The Continuum in Special Relativity

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A relativistically covariant formalism describing the behavior of a continuous medium is presented. A method is developed for resolving covariant expressions into space and time components. One obtains by this method from a symmetric energy-momentum tensor, a scalar invariant energy density, a vector heat flux, and the stress tensor. The laws of thermodynamics appear as time components and the laws of dynamics as space components of the same relativistic equations. If the divergence of the energy-momentum tensor vanishes, then the first law of thermodynamics and Newton's second law of dynamics (including thermal effects) follow. Reversible processes are considered, and the second law of thermodynamics is formulated with the use of scalar invariant temperature and entropy.

THE equations of motion of a continuous medium in special relativity theory can, as is well known,[1] be expressed as the vanishing of the divergence of a symmetric, second-order, four-dimensional tensor, called the energy-momentum tensor. We consider here some dynamical and thermodynamic consequences of such a formulation.

A. MATHEMATICAL BASIS

Following Abraham and Becker[2] we take as transformation matrix for a Lorentz transformation the hermitian matrix

$$a_{\sigma\tau}=\begin{bmatrix} 1+\hat{u}_1^2/(1-i\hat{u}_4), & \hat{u}_1\hat{u}_2/(1-i\hat{u}_4), & \hat{u}_1\hat{u}_3/(1-i\hat{u}_4), & -i\hat{u}_1 \\ \hat{u}_1\hat{u}_2/(1-i\hat{u}_4), & 1+\hat{u}_2^2/(1-i\hat{u}_4), & \hat{u}_2\hat{u}_3/(1-i\hat{u}_4), & -i\hat{u}_2 \\ \hat{u}_1\hat{u}_3/(1-i\hat{u}_4), & \hat{u}_2\hat{u}_3/(1-i\hat{u}_4), & 1+\hat{u}_3^2/(1-i\hat{u}_4), & -i\hat{u}_3 \\ i\hat{u}_1, & i\hat{u}_2, & i\hat{u}_3, & -i\hat{u}_4 \end{bmatrix}. \tag{1}$$

The determinant of this matrix equals unity. If we represent the four coordinates by x_1, x_2, x_3, $x_4=ict$, then the quantities $\hat{u}_\sigma$ in the matrix are components of the unit time vector

$$\hat{u}_\sigma=(1/c)(dx_\sigma/d\tau),\quad \hat{u}_\sigma^2=-1, \tag{2}$$

where c is the speed of light and τ is the proper time. (Throughout this paper Greek indices run from 1 to 4; Latin indices from 1 to 3; all repeated indices are to be summed.) With this transformation matrix

$$P_\sigma=a_{\sigma\mu}P_\mu{}^0 \quad\text{and}\quad \psi_{\sigma\tau}=a_{\sigma\mu}a_{\tau\nu}\psi_{\mu\nu}{}^0, \tag{3}$$

where P_σ and $\psi_{\sigma\tau}$ are tensor components measured in coordinate system x_σ, and $P_\sigma{}^0$ and $\psi_{\sigma\tau}{}^0$ are measured in coordinate system $x_\sigma{}^0$, which moves relatively to x_σ with velocity u_σ:

$$u_\sigma=c\hat{u}_\sigma=dx_\sigma/d\tau,\quad u_\sigma^2=-c^2. \tag{4}$$

In describing the motion of an element of a continuum, we shall take the coordinate system $x_\sigma{}^0$ as attached to the element at a given point in its trajectory, so that $x_\sigma{}^0$ is the momentary rest frame (at that given point) for which $u_\sigma{}^0=(0,0,0,ic)$. The coordinate system x_σ is thus one relative to which the element momentarily moves with velocity u_σ.

Define

$$\bar{\delta}_{\sigma\tau}=\delta_{\sigma\tau}+\hat{u}_\sigma\hat{u}_\tau, \tag{5}$$

where $\delta_{\sigma\tau}$ is the Kronecker delta. In the rest frame all components of $\bar{\delta}_{\sigma\tau}{}^0$ vanish except

$$\bar{\delta}_{11}{}^0=\bar{\delta}_{22}{}^0=\bar{\delta}_{33}{}^0=1, \tag{6}$$

so that $\bar{\delta}_{\sigma\tau}$ is a unit space tensor. A scalar product of two unit space tensors is a unit space tensor. Also, we have

$$\hat{u}_\sigma\bar{\delta}_{\sigma\tau}=0. \tag{7}$$

We may use the unit time vector $\hat{u}_\sigma$ and unit space tensor $\bar{\delta}_{\sigma\tau}$ to produce new covariant quantities by resolving known covariant tensors into space and time components. Any vector P_σ can be resolved into two components, A_σ and B_σ,

$$P_\sigma=A_\sigma+B_\sigma,\quad A_\sigma=\bar{\delta}_{\sigma\tau}P_\tau,\quad B_\sigma=-\hat{u}_\sigma\hat{u}_\tau P_\tau, \tag{8}$$

where, in the rest frame, $A_\sigma{}^0=(P_1{}^0, P_2{}^0, P_3{}^0, 0)$ and $B_\sigma{}^0=(0,0,0,P_4{}^0)$. Clearly, it is also the case that

$$\begin{gathered} A_\sigma B_\sigma=0, \\ \bar{\delta}_{\sigma\tau}A_\tau=A_\sigma,\quad \bar{\delta}_{\sigma\tau}B_\tau=0, \\ \hat{u}_\sigma A_\sigma=0,\quad \hat{u}_\sigma B_\sigma=\hat{u}_\sigma P_\sigma=iP_4{}^0. \end{gathered} \tag{9}$$

A_σ is a space vector and B_σ, a time vector. $P_4{}^0$, we see, equals a scalar invariant.

[1] M. Abraham and R. Becker, *Theorie der Elektrizität* (B. G. Teubner, Berlin, 1933), sixth edition, Vol. II, p. 356. R. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford University Press, New York, 1934), p. 115. G. Y. Rainich, *Mathematics of Relativity*, (John Wiley and Sons, Inc., New York, 1950), p. 64.
[2] See reference 1, p. 286.

Any second-order tensor $\psi_{\sigma\tau}$ can also be resolved into components,

$$\psi_{\sigma\tau}=\phi_{\sigma\tau}+\chi_{\sigma\tau}, \tag{10}$$

where

$$\phi_{\sigma\tau}=\bar{\delta}_{\sigma\mu}\bar{\delta}_{\tau\nu}\psi_{\mu\nu}, \tag{11}$$

$$\chi_{\sigma\tau}=-(\mathfrak{U}_\sigma\mathfrak{U}_\mu\psi_{\mu\tau}+\mathfrak{U}_\tau\mathfrak{U}_\nu\psi_{\sigma\nu}+\mathfrak{U}_\sigma\mathfrak{U}_\tau\mathfrak{U}_\mu\mathfrak{U}_\nu\psi_{\mu\nu}). \tag{12}$$

In the rest frame $\phi_{\sigma\tau}{}^0$ gives the pure space components of $\psi_{\sigma\tau}{}^0$, and $\chi_{\sigma\tau}{}^0$ gives the other components:

$$\phi_{\sigma\tau}{}^0=\begin{Bmatrix}\psi_{11}{}^0, & \psi_{12}{}^0, & \psi_{13}{}^0, & 0\\ \psi_{21}{}^0, & \psi_{22}{}^0, & \psi_{23}{}^0, & 0\\ \psi_{31}{}^0, & \psi_{32}{}^0, & \psi_{33}{}^0, & 0\\ 0, & 0, & 0, & 0\end{Bmatrix},$$

$$\chi_{\sigma\tau}{}^0=\begin{Bmatrix}0, & 0, & 0, & \psi_{14}{}^0\\ 0, & 0, & 0, & \psi_{24}{}^0\\ 0, & 0, & 0, & \psi_{34}{}^0\\ \psi_{41}{}^0, & \psi_{42}{}^0, & \psi_{43}{}^0, & \psi_{44}{}^0\end{Bmatrix}. \tag{13}$$

Clearly, we have

$$\phi_{\sigma\tau}B_\tau=\phi_{\sigma\tau}B_\sigma=0, \tag{14}$$

$$\psi_{\sigma\tau}\mathfrak{U}_\tau=\chi_{\sigma\tau}\mathfrak{U}_\tau,\quad \psi_{\sigma\tau}\mathfrak{U}_\sigma=\chi_{\sigma\tau}\mathfrak{U}_\sigma. \tag{15}$$

The component $\phi_{\sigma\tau}$ is a pure space tensor, but $\chi_{\sigma\tau}$ is a mixed space-time tensor which we may resolve further. Let

$$Q_\sigma{}^{(1)}=c\bar{\delta}_{\sigma\rho}\psi_{\rho\tau}\mathfrak{U}_\tau=c\bar{\delta}_{\sigma\rho}\chi_{\rho\tau}\mathfrak{U}_\tau,\quad Q_\sigma{}^{(2)}=c\bar{\delta}_{\sigma\rho}\psi_{\tau\rho}\mathfrak{U}_\tau=c\bar{\delta}_{\sigma\rho}\chi_{\tau\rho}\mathfrak{U}_\tau. \tag{16}$$

In the rest frame, we have

$$Q_\sigma{}^{(1)0}=(ic\psi_{14}{}^0,\ ic\psi_{24}{}^0,\ ic\psi_{34}{}^0,\ 0),\quad Q_\sigma{}^{(2)0}=(ic\psi_{41}{}^0,\ ic\psi_{42}{}^0,\ ic\psi_{43}{}^0,\ 0). \tag{17}$$

$Q_\sigma{}^{(1)}$ and $Q_\sigma{}^{(2)}$ are thus space vectors, so that

$$Q_\sigma{}^{(1)}\mathfrak{U}_\sigma=Q_\sigma{}^{(2)}\mathfrak{U}_\sigma=0. \tag{18}$$

If we also write

$$\mathfrak{U}_\sigma\mathfrak{U}_\tau\psi_{\sigma\tau}=\mathfrak{U}_\sigma\mathfrak{U}_\tau\chi_{\sigma\tau}=-\psi_{44}{}^0=-\chi_{\rho\rho}, \tag{19}$$

which shows that $\psi_{44}{}^0$ is equal to the scalar invariant trace $\chi_{\rho\rho}$ of the tensor $\chi_{\sigma\tau}$, then we may write Eq. (12) as

$$\chi_{\sigma\tau}=-[(\mathfrak{U}_\tau Q_\sigma{}^{(1)}/c)+(\mathfrak{U}_\sigma Q_\tau{}^{(2)}/c)]-\mathfrak{U}_\sigma\mathfrak{U}_\tau\chi_{\rho\rho}. \tag{20}$$

Here $\chi_{\sigma\tau}$ appears resolved into the pure time tensor, $-\mathfrak{U}_\sigma\mathfrak{U}_\tau\chi_{\rho\rho}$, and the mixed space-time tensor, $-[(\mathfrak{U}_\tau Q_\sigma{}^{(1)}/c)+(\mathfrak{U}_\sigma Q_\tau{}^{(2)}/c)]$.

We shall be interested only in the case for which the original tensor $\psi_{\sigma\tau}$ is symmetric. For this case $\phi_{\sigma\tau}$ and $\chi_{\sigma\tau}$ are symmetric and

$$\chi_{\sigma\tau}=-(1/c^2)(u_\tau Q_\sigma+u_\sigma Q_\tau+u_\sigma u_\tau\chi_{\rho\rho}), \tag{21}$$

where we have written

$$Q_\sigma=Q_\sigma{}^{(1)}=Q_\sigma{}^{(2)}. \tag{22}$$

The explicit expression for any covariant tensor in terms of its components in the rest frame is obtained from Eq. (3). For the symmetric case, we have

$$Q_k=ic\psi_{k4}{}^0+iu_ku_m\psi_{m4}{}^0/c(1-iu_4/c),\quad Q_4=-u_m\psi_{m4}{}^0, \tag{23}$$

and

$$\begin{aligned}\phi_{mk}&=\psi_{mk}{}^0+[(u_m\psi_{kn}{}^0+u_k\psi_{mn}{}^0)u_n/c^2(1-iu_4/c)]\\&\quad+u_mu_ku_nu_l\psi_{nl}{}^0/c^4(1-iu_4/c)^2,\\ \phi_{k4}&=(iu_m\psi_{mk}{}^0/c)+iu_ku_mu_n\psi_{mn}{}^0/c^3(1-iu_4/c),\\ \phi_{44}&=-u_mu_n\psi_{mn}{}^0/c^2.\end{aligned} \tag{24}$$

It will be noted that although the various covariant quantities $\phi_{\sigma\tau}$, Q_σ, etc., were obtained by operation with $\mathfrak{U}_\sigma$ and $\bar{\delta}_{\sigma\tau}$ on a single tensor $\psi_{\sigma\tau}$, nevertheless, each quantity transforms independently of the others in the Lorentz transformation and thus preserves whatever physical meaning is attached to it, unalloyed by contributions from the other quantities, regardless of the state of motion of the observer measuring them.

Identifying the symmetric tensor $\psi_{\sigma\tau}$ with the energy-momentum tensor associated with an element of continuum (as measured by an observer relative to whom the element moves with velocity u_σ), we write the equations of motion

$$\partial_\sigma\psi_{\sigma\tau}=0, \tag{25}$$

the symbol ∂_σ indicating partial differentiation with respect to coordinate x_σ. From Eqs. (10) and (21) we find

$$(1/c^2)\partial_\sigma(\chi_{\rho\rho}u_\sigma u_\tau+u_\sigma Q_\tau+u_\tau Q_\sigma)=\partial_\sigma\phi_{\sigma\tau}. \tag{26}$$

The vectors on the two sides of this equation can be resolved into space and time components by the method of Eq. (8), and Eq. (26) will hold for the space components and the time components separately. The equation on the space components will appear as a basic equation in dynamics; the equation on the time components, as the first law of thermodynamics.

B. THERMODYNAMICS

Multiplying Eq. (26) by u_τ and using Eqs. (4), (14), and (18), we obtain

$$\partial_\sigma\chi_{\rho\rho}u_\sigma+\partial_\sigma Q_\sigma+(Q_\sigma/c^2)(du_\sigma/d\tau)=\phi_{\sigma\tau}\partial_\sigma u_\tau, \tag{27}$$

where

$$d/d\tau=u_\sigma\partial_\sigma. \tag{28}$$

Since, according to Eq. (21),

$$\chi_{\sigma\tau}\partial_\sigma u_\tau=-(Q_\sigma/c^2)(du_\sigma/d\tau), \tag{29}$$

we may also write Eq. (27) as

$$\partial_\sigma\chi_{\rho\rho}u_\sigma+\partial_\sigma Q_\sigma=\psi_{\sigma\tau}\partial_\sigma u_\tau. \tag{30}$$

If we interpret $\chi_{\rho\rho}$ as the scalar invariant energy density, Q_σ as the vector heat flux, and $\phi_{\sigma\tau}$ as the stress tensor, then Eq. (27) becomes the statement of the first law of thermodynamics. Note that Eqs. (17) and (13) for the symmetric case show that Q_σ and $\phi_{\sigma\tau}$ have properties which make them acceptable to a rest observer as the heat flux and stress, respectively. The rate at which the surroundings do thermodynamic work on the element (per unit volume) is defined as the scalar invariant term $\phi_{\sigma\tau}\partial_\sigma u_\tau$. We may write

$$\phi_{\sigma\tau}\partial_\sigma u_\tau = \phi_{\sigma\tau}\Sigma_{\sigma\tau}, \tag{31}$$

where

$$\Sigma_{\sigma\tau} = \bar{\delta}_{\sigma\mu}\bar{\delta}_{\tau\nu}(\tfrac{1}{2})(\partial_\mu u_\nu + \partial_\nu u_\mu) \tag{32}$$

is the symmetric rate of strain tensor encountered in hydrodynamics. In the rest frame, as a pure space tensor, $\Sigma_{km}{}^0$ has precisely the classical form and $\Sigma_{\sigma 4}{}^0 = 0$. In our definition, thermodynamic work is done on an element only if distortion (strain) occurs. The rate of thermodynamic work should not, however, be confused with the rate of dynamic work, $\bar{\delta}_{4\tau}\partial_\rho\phi_{\tau\rho}$, which contributes to the rate of increase of kinetic energy [see Eq. (73) subsequently]. For a rest observer Eq. (27) becomes

$$(\partial\chi_{\rho\rho}/\partial t^0) + \partial_k{}^0\chi_{\rho\rho}u_k{}^0 + \partial_k{}^0 Q_k{}^0 + 2(Q_k{}^0/c^2)(\partial u_k{}^0/\partial t^0) = \phi_{mk}{}^0\partial_m{}^0 u_k{}^0. \tag{33}$$

Apart from a term of order $1/c^2$ which disappears if the element is unaccelerated, this is just the required statement of conservation of energy in the rest frame. The term $-\partial_k{}^0 Q_k{}^0$ is the rate at which heat is transferred to the element in the rest frame, which justifies our designation of Q_σ as the heat flux,; $\partial_k{}^0 Q_k{}^0$ is not invariant, however, requiring the addition of $(Q_k{}^0/c^2)(\partial u_k{}^0/\partial t^0)$ for invariancy. The subject of the scalar invariant nature of the energy density we reserve for discussion at the end of this section on thermodynamics.

We now write

$$\chi_{\rho\rho} = \rho(U + c^2), \tag{34}$$

where ρ is the scalar invariant mass density which must satisfy the equation of continuity,

$$\partial_\sigma \rho u_\sigma = 0. \tag{35}$$

Thus, we may define ρu_σ as that portion of $\chi_{\rho\rho}u_\sigma/c^2$ for which the divergence vanishes in the absence of chemical reaction. Equation (35) expresses the chemical fact that a portion of the energy is inert to transformation into other forms.[3] Equations (27) and (33) now become

$$\rho(dU/d\tau) + \partial_\sigma Q_\sigma + (Q_\sigma/c^2)(du_\sigma/d\tau) = \phi_{\sigma\tau}\partial_\sigma u_\tau, \tag{36}$$

$$\rho(\partial U/\partial t^0) + \partial_k{}^0 Q_k{}^0 + 2(Q_k{}^0/c^2)(\partial u_k{}^0/\partial t^0) = \phi_{km}{}^0\partial_k{}^0 u_m{}^0. \tag{37}$$

The tensor $\phi_{\sigma\tau}$ we have taken to represent the stress on the element exerted by its surroundings. In order to formulate the second law of thermodynamics, it is necessary to assume the existence of a stress system independent of the surroundings and defined by the nature of the element itself. We represent this stress by the tensor $\phi_{\sigma\tau}'$ and call it the stress in the element, as distinguished from $\phi_{\sigma\tau}$, the stress in the surroundings. (This corresponds to the distinction customarily made in a thermodynamic treatment between the "pressure in the system" and the "pressure in the surroundings".) We assume a basic relation between the stress in the element and the properties of the element

$$(1/\rho)(\partial_\sigma\phi_{\sigma\tau}') = T\partial_\sigma S - \partial_\sigma U. \tag{38}$$

This equation giving $\partial_\sigma\phi_{\sigma\tau}'$ should be compared with Eq. (26). Just as the space and time component equations of Eq. (26) appear as laws of dynamics and thermodynamics, respectively, so will the space and time component equations of Eq. (38). We discuss the time component equations here. Multiplication by ρu_τ gives the thermodynamic equation

$$\rho dU/d\tau = \rho T(dS/d\tau) + \phi_{\sigma\tau}'\Sigma_{\sigma\tau}. \tag{39}$$

The scalar invariant quantities T and S introduced here have the properties of the Kelvin temperature and the specific Clausius entropy, respectively. In fact, comparison of Eqs. (36) and (39) gives

$$T\rho(dS/d\tau) + \partial_\sigma Q_\sigma + (Q_\sigma/c^2)(du_\sigma/d\tau) = (\phi_{\sigma\tau} - \phi_{\sigma\tau}')\Sigma_{\sigma\tau}, \tag{40}$$

so that in a process in which either $\phi_{\sigma\tau} = \phi_{\sigma\tau}'$ or $\Sigma_{\sigma\tau} = 0$ (corresponding to the two possibilities that either the stress in the surroundings equals that in the element, in which case whatever work is done by one stress system is done against the other; or there is no rate of strain of the element, in which case neither stress system does any thermodynamic work), we find

$$T\rho dS/d\tau = -\partial_\sigma Q_\sigma - (Q_\sigma/c^2)(du_\sigma/d\tau). \tag{41}$$

In the rest system this becomes

$$T\rho\partial S/\partial t^0 = -\partial_k{}^0 Q_k{}^0 - 2(Q_k{}^0/c^2)(\partial u_k{}^0/\partial t^0); \tag{42}$$

i.e., the heat absorbed by the element (plus a term of order $1/c^2$, which disappears in an unaccelerated element) is equal to the temperature times the entropy increase of the element.

Let the entropy flux vector be defined as

$$S_\sigma = Q_\sigma/T. \tag{43}$$

Then Eq. (40) gives

$$T[\rho(dS/d\tau) + \partial_\sigma S_\sigma + (S_\sigma/c^2)(du_\sigma/d\tau)] = -S_\sigma\partial_\sigma T + (\phi_{\sigma\tau} - \phi_{\sigma\tau}')\Sigma_{\sigma\tau}. \tag{44}$$

Define the scalar invariant specific Helmholtz free energy

$$A = U - TS. \tag{45}$$

Then, according to Eqs. (38) and (39), we have

$$\partial_\sigma A = -S\partial_\sigma T - (1/\rho)(\partial_\tau\phi_{\sigma\tau}'), \tag{46}$$

$$\rho dA/d\tau = -\rho S(dT/d\tau) + \phi_{\sigma\tau}'\Sigma_{\sigma\tau}, \tag{47}$$

[3] See C. Eckart, Phys. Rev. 58, 919 (1940).

so that Eq. (44) becomes

$$T[\rho(dS/d\tau)+\partial_\sigma S_\sigma+(S_\sigma/c^2)(du_\sigma/d\tau)] = -(S_\sigma+\rho S u_\sigma)\partial_\sigma T-\rho(dA/d\tau)+\phi_{\sigma\tau}\Sigma_{\sigma\tau}. \quad (48)$$

The entropy quantities S and S_σ can be collected into a single entropy tensor given by

$$S_{\sigma\tau}=-(1/c^2)(\rho S u_\sigma u_\tau+u_\sigma S_\tau+u_\tau S_\sigma). \quad (49)$$

It is evident, in fact, that by introduction of various covariant quantities the tensor $\chi_{\sigma\tau}$ of Eqs. (21) and (13) can be written

$$\chi_{\sigma\tau}=-(1/c^2)[\rho(c^2+A)u_\sigma u_\tau]+TS_{\sigma\tau}. \quad (50)$$

$S_{\sigma\tau}$ is thus a tensor with the same transformation properties as $\chi_{\sigma\tau}$. It is a symmetric tensor whose nine space components vanish in the rest frame, the fourth row and column consisting of

$$S_{k4}{}^0=-iS_k{}^0/c \quad \text{and} \quad S_{44}{}^0=\rho S, \quad (51)$$

where $S_{44}{}^0$ equals a scalar invariant, $S_{\rho\rho}$. Also, we have

$$u_\tau S_{\sigma\tau}=S_\sigma+\rho S u_\sigma, \quad (52)$$

$$u_\tau\partial_\sigma S_{\sigma\tau}=\rho(dS/d\tau)+\partial_\sigma S_\sigma+(S_\sigma/c^2)(du_\sigma/d\tau), \quad (53)$$

so that Eq. (48) becomes simply

$$Tu_\tau\partial_\sigma S_{\sigma\tau}=-u_\tau S_{\sigma\tau}\partial_\sigma T-\rho(dA/d\tau)+\phi_{\sigma\tau}\Sigma_{\sigma\tau}. \quad (54)$$

The second law of thermodynamics can now be written in covariant form as

$$u_\tau\partial_\sigma S_{\sigma\tau}\geq 0. \quad (55)$$

For the rest observer, according to Eq. (53), this gives

$$\rho(\partial S/\partial t^0)+\partial_k{}^0 S_k{}^0+2(S_k{}^0/c^2)(\partial u_k{}^0/\partial t^0)\geq 0, \quad (56)$$

which states that the local entropy production rate, plus a term of order $1/c^2$ which vanishes in an unaccelerated element, is nonnegative. $Tu_\tau\partial_\sigma S_{\sigma\tau}$ is therefore the energy dissipation. We have therefore from Eqs. (44) and (54)

$$-S_\sigma\partial_\sigma T+(\phi_{\sigma\tau}-\phi_{\sigma\tau}')\Sigma_{\sigma\tau}\geq 0, \quad (57)$$

$$-(\rho S u_\sigma+S_\sigma)\partial_\sigma T-\rho(dA/d\tau)+\phi_{\sigma\tau}\Sigma_{\sigma\tau}\geq 0. \quad (58)$$

Consider now a continuum which is isotropic even when in motion. We may write

$$\phi_{\sigma\tau}'=-p\bar{\delta}_{\sigma\tau}, \quad p=-\phi_{\rho\rho}'/3, \quad (59)$$

where p is the scalar invariant hydrostatic pressure in the element. In this case we have

$$\phi_{\sigma\tau}'\Sigma_{\sigma\tau}=-p\partial_\sigma u_\sigma=-\rho p d(1/\rho)/d\tau, \quad (60)$$

so that Eq. (57) becomes essentially identical with a previously published result in which it is shown that $(\phi_{\sigma\tau}-p\delta_{\sigma\tau})\partial_\sigma u_\tau$, the Rayleigh dissipation function for viscous flow of isotropic fluid, must be nonnegative in isothermal flow as a consequence of the second law of thermodynamics.[4] Equation (60) together with Eqs. (39) and (47) gives, for the isotropic case,

$$dU/d\tau=T(dS/d\tau)-pd(1/\rho)/d\tau, \quad (61)$$

$$dA/d\tau=-S(dT/d\tau)-pd(1/\rho)/d\tau, \quad (62)$$

in agreement with the usual thermodynamic equations for this case. It may be noted that two independent variables suffice to specify the thermodynamic properties of an isotropic, single pure substance. If the pressure is given as the isotropic part of the total stress system of the substance, then all thermodynamic work appears as "pressure-volume" work. There is in principle, therefore, no necessity to introduce numerous work terms in the usual thermodynamic presentation of the isotropic case.

When the equality holds in Eq. (55) the process which occurs is nondissipative or "reversible". The equality will also hold in Eqs. (57) and (58). If we require that

$$(\phi_{\sigma\tau}-\phi_{\sigma\tau}')\Sigma_{\sigma\tau}=0 \quad (63)$$

and

$$S_\sigma\partial_\sigma T=S_k{}^0\partial_k{}^0 T=0, \quad (64)$$

then these conditions are satisfied. It will be noted that Eq. (63) was a requirement for Eq. (41), but a process in which Eq. (41) is valid is not necessarily reversible unless the condition of Eq. (64) holds also. Equation (41) describes processes of heat transfer which may be dissipative unless Eq. (64) holds. This fact is not usually evident in classical thermodynamic discussions in which temperature gradients are not allowed in the static, homogeneous phases considered to constitute a thermodynamic system.

Equation (63) is satisfied by either $\phi_{\sigma\tau}=\phi_{\sigma\tau}'$ or $\Sigma_{\sigma\tau}=0$; Eq. (64), by either $S_\sigma=0$ or $\bar{\delta}_{\sigma\tau}\partial_\tau T=0$. There is evidence that these conditions are not independent. Thus, in Fourier's law of heat conduction, which may be written covariantly as

$$TS_\sigma=-\lambda_{\sigma\rho}\bar{\delta}_{\rho\tau}\partial_\tau T, \quad (65)$$

where $\lambda_{\sigma\rho}$ are coefficients of thermal conductivity, and in the linear relations assumed in the theory of viscous flow,

$$\phi_{\sigma\tau}-\phi_{\sigma\tau}'=\eta_{\sigma\tau\mu\nu}\Sigma_{\mu\nu}, \quad (66)$$

where $\eta_{\sigma\tau\mu\nu}$ are coefficients of viscosity, the vanishing of the gradient of temperature and of the components of stress, $\phi_{\sigma\tau}-\phi_{\sigma\tau}'$ implies the vanishing of the heat flux and of the components of rate of strain. Because of these relationships, we shall specify for reversible processes that

$$\bar{\delta}_{\sigma\tau}\partial_\tau T=0 \quad \text{and} \quad \phi_{\sigma\tau}=\phi_{\sigma\tau}'. \quad (67)$$

From Eq. (46) we have for a reversible process

$$\bar{\delta}_{\sigma\tau}\partial_\rho\phi_{\tau\rho}=-\bar{\delta}_{\sigma\tau}\rho\partial_\sigma A; \quad (68)$$

i.e., the gradient of the specific Helmholtz free energy balances the force exerted by the external stress; the specific Helmholtz free energy acts as potential func-

[4] B. Leaf, Phys. Rev. **70**, 749 (1946), Eq. (10); and 752 (1946), Eq. (40b).

tion for the stress $\phi_{\sigma\tau}'$ in the element. In the isotropic case Eq. (59) gives

$$\bar{\delta}_{\sigma\tau}\partial_\rho\phi_{\sigma\rho} = -\bar{\delta}_{\sigma\tau}[\partial_\sigma p + (p/c^2)(du_\sigma/d\tau)], \tag{69}$$

so that the pressure gradient in the element (plus a term of order $1/c^2$ which disappears in an unaccelerated element) balances the force of the external stress.

It should be noted that reversible or nondissipative processes need not be quasi-static, despite the common statement to the contrary in the literature. It is well known that much of classical dynamics contemplates nonquasi-static processes which are thermodynamically reversible, dissipative effective not being considered. [See Eq. (77) below.]

We conclude this section on thermodynamics by considering the scalar invariant nature of the densities of mass, energy, entropy, etc., which have appeared in the preceding discussion. Now

$$dVdx_4 = dx_1dx_2dx_3dx_4 \tag{70}$$

is a scalar invariant. But $d\tau$ is also scalar invariant, where

$$d\tau = dx_4/u_4. \tag{71}$$

Therefore, $(u_4/ic)dV$ is scalar invariant. Hence, the volume element in the rest frame,

$$dV^0 = (u_4/ic)dV, \tag{72}$$

is invariant. The integral of ρ over the finite rest volume of a given region gives therefore an invariant mass $\int \rho dV^0$. Similarly, the internal energy $\int \rho U dV^0$, the Helmholtz free energy $\int \rho A dV^0$, the entropy $\int \rho S dV^0$, for a region of finite extent are all invariant. Since the temperature T and the isotropic pressure p are also invariant in our presentation, we have found that all the usual thermodynamic variables except volume are invariant. That the invariance of temperature is consistent with that of energy density is illustrated by the case of radiation for which the energy density equals αT^4, where α is the universal Stefan-Boltzmann constant. These results may be compared with those of Planck[5] who also finds scalar invariant pressure and entropy but who concludes that temperature, as well as volume, is diminished by the factor ic/u_4 as compared with the rest value when measured by an observer with respect to whom the system moves with velocity u_σ.

C. DYNAMICS

The dynamical results of our formulation proceed from the same equations as did the thermodynamic results. The time components of Eq. (25) and Eq. (38) [or its equivalent, Eq. (46)] yielded thermodynamic relations; the space components give dynamical relations. Multiplying Eqs. (26) and (46) by $\bar{\delta}_{\sigma\tau}$, we have

$$\bar{\delta}_{\sigma\tau}\partial_\rho\phi_{\rho\tau} = (1/c^2)[\chi_{\rho\rho}(du_\sigma/d\tau) + Q_\sigma\partial_\rho u_\rho + Q_\rho\partial_\rho u_\sigma + \bar{\delta}_{\sigma\tau}dQ_\tau/d\tau], \tag{73}$$

$$\bar{\delta}_{\sigma\tau}\partial_\rho\phi_{\rho\tau}' = -\bar{\delta}_{\sigma\tau}\rho(\partial_\tau A + S\partial_\tau T). \tag{74}$$

Equation (73) is the relativistic formulation of Newton's second law of motion for an element of continuum, including thermal effects. In the rest frame it consists of only three equations of motion associated with the three space coordinates,

$$(1/c^2)\chi_{\rho\rho}du_k^0/dt^0 = \partial_m^0\phi_{mk}^0 + (\phi_{mk}^0/c^2)(\partial u_m^0/\partial t^0) - (1/c^2)(Q_k^0\partial_m^0 u_m^0 + Q_m^0\partial_m^0 u_k^0 + dQ_k^0/dt^0). \tag{75}$$

The fourth equation, both sides of which vanish in the rest frame, is the kinetic energy equation, which may be written

$$-(u_4/c^2)(\chi_{\rho\rho}du_4/d\tau) = u_k\partial_\sigma\phi_{k\sigma} - (u_k^2/c^2)(\phi_{\sigma\tau}\partial_\sigma u_\tau) + (u_4/c^2)(Q_4\partial_\sigma u_\sigma + Q_\sigma\partial_\sigma u_4 + \bar{\delta}_{4\sigma}dQ_\sigma/d\tau). \tag{76}$$

In a reversible process, Eqs. (67), (73), (74), and (65) give

$$(1/c^2)(\chi_{\rho\rho}du_\sigma/d\tau) = -\bar{\delta}_{\sigma\tau}\rho\partial_\tau A, \tag{77}$$

so that the element moves like a particle in a field of potential A with a total mass density $\chi_{\rho\rho}/c^2$. If the process is not reversible, however, then there must be added to the right side of Eq. (77) the terms

$$\bar{\delta}_{\sigma\tau}\partial_\rho(\phi_{\rho\tau} - \phi_{\rho\tau}') - \bar{\delta}_{\sigma\tau}\rho S\partial_\tau T - (1/c^2)(Q_\sigma\partial_\rho u_\rho + Q_\rho\partial_\rho u_\sigma + \bar{\delta}_{\sigma\tau}dQ_\tau/d\tau) \tag{78}$$

the first of these representing the force produced by viscous stress [see Eq. (66)]; the second, the force exerted on matter in a temperature gradient as a result of the association of entropy with matter. The second term must be considered in nonisothermal hydrodynamic processes.

If we write

$$(\chi_{\rho\rho}/c^2)(du_\sigma/d\tau) = k_\sigma, \tag{79}$$

where k_σ represents the total force density given by the right side of Eq. (77) together with the terms of Eq. (78), we have, in view of Eq. (72),

$$\int(\chi_{\rho\rho}/c^2)(du_\sigma/d\tau)dV^0 = \int(k_\sigma u_4/ic)dV. \tag{80}$$

In the motion of a particle every point of which has the same acceleration, if we let the mass be given by the invariant quantity

$$m = \int(\chi_{\rho\rho}/c^2)dV^0 \tag{81}$$

then the equation of motion for the particle becomes

$$mdu_\sigma/d\tau = K_\sigma, \tag{82}$$

[5] M. Planck, Ann. Physik **26**, 1 (1908).

where K_σ is the Minkowski force on the particle

$$K_\sigma = \int (k_\sigma u_4/ic)dV. \tag{83}$$

The mass m though invariant in a Lorentz transformation is not necessarily a constant of the motion, even if the density ρ of the particle is constant, since m includes the thermodynamic internal energy [see Eq. (34)]. Included in the Minkowski force is a term which predicts a force on a particle when it exists in a temperature gradient, a force associated with the entropy the particle possesses. If $\bar{\delta}_{\sigma\tau}\partial_\tau T$ is constant throughout the volume of the particle, then this force is $-\mathcal{S}\bar{\delta}_{\sigma\tau}\partial_\tau T$, where $\mathcal{S}$ is the total entropy of the particle.

Equation (25) may be written in a form which facilitates comparison with the methods of field theory, namely,

$$\partial_\sigma \rho u_\sigma u_\tau = \partial_\sigma T_{\sigma\tau}, \tag{84}$$

where

$$T_{\sigma\tau} = -(1/c^2)[\rho(A+TS)u_\sigma u_\tau + u_\sigma Q_\sigma + u_\tau Q_\sigma] + \phi_{\sigma\tau}. \tag{85}$$

Equation (84) describes motion of a "naked" element, i.e., an element devoid of thermodynamic structure, through a field described by the energy-momentum tensor $T_{\sigma\tau}$. If we take $T_{\sigma\tau}$ to be the electromagnetic tensor,

$$T_{\sigma\tau} = (1/4\pi)(F_{\sigma\rho}F_{\rho\tau} + \delta_{\sigma\tau}F_{\rho\nu}{}^2/4), \tag{86}$$

where $F_{\sigma\tau} = \partial_\sigma A_\tau - \partial_\tau A_\sigma$ and A_σ is the four-vector potential, then $Q_\sigma{}^0$ coincides with the Poynting flux, the energy density $\rho U = (E^{0\,2} + H^{0\,2})/8\pi$, where E^0 and H^0 are the electric and magnetic intensities in the rest frame, and the stress tensor

$$4\pi\phi_{\sigma\tau} = -(F_{\sigma\rho} + u_\sigma u_\mu F_{\mu\rho}/c^2)(F_{\tau\rho} + u_\tau u_\nu F_{\nu\rho}/c^2) + \bar{\delta}_{\sigma\tau}F_{\rho\nu}{}^2/4, \tag{87}$$

which reduces in the rest frame to the maxwell stress tensor in its nine space components and to zero in its fourth row and column.

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Bound States in Quantum Field Theory

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The relativistic two-body equation of Bethe and Salpeter is derived from field theory. It is shown that the Feynman two-body kernel may be written as a sum of wave functions over the states of the system. These wave functions depend exponentially on the energies of the states to which they correspond and therefore provide a means of calculating energy levels of bound states.

I. INTRODUCTION

SEVERAL attempts have been made to calculate the energy levels of bound systems of two particles that interact through a quantized field. The standard method[1] has been to calculate an effective potential energy function and to insert that function into some two-particle Schroedinger or Dirac equation. In a case where the major effects of the interaction are obviously in the nonrelativistic region (e.g., the hydrogen atom), such a procedure seems reasonable (although even here higher order effects may not be describable by a potential).[2] However, in the treatment of nuclear problems, one may have to deal with specifically relativistic interactions and singular forces for which methods successful in the atomic domain may fail entirely.

Recently, Dancoff[3] has used an approximate method based directly on field theory. He has solved the Schroedinger equation for the state vector with the requirement that it contain no particle-pairs and only one field quantum. The formal extension of his method to include higher approximations is difficult on account of the necessity of separating divergences in a noncovariant way. Furthermore, it appears impossible in his framework to make use of the elegant techniques developed by Feynman[4] and Dyson.[5]

Bethe and Salpeter have proposed an equation[6] for a two-body "wave function"; their equation is covariant in form and permits the separation of divergences as in the S-matrix theory. Their reasoning, however, is based on an analogy to that in Feynman's "Theory of positrons"[4] and the demonstration of equivalence to

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[3] S. M. Dancoff, Phys. Rev. **78**, 382 (1950).
[4] R. P. Feynman, Phys. Rev. **76**, 749 (1949); Phys. Rev. **76**, 769 (1949).
[5] F. J. Dyson, Phys. Rev. **75**, 486 (1949); Phys. Rev. **75**, 1736 (1949).
[6] H. A. Bethe and E. E. Salpeter, Phys. Rev. **82**, 309 (1951). We are indebted to Drs. Bethe and Salpeter for communicating their results to us prior to publication. We understand that this equation has been treated by Schwinger in his lectures at Harvard.