalar field. Couplings of the type introduced bring many high order aspects of the usually treated couplings much closer to the surface of the theory. Among these are higher order corrections to nuclear forces and the multiple production of mesons. We may hope, by treating these processes to gain some insight which the usual theory has not granted us.

[1] J. Schwinger, Phys. Rev. 74, 1439 (1948).
[2] J. Schwinger, Phys. Rev. 75, 651 (1949).
[3] F. Bloch and A. Nordsieck, Phys. Rev. 52, 54 (1937).
[4] F. J. Dyson, Phys. Rev. 73, 929 (1948).

II. RADIATION OF PHOTONS BY CLASSICAL CURRENTS

The infrared catastrophe causes low frequency divergences in the calculation of the radiative corrections to any process involving the sudden acceleration of charge, e.g., scattering in a potential field, Compton effect, pair production, etc. Bloch and Nordsieck treated the scattering of an electron by a potential by introducing several approximations, principally the neglect of pair effects and of the electron's recoil in photon emission. These approximations, which are justified by the very low energy of the photons involved, may be epitomized by saying that only the classical properties of the electron current are important. The general class of problems for which this property holds may be treated by considering the interaction of the quantized electromagnetic vector potential $A_\mu(x)$ ($\mu=1\cdots 4$, $x_\mu=\mathbf{r}$, ict) with a classical current distribution $j_\mu(x)$, prescribed as a function of space and time.[5] The state vector $\Psi(\sigma)$ of the system on a space-like surface σ in the interaction representation obeys the Schrödinger equation[6]

$$i\delta/\delta\sigma(x)\Psi=H(x)\Psi, \tag{1}$$

where $H(x)$ is the interaction hamiltonian,

$$H(x)=-j_\mu(x)A_\mu(x). \tag{2}$$

The quantities we shall be interested in calculating are the probability amplitudes for the real emission of any specified number of photons. We may assume that the interaction began in the remote past when the system was in a state with no real photons present. The

[5] A treatment of the infrared catastrophe using classical currents has just been published by W. Thirring and B. Touschek, Phil. Mag. 42, 244 (1951).
[6] We use units in which $\hbar=1$, $c=1$.

specification of the functions $j_\mu(x)$ may remain to the last step of the calculation.

Equations (1) and (2) are readily solved in two steps. The first is the familiar contact transformation due to Schwinger.[2] We define a new state vector $\Psi'(\sigma)$ by the relation,

$$\Psi(\sigma)=\exp\left(i\int_{-\infty}^{\sigma} j_\mu A_\mu dx\right)\Psi'(\sigma) \tag{3}$$

(in which $dx=dx_0dx_1dx_2dx_3$). Substituting into (1) and using the multiple commutator expansion to evaluate the transformed operators we find the equation of motion for Ψ'

$$\begin{aligned} i\frac{\delta}{\delta\sigma(x)}\Psi'(\sigma) &= -\frac{i}{2}\int_{-\infty}^{\sigma} j_\mu(x)[A_\mu(x), A_\nu(x')]j_\nu(x')dx'\Psi'(\sigma) \\ &= \frac{1}{2}\int_{-\infty}^{\sigma} j_\mu(x)D(x-x')j_\mu(x')dx'\Psi'(\sigma). \end{aligned} \tag{4}$$

This simple equation is exact since j_μ is a c-number. No quantized quantities are present in the transformed interaction, and the time dependence of $\Psi'(\sigma)$ therefore contains no real transitions. The information sought is already implicit in the part of the S-matrix given by (3). While the S-matrix is usually defined by the equation $\Psi(\infty)=S\Psi(-\infty)$, we shall find it briefer to work with an effective S-matrix defined from (3) as

$$S=\exp\left(i\int_{-\infty}^{\infty} j_\mu A_\mu dx\right). \tag{5}$$

This lacks merely the phase factor

$$\exp\left\{\tfrac{1}{2}i\int j_\mu(x)\bar{D}(x-x')j_\mu(x')dxdx'\right\}$$

which is due to the current's energy of self-interaction. A phase factor does not affect proton emission probabilities, since the latter are expressed as the absolute value squared of certain matrix elements of S. The calculation of these matrix elements occupies us next.

The operator S defined by (5) has matrix elements for the creation of any number of photons. A term which creates n photons is evidently given by the nth order term in the power series expansion of the exponential. However, n real photons may also be created by any term of order $n+2p$, where p is a positive integer. Such terms correspond to the creation of n real photons together with the virtual emissions and reabsorptions of p additional photons. Rather than consider the expansion term by term it will be convenient to develop general methods for separating real effects from their corrections due to virtual effects. We may do this in fact for functions of boson fields more general than the exponential of (5).

Let $\mathcal{A}$ be a hermitian operator which, like $A_\mu(x)$ contains a sum of matrix elements for the creation and destruction of all possible photons. Such an operator may be a linear combination of the A_μ's evaluated at different places, or in particular the integral in the exponent of (5). Consider a power of $\mathcal{A}$, say $\mathcal{A}^m$. We ask now for the matrix element of $\mathcal{A}^m$ for the creation of n real photons, where $n\leqq m$. The n factors of $\mathcal{A}$ which are allowed to create the real photons may be chosen from the product in the binomial coefficient $\binom{m}{n}$ different ways, the remaining $m-n$ operators in each case being allowed to carry out virtual transitions. The desired matrix element, which we shall write as $\langle\mathcal{A}^m\rangle_n$ must of course vanish unless $m-n$ is even. Setting aside for the moment some questions of commutation, if the real operators are separated from the virtual ones without changing the order of the latter, each of the partitions of the $\mathcal{A}$'s should give the same matrix element. Their sum is

$$\langle\mathcal{A}^m\rangle_n=\binom{m}{n}\mathcal{A}^n\langle\mathcal{A}^{m-n}\rangle_0, \tag{6}$$

in which the bracket $\langle\ \rangle_0$ retains its usual significance as the vacuum expectation value. The noncommutation of operators in the different factors of $\mathcal{A}$ might be thought to add further terms to (6). Such terms though would only have the effect of correcting (6) for the cases in which some of the virtual photons are identical to the real ones. Since the density of photon states is enormous the contribution of such exact coincidences to the summations over all virtual photons implicit in the vacuum expectation is vanishingly small.[7] Equation (6) is in effect exactly correct in a quantized field theory. We may rewrite it in the symbolic form,

$$\langle\mathcal{A}^m\rangle_n=\frac{1}{n!}\mathcal{A}_n\left\langle\frac{d^n}{d\mathcal{A}^n}\mathcal{A}^m\right\rangle_0. \tag{7}$$

It is now evident that for any function $f(\mathcal{A})$ expansible in a power series

$$\langle f(\mathcal{A})\rangle_n=(1/n!)\mathcal{A}^n\langle f^{(n)}(\mathcal{A})\rangle_0, \tag{8}$$

where $f^{(n)}(\mathcal{A})$ signifies the nth derivative.

Returning to (5) and identifying $\mathcal{A}$ with $\int j_\mu A_\mu dx$; the effective matrix element for the creation of n photons becomes

$$\langle S\rangle_n=(1/n!)\left(i\int j_\mu A_\mu dx\right)^n\left\langle\exp\left(i\int j_\mu A_\mu dx\right)\right\rangle_0. \tag{9}$$

[7] If the quantization is carried out in a region of finite volume the corrections are easily seen to vanish as the reciprocal of the volume. Equation (6) also holds when the free factors of $\mathcal{A}$ are used to absorb real photons. However, since we must assume that the ordering of these factors is immaterial they may not be used to emit and reabsorb the same photon. Such processes are already correctly accounted for among the vacuum fluctuations.

We require still an evaluation of the vacuum expectation of the S-matrix. To find this, we consider the quantity $\langle\exp(\lambda\mathfrak{A})\rangle_0$ as a function of the variable λ. Its derivative is

$$d/d\lambda\langle\exp(\lambda\mathfrak{A})\rangle_0=\langle\mathfrak{A}\exp(\lambda\mathfrak{A})\rangle_0$$
$$=(\Psi_{\text{vac}},\,\mathfrak{A}\exp(\lambda\mathfrak{A})\Psi_{\text{vac}}),\quad(10)$$

where the state vector for the vacuum has been written in explicitly. The only part of the operator $\exp(\lambda\mathfrak{A})$ which can contribute to the expectation value a term which does not vanish because of orthogonality is the part which creates a single photon, aside from virtual effects. This photon may then be annihilated by the remaining factor of $\mathfrak{A}$ in order to secure again the original vacuum state. Hence we may substitute $\langle\exp(\lambda\mathfrak{A})\rangle_1=\lambda\mathfrak{A}\langle\exp(\lambda\mathfrak{A})\rangle_0$ for the exponential on the right side of (10). We obtain thus a differential equation for the original expectation value

$$d/d\lambda\langle\exp(\lambda\mathfrak{A})\rangle_0=\lambda\langle\mathfrak{A}^2\rangle_0\langle\exp(\lambda\mathfrak{A})\rangle_0.\quad(11)$$

The solution, which must reduce to unity for $\lambda=0$ is evidently

$$\langle\exp(\lambda\mathfrak{A})\rangle_0=\exp(\tfrac{1}{2}\lambda^2\langle\mathfrak{A}^2\rangle_0).\quad(12)$$

Vacuum expectations of more general functions of $\mathfrak{A}$ may be found by expanding them as fourier or laplace integrals and applying (12) to the integrands.

By employing (12) the expression (9) for $\langle S\rangle_n$ may be evaluated explicitly,

$$\langle S\rangle_n=(1/n!)\left(i\int j_\mu A_\mu dx\right)^n e^{-\frac{1}{2}W},\quad(13)$$

where W is the integral

$$W=\int j_\mu(x)\langle A_\mu(x)A_\nu(x')\rangle_0 j_\nu(x')dxdx'\quad(14)$$

$$=\tfrac{1}{2}\int j_\mu(x)\langle A_\mu(x)A_\nu(x')+A_\nu(x')A_\mu(x)\rangle_0 j_\nu(x')dxdx'\quad(15)$$

$$=\tfrac{1}{2}\int j_\mu(x)D^{(1)}(x-x')j_\mu(x')dxdx'.\quad(16)$$

Use has been made, in going from (15) to (16), of Schwinger's evaluation[2] of the vacuum expectation of the symmetrized product of two A's. The function $D^{(1)}(x)$ is defined by the integral

$$D^{(1)}(x)=(2\pi)^{-3}\int e^{ikx}\delta(k^2)(dk).\quad(17)$$

Having evaluated the required matrix elements we are in a position to find the emission probabilities. The final state vector of the system $\Psi(\infty)$ is a sum of mutually orthogonal components $\Psi_n(\infty)$, each of which corresponds to a different number of photons present. Apart from a phase factor,

$$\Psi_n(\infty)=\langle S\rangle_n\Psi(-\infty)\quad(18)$$
$$=\langle S\rangle_n\Psi_{\text{vac}}.\quad(19)$$

The probability w_n for the emission of n photons is just the absolute magnitude squared of $\Psi_n(\infty)$.

$$w_n=(\Psi_n(\infty),\,\Psi_n(\infty))\quad(20)$$
$$=\langle\langle S\rangle_n\langle S\rangle_n\rangle_0.\quad(21)$$

The latter expression for w_n allows the summation over real final states of the system by employing the same formalism as is used for vacuum fluctuations. Employing the expression (13) for $\langle S\rangle_n$ we have

$$w_n=(n!)^{-2}\left\langle\left(\int j_\mu A_\mu dx\right)^n;\left(\int j_\mu A_\mu dx\right)^n\right\rangle_0 e^{-W}.\quad(22)$$

A semicolon has been used to separate the two factors in the expectation bracket in order to indicate that n photons must be created by the factor on the right and annihilated by the one on the left. No A's may be paired within a single factor since this would amount to using real photon operators to carry out vacuum fluctuations. These are already accounted for correctly in the factor e^{-W}. There are $n!$ ways of pairing the A's on either side of the semicolon and each contributes a factor W^n to the expectation value. Hence the emission probabilities form a poisson distribution,

$$w_n=W^n e^{-W}/n!.\quad(23)$$

The probability distribution assumes this simple form for the radiation from all classical systems since the successive photon emissions are statistically independent. This independence is destroyed, for example, when recoil effects are not negligible. Had a first-order perturbation calculation been made of w_1 the result would have been $w_1=W$. Evidently, the perturbation theory calculates in general in this approximation not the single photon emission probability, but the expectation value of the number emitted.

For most purposes $j_\mu(x)$ may be taken as simply the current due to a moving point charge. When the charged particle suffers a sudden change of velocity W is easily shown to be the infrared divergent integral familiar from the work of Bloch and Nordsieck, and of Pauli and Fierz.[8] The foregoing methods are also applicable to the discussion of Čerenkov radiation, for which purpose the D-functions may be generalized in a way appropriate to dispersive media.

III. MULTIPLE MESON EFFECTS

Hamiltonians for the interaction of nucleon and meson fields are usually taken, by analogy with quantum electrodynamics, to be linear in the meson field. More general types of interactions depending

[8] W. Pauli and M. Fierz, Nuovo cimento 15, 167 (1938).

nonlinearly on the meson field may, however, arise either as new assumptions, or as transformed versions of the familiar linear couplings. Calculations with such hamiltonians may be greatly simplified by some of the rules established in the previous section. As an exploratory example which, it develops, has particularly simple mathematical properties, we shall consider interactions of the form,

$$H(x)=\lambda\bar{\psi}(x)e^{\beta\varphi(x)}\psi(x), \qquad (24)$$

between the spinor nucleon field $\psi(x)$ and the neutral scalar meson field $\varphi(x)$. The interaction is characterized by two coupling parameters, λ and β. The constant λ gives the over-all strength of the nucleon-meson coupling, while β determines the average number of mesons taking part in an elementary process. A corresponding pseudoscalar theory may be considered without changing notations by letting $\varphi(x)$ be a pseudoscalar field and letting β represent the matrix $i\eta\gamma_5$, where η is again a new coupling constant.

An interesting feature of interactions like (24) is that elementary events may involve high multiplicities of mesons (real and virtual) even when the coupling is weak (i.e., λ is small). Many effects of a type characteristic of the high orders of perturbation theory for the usual couplings are already present in the earliest terms of an expansion in powers of λ. We shall investigate some of the properties of such an expansion performed without specializing the parameter β.

The exponential hamiltonian (24) is of particular interest in the pseudoscalar case because of its coincidence with the result of a contact transformation performed on the familiar (linear) pseudovector coupling of the pseudoscalar field. The latter coupling is

$$H(x)=(1/\mu)j_\nu'(x)\partial\varphi(x)/\partial x_\nu+1/2\mu^2(\eta_\nu j_\nu)^2, \qquad (25)$$

where

$$j_\mu'(x)=ig\bar{\psi}(x)\gamma_5\gamma_\mu\psi(x). \qquad (26)$$

η_ν is the unit normal vector to the space-like surface, and u and M are the meson and nucleon masses, respectively. Dyson has shown[4] that the hamiltonian (25) may be brought to a form containing no derivatives of the meson field by a contact transformation which suitably redefines the state vector $\Psi(\sigma)$. The new state vector $\Psi'(\sigma)$ is defined by the relation,

$$\Psi(\sigma)=e^{-iS(\sigma)}\Psi'(\sigma), \qquad (27)$$

where

$$S(\sigma)=(1/\mu)\int_\sigma j_\mu'(x')\varphi(x')d\sigma_\nu', \qquad (28)$$

the integration being carried out over the contemporary surface σ. ($d\sigma_\nu$ is a directed surface element parallel to η_ν.) It is not difficult to show that the Schrödinger equation satisfied by $\Psi'(\sigma)$ has the interaction hamiltonian given exactly by

$$H'(x)=M\bar{\psi}(x)\{\exp[(2ig/\mu)\gamma_5\varphi(x)]-1\}\psi(x). \qquad (29)$$

The similarity of (29) to the pseudoscalar form of (24) is obvious when we note that the subtraction of a term $\lambda\bar{\psi}\psi$ from (24) would affect only the nucleon self-energy. The results to be derived from (24), therefore, include many of the higher order effects of the pseudovector coupling (25), but the approximation used, treating λ as small, is rigorously applicable to (25) only when M is small, i.e., $\mu\gg M$. Although such a limitation on the masses might be of use in treating a conjectured coupling between electrons and mesons, its effect for nucleon-meson systems is naturally quite unrealistic. The exact correspondence of (25) or (29) with the pseudoscalar form of (24) is, therefore, mainly of interest as a limiting case.

The simplest of the physical effects contained in the hamiltonian (24) is the nucleon self-energy. This is found by taking the vacuum expectation of the exponential factor.

$$\delta M=\lambda\langle e^{\beta\varphi(x)}\rangle_0, \qquad (30)$$

which is evaluated, according to (11), as

$$\delta M=\lambda\exp[\tfrac{1}{2}\beta^2\langle\varphi^2(x)\rangle_0]. \qquad (31)$$

To evaluate vacuum expectations of quantities quadratic in the meson field, we recall the relation,

$$\langle\varphi(x)\varphi(x')+\varphi(x')\varphi(x)\rangle_0=\Delta^{(1)}(x-x'). \qquad (32)$$

The quantity $\beta^2\Delta^{(1)}(0)$ is quadratically divergent, and positive in the scalar case, negative in the pseudoscalar case. In either instance, however, its infinite effects may be removed by a redefinition, in effect a renormalization, of the constant λ. This is most directly illustrated by considering the matrix element for the collision of a meson with a nucleon, leading to the production of n mesons (including the original one). The desired matrix element is given, according to (8) by

$$\langle H\rangle_{n+1}=\lambda\bar{\psi}(x)\psi(x)\frac{(\beta\varphi(x))^{n+1}}{(n+1)!}\langle e^{\beta\varphi(x)}\rangle_0$$
$$=\lambda'\bar{\psi}(x)\psi(x)\frac{(\beta\varphi(x))^{n+1}}{(n+1)!}, \qquad (33)$$

where

$$\lambda'=\lambda\exp[\tfrac{1}{4}\beta^2\Delta^{(1)}(0)]. \qquad (34)$$

Evidently if we consider the product $\lambda\exp[\frac{1}{4}\beta^2\Delta^{(1)}(0)]$ as a renormalized coupling constant λ', and regard the latter quantity as finite, the remaining effects are finite as well. This type of renormalization may be shown, by a somewhat simpler example (see Appendix) to be of use throughout the theory. It will be the only renormalization needed in order to obtain, in the lowest powers of λ, the physical effects of immediate interest.

The probabilities of the multiple production of mesons in meson-nucleon encounters[9] are readily cal-

[9] Meson-nucleon collisions, with exponential pseudoscalar coupling are also considered in a note by K. Sawada and K. Takagi, Prog. Theor. Phys. 4, 239 (1949).

culated by using the matrix element (33). Recursion relations for increasing numbers of mesons provide a convenient way of finding the required multiple integrals over the momentum spaces of the emergent particles. The results are most compactly stated for extremely energetic collisions, i.e., letting Pp be the scalar product, $\mathbf{P}\cdot\mathbf{p}-P_0p_0$, of the nucleon and meson momentum-energy vectors, we assume $-Pp\gg M^2$. In this limit, the total cross section, expressed as a sum over the possible numbers of emerging mesons is

$$\sigma=\frac{\lambda'^2\beta^4}{16\pi}\sum_{n=1}^{n_{\max}}\frac{1}{(n+1)!n!(n-1)!}\left(\frac{-\beta^2Pp}{8\pi^2}\right)^{n-1}. \quad (35)$$

The upper limit of summation is given by $n_{\max}\sim -Pp/M\mu$. The most probable multiplicity (and asymptotically the average) is

$$n\sim(-\beta^2Pp/8\pi^2)^{\frac{1}{3}}. \quad (36)$$

The meson emission is spherically symmetrical in the center-of-gravity system. Nucleon-anti-nucleon annihilation with the production of mesons is also contained in the matrix elements (33). The average multiplicity is again given by (36), expressed in terms of the scalar product of nucleon momenta.

To investigate two-nucleon processes more generally we must consider effects quadratic in λ. The two-nucleon part of the second-order hamiltonian, found by contact transformation, is an expression analogous to the hamiltonian in (4)

$$H^{(2)}(x)=\tfrac{1}{2}\int_{-\infty}^{\infty}\bar{\psi}(x)\psi(x)\mathcal{W}(x,x')\bar{\psi}(x')\psi(x')dx', \quad (37)$$

where

$$\mathcal{W}(x,x')=-\tfrac{1}{2}i\lambda^2[e^{\beta\varphi(x)},e^{\beta'\varphi(x')}]\epsilon(x-x'), \quad (38)$$

in which $\epsilon(x)$ is the familiar time-reversal function $x_0/|x_0|$. A distinction has been drawn between the two occurrences of β by priming the second since, in the pseudoscalar case, these are γ_5 matrices operating between different pairs of wave functions. In the scalar case they are of course identical constants. A simple theorem concerning exponential functions of operators enables us to evaluate the commutator in (38). Let a and b be any two operators whose commutator commutes with both of them. Then

$$e^ae^b=e^{a+b+\frac{1}{2}[a,b]}. \quad (39)$$

This is proved by considering the expression $e^{\zeta a}e^{\zeta b}e^{-\zeta(a+b)}$ as a function of the parameter ζ, and differentiating

$$\frac{d}{d\zeta}e^{\zeta a}e^{\zeta b}e^{-\zeta(a+b)}=e^{\zeta a}[a,e^{\zeta b}]e^{-\zeta(a+b)}=\zeta[a,b]e^{\zeta a}e^{\zeta b}e^{-\zeta(a+b)}.$$

The solution of this differential equation leads immediately to (39).

Using (39) to evaluate the commutator in (38) the interaction $\mathcal{W}(x,x')$ is seen to be the operator,

$$\begin{aligned}\mathcal{W}(x,x')&=i\lambda^2e^{\beta\varphi(x)+\beta'\varphi(x')}\\&\quad\times\sinh\{\tfrac{1}{2}\beta\beta'[\varphi(x),\varphi(x')]\epsilon(x-x')\}\\&=-\lambda^2e^{\beta\varphi(x)+\beta'\varphi(x')}\sin\{\beta\beta'\bar{\Delta}(x-x')\}.\end{aligned} \quad (40)$$

In the limit of very small β, which corresponds to the usual treatment of linear couplings, the interaction has the form $-\lambda^2\beta\beta'\bar{\Delta}(x-x')$, which is responsible for the familiar form of nuclear forces. For unrestricted values of β, $\mathcal{W}(x,x')$ contains not only a rather different force between nucleons, but matrix elements as well for the creation of arbitrarily many mesons in a nucleon-nucleon collision. We consider first the nuclear forces.

The analogue of the Møller interaction between nucleons is the part of $\mathcal{W}(x,x')$ which emits no real mesons, i.e., its vacuum expectation,

$$V(x-x')=-\lambda^2\langle e^{\beta\varphi(x)+\beta'\varphi(x')}\rangle_0\sin\{\beta\beta'\bar{\Delta}(x-x')\}$$

which, by using (11) and (32), is found to have the value

$$\begin{aligned}V(x-x')&=-\lambda^2\exp[\tfrac{1}{2}\langle(\beta\varphi(x)+\beta'\varphi(x'))^2\rangle_0]\\&\qquad\times\sin\{\beta\beta'\bar{\Delta}(x-x')\}\\&=-\lambda^2\exp[\tfrac{1}{2}\{\beta^2\Delta^{(1)}(0)+\beta\beta'\Delta^{(1)}(x-x')\}]\\&\qquad\times\sin\beta\beta'\bar{\Delta}(x-x').\end{aligned}$$

Once again the renormalization (34) may be used to remove the ambiguous factor. The result is more compactly expressed in terms of the Feynman function $\Delta_F=\Delta^{(1)}-2i\bar{\Delta}$,

$$V(x-x')=\lambda'^2\mathcal{J}\exp[\tfrac{1}{2}\beta\beta'\Delta_F(x-x')], \quad (41)$$

where $\mathcal{J}$ means the imaginary part.

The applicability of (41) to bound state problems may be decided, in the scalar case, by considering, in the nonrelativistic limit, the static potential between two nucleons. Recoil and retardation effects are neglected by assuming that the nucleon wave packets in (37) are slowly varying and may be evaluated at the same time. The time integration in (37) is then carried out approximately by just integrating $\mathcal{W}(x,x')$, or in the present case $V(x-x')$. In effect, the time integral of $V(x)$ is just the nuclear potential. (For linear couplings it is easily verified that $\int_{-\infty}^{\infty}\bar{\Delta}(x)dx_0$ is the Yukawa potential.) The integration may be carried out by first expanding V in series, and gives for the first terms of the potential[10]

$$U(r)=\frac{-\lambda'^2\beta\beta'}{4\pi}\left\{\frac{e^{-\mu r}}{r}+\frac{\beta\beta'\mu}{(2\pi)^2r^2}K_1(2\mu r)+\cdots\right\}. \quad (42)$$

Near the origin the second term diverges as r^{-3}. The

[10] The bessel function $K_1(x)$ is defined by Watson, *A Treatise on the Theory of Bessel Functions* (Macmillan Company, New York, 1948).

further terms have the same signs and diverge successively as r^{-5}, r^{-7}, etc. The potential is therefore, much too strongly singular at the origin to permit a bound state.[11] In the pseudoscalar case, relativistic effects are not negligible, and the applicability of (41) can be decided rigorously only by much more detailed considerations. It appears, however, very unlikely that the interaction has bound states.

When two nucleons collide, mesons may be emitted by either one. The matrix element for the production of n mesons, in all, is given according to (8) and (40), by

$$\langle \mathcal{W}(x,x')\rangle_n = \frac{\{\beta\varphi(x)+\beta'\varphi(x')\}^n}{n!}\langle \mathcal{W}(x,x')\rangle_0 = \frac{\{\beta\varphi(x)+\beta'\varphi(x')\}^n}{n!}V(x-x'). \quad (43)$$

the special case of which, $n=0$, has already been considered. The form of the matrix element is a particularly simple one. Mesons of the group created by each nucleon are emitted at the same point in space-time and therefore independently of one another in momentum-space. A correlation between the momenta of the two groups is introduced, however, by the nuclear potential, and by the over-all conservation conditions for momentum and energy. In effect, the operator (43) preserves, in so far as possible, the statistical independence of the type mentioned in part II. Similar operators have already been treated in this connection by Lewis, Oppenheimer, and Wouthuysen.[12] An approximation they make to simplify their calculations, neglect of the momentum dependence in the fourier transform of the nuclear interaction, amounts to the replacement of $V(x-x')$ by a four-dimensional delta-function.[13] With this assumption, substitution of (43) into (37) gives a very simple matrix element, similar in its dependence on the meson field to (33), which was used for meson-nucleon collisions. The resulting meson production is once again spherically symmetric in the center-of-gravity system, and has an energy-dependent multiplicity proportional to (36). Experimentally observed angular distributions of mesons show a tendency to cluster around the axis of the collision. To investigate such effects with the operator (43) it will be necessary to consider interactions less singular than delta-functions.

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APPENDIX

It may be of interest to note that for the coupling

$$H(x)=\lambda\rho(x)e^{\beta\varphi(x)}, \quad (A.1)$$

where $\rho(x)$ is a classical scalar nucleon density, the S-matrix may be found exactly, and contains no infinities after the renormalization (34). The nth term of the S-matrix is given by

$$S^{(n)}=\frac{(-i\lambda)^n}{n!}\int P\{e^{\beta\varphi_1}, e^{\beta\varphi_2}\cdots e^{\beta\varphi_n}\}\prod_{j=1}^{n}\rho_j dx_j, \quad (A.2)$$

where P is Dyson's time-ordering symbol, and φ_j and ρ_j are these functions evaluated at x_j. The reduction (39) of the product of two exponentials to a single one may be used to reduce the entire P-bracket.

$$P\{e^{\beta\varphi_1}\cdots e^{\beta\varphi_n}\}=\exp[\beta\Sigma\varphi_j - i\beta^2\sum_{j<k}\bar{\Delta}(x_j-x_k)]. \quad (A.3)$$

The matrix elements of $S^{(n)}$, because of (8), all involve the vacuum expectation of this expression, which may be written in renormalized form as

$$\lambda^n\langle P\{e^{\beta\varphi_1}\cdots e^{\beta\varphi_n}\}\rangle_0=\lambda'^n\exp[\tfrac{1}{2}\beta^2\sum_{j<k}\Delta_F(x_j-x_k)]. \quad (A.4)$$

[11] Ironically, however, for $\beta=-\beta'$ (dissimilar nucleons oppositely charged), the potential (42), although repulsive, is quite well-behaved. The series, summed over all terms, is less singular than $1/r$ near the origin (along the real axis) and may be shown to approach a finite value there for $\mu=0$.

[12] Lewis, Oppenheimer, and Wouthuysen, Phys. Rev. **73**, 127 (1948); H. Fukuda and G. Takeda, Prog. Theor. Phys. **5**, 957 (1950).

[13] Treatments to-date of multiple production by two nucleons have overlooked the interference between amplitudes for the creation of different numbers of mesons by each nucleon (the total number of mesons remaining fixed). For $V(x)\sim\delta^{(4)}(x)$ the interference effects are as important as the direct ones. Their inclusion increases the n-fold emission probability by a factor 2^n and the average multiplicity by $2^{\frac{1}{2}}$.