Our choice of statistics in the nucleus is not so arbitrary as might be thought, since it amounts primarily to a choice of normalization of $P(z_{Av})$.

To interpret $P(z_{Av})$ and $P_D(z_{Av}^D)$ further, we set $z_{Av}^D=0$. Then

$$P_D(0)=|\psi_D(0)|^2,$$

where $\psi_D(0)$ is the deuteron wave function for zero separation of the neutron and proton. $P_D(0)$ is just the probability of finding them in contact. We write

$$P(z_{Av})=f/(4\pi/3)a_0^3A \tag{48}$$

or a correlation factor divided by the nuclear volume. $f=1$ would correspond to random spacing of particles in a box of nuclear volume. Using the Chew-Goldberger[23] wave function for the deuteron, we have

$$P(z_{Av})/P_D(0)=0.82f/A. \tag{49}$$

For C^{12}, Eq. (47) yields $\Gamma=0.28f$. Choosing the reasonable value $\Gamma\simeq 10$, we have

$$f\simeq 35.0. \tag{50}$$

This would seem to indicate a reasonably strong degree of correlation in nuclear structure. Such a conclusion appears quite compatible with the evidence presented by several authors from high energy $p-p$ scattering[24,25] for strong nuclear interactions at close distances. It is also compatible with the evidence concerning nuclear structure which was given by Chew and Goldberger[23] on the basis of York's [26] measurement of high energy $(n-d)$ processes (see also the discussion in I on this point).

The analysis of π^+ absorption can be carried through in the same manner. As mentioned in Sec. II, we have reason to expect the absorption of π^+ and π^- mesons to be similar.

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[24] R. Christian and H. P. Noyes, Phys. Rev. **79**, 89 (1950); R. Jastrow, Phys. Rev. **81**, 165 (1951).

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PHYSICAL REVIEW VOLUME 84, NUMBER 2 OCTOBER 15, 1951

On the Polarization of High Energy Bremsstrahlung and of High Energy Pairs

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(Received July 2, 1951)

The polarization of bremsstrahlung due to electrons with initial energies much larger than $137Z^{-\frac{1}{3}}\ mc^2$ is calculated under relativistic, small angles approximations. The cross section for photons polarized normally to the plane containing the initial direction of the electron and the direction of the photon is found to be larger than for photons polarized in that plane. A similar calculation shows that the plane containing one of a pair produced by a polarized photon together with the direction of that photon tends to lie parallel to the plane of polarization rather than normal to it, except for one special case. The effect of the deviation due to multiple scattering of electrons in the target upon the angular dependence of the polarization is considered.

IN this note we shall investigate the polarization of bremsstrahlung due to electrons of energy $E_0\gg 137Z^{-\frac{1}{3}}\ mc^2$, where Z is the atomic number of the target material. We shall then carry out analogous calculations for pairs produced by high energy, polarized γ-rays and obtain a preferred azimuth of the plane of the pairs relative to the plane of polarization. In the last part of the paper, we shall consider the effect of multiple scattering of electrons in the target upon our results for the polarization of bremsstrahlung. This note has been written in confirmation and in partial extension of previous results[1,2] obtained by using the method of virtual quanta.[3]

[1] G. C. Wick, Phys. Rev. **81**, 467 (1951).

[2] M. May and G. C. Wick, Phys. Rev. **81**, 628 (1951).

[3] C. F. v Weizsäcker, Z. Physik **88**, 612 (1934); E. J. Williams, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **13**, 4 (1935).

I. BREMSSTRAHLUNG

Let us consider an electron of total energy E_0, momentum $\mathbf{p}_0$, deflected by a nucleus of charge Ze. Let a quantum of momentum $\mathbf{k}$ (we take $c=1$ from here on) be radiated at an angle θ_0 with the initial direction of the electron. (See Fig. 1.) After radiation, let E be the total energy and $\mathbf{p}$ the momentum of the deflected electron, and let its direction make an angle θ with the direction of the emitted quantum. Call ψ the angle between the $\mathbf{p}_0\mathbf{k}$ plane (plane of emission) and the $\boldsymbol{\varepsilon}\mathbf{k}$ plane, where $\boldsymbol{\varepsilon}$ is the polarization vector of the photon; call φ the angle between the $\mathbf{p}_0\mathbf{k}$ and the $\mathbf{pk}$ planes, and φ_0 the angle between the $\mathbf{p}_0\mathbf{k}$ plane and some fixed plane. If $\mathbf{q}$ is the momentum given up to the nucleus, the conservation conditions read:

$$\mathbf{q}=\mathbf{p}_0-\mathbf{p}-\mathbf{k};\quad E_0=E+k.$$

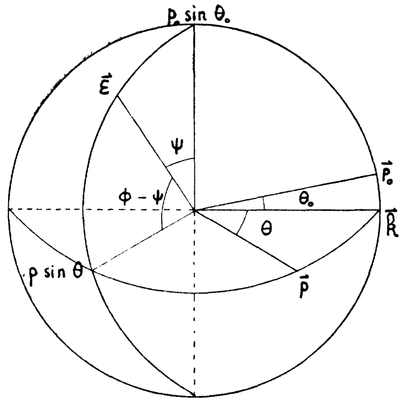

FIG. 1. Relationships between the angles involved in Eqs. (1).

Writing all energies and momenta in units of the rest energy of the electron, the differential cross section for the process described is[4]

$$d\sigma=\frac{\bar{\Phi}}{4\pi^2}\frac{p}{p_0}\frac{dk}{k}d\Omega_0 d\Omega\frac{1}{q^4}$$
$$\times\left\{\left[2E_0\frac{\beta\sin\theta\cos(\varphi-\psi)}{1-\beta\cos\theta}-2E\frac{\beta_0\sin\theta_0\cos\psi}{1-\beta_0\cos\theta_0}\right]^2\right.$$
$$+q^2\left[\frac{\beta\sin\theta\cos(\varphi-\psi)}{1-\beta\cos\theta}-\frac{\beta_0\sin\theta_0\cos\psi}{1-\beta_0\cos\theta_0}\right]^2$$
$$\left.+k^2\frac{p^2\sin^2\theta+p_0^2\sin^2\theta_0-2p_0p\sin\theta_0\sin\theta\cos\varphi}{E_0E(1-\beta\cos\theta)(1-\beta_0\cos\theta_0)}\right\},\quad(1)$$

where

$$\bar{\Phi}=Z^2e^4/137,\quad \beta_0=p_0/E_0,\quad \beta=p/E,$$
$$d\Omega_0=\sin\theta_0 d\theta_0 d\varphi_0,\quad d\Omega=\sin\theta d\theta d\varphi,$$
$$q^2=p^2+p_0^2+k^2-2p_0k\cos\theta_0+2pk\cos\theta$$
$$-2p_0p(\cos\theta_0\cos\theta+\sin\theta_0\sin\theta\cos\varphi).$$

This formula is valid for a pure coulomb field. However, as stated above, we are considering electrons with initial energies large compared with $137Z^{-\frac{1}{3}}m$ (which is about 16 Mev for lead or for platinum). For given E_0, E, and k, the cross section reaches a sharp maximum when $\mathbf{p}_0$, $\mathbf{p}$, and $\mathbf{k}$ are parallel, that is, when

$$q=q_{\text{minimum}}\cong\mathbf{m}\mathbf{k}/2E_0E\quad\text{(in units of } m)$$

at relativistic energies. Therefore, under our assumption,

$$q_{\text{minimum}}\ll\hbar/ma_0Z^{-\frac{1}{3}}\quad(a_0=\text{radius of H atom})$$

and screening is effective. Classically speaking, the most effective impact parameter is large compared with the atomic radius. Accordingly, we choose for the potential due to the screened nucleus[5]

$$V(r)=(Ze/r)\exp(-r/a),\qquad(2)$$

where $a=108\hbar Z^{-\frac{1}{3}}/m$, and is so determined that in the high energy limit the integral cross section agrees with that obtained using the numerical values of the form factor for a Fermi atom. Obviously, the more complete the screening, i.e., the higher the primary energy, the more justifiable is our choice. Its effect is to replace $1/q^4$ in Eq. (1) by $1/(g^2+q^2)^2$, where $g=Z^{\frac{1}{3}}/108$.

Furthermore, at high energies, as is well known, the radiated photon and the deflected electron will both go mainly in the forward direction, the cross section decreasing rapidly outside of cones defined by $\theta_0=1/E_0$, $\theta=1/E$. We shall assume that the energies are such that we can neglect both $1-\beta_0^2$ and $1-\beta^2$ compared with 1, and by the same token we shall write

$$\cos\theta_0\cong1-\tfrac{1}{2}\sin^2\theta_0,\quad\cos\theta\cong1-\tfrac{1}{2}\sin^2\theta.$$

We then find that

$$q^2=p_0^2\sin^2\theta_0+p^2\sin^2\theta-2p_0p\sin\theta_0\sin\theta\cos\varphi$$
$$+q^2_{\text{min}}\ll E_0^2,\ E^2,\ k^2.$$

We have thus eliminated from consideration the very soft quanta, with energies of order $(1-\beta_0^2)^{\frac{1}{2}}E_0\cong m$, and the very hard ones, for which the energy of the deflected electron E is of order m. An additional and more stringent restriction on the upper end of the γ-ray spectrum is imposed by the screening assumption, since screening is not effective in the production of photons of energy:

$$k>E_0[E_0Z^{\frac{1}{3}}/137-1]^{-1}.$$

We may now neglect the term proportional to q^2 in Eq. (1) and integrate over φ.[6] The result is divided into two parts, one, $d\sigma_{\parallel}$ defined as the cross section for radiation polarized parallel to the plane of emission ($\psi=0$), the other, $d\sigma_{\perp}$, defined as the cross section for radiation polarized normally to the plane of emission ($\psi=\pi/2$). If we introduce the dimensionless variables,

$$x_0=E_0^2\sin^2\theta_0;\quad x=E^2\sin^2\theta,$$

we obtain under our approximations

$$d\sigma_{\perp}=\frac{\bar{\Phi}}{2\pi}\frac{p}{p_0}\frac{dk}{k}d\varphi_0dx_0dx\left\{\frac{1}{x_0(1+x)^2}\left(-1+\frac{x+x_0+f^2}{\sqrt{X}}\right)\right.$$
$$\left.+\frac{k^2}{E_0E}\frac{(x-x_0)^2+f^2(x+x_0)}{(1+x_0)(1+x)X^{\frac{3}{2}}}\right\},\quad(3)$$

[4] The cross section is given after integration over the polarization variable by, e.g., W. Heitler *The Quantum Theory of Radiation* (Oxford University Press London, 1944), second edition, p. 164.

[5] H. A. Bethe, Proc. Cambridge Phil. Soc. **30**, 538 (1934).

[6] The integration can easily be carried out without neglecting the term proportional to q^2, but the extra terms obtained are cumbersome and of order $1-\beta_0^2$ compared with the terms in Eqs. (3) and (4).

$$d\sigma_{\parallel}=\frac{\bar{\phi}}{2\pi}\frac{p}{p_0}\frac{dk}{k}d\varphi_0dx_0dx\left\{\frac{1}{x_0(1+x)^2}\left(1-\frac{x+x_0+f^2}{\sqrt{X}}\right)+\frac{k^2}{E_0E}\frac{(x-x_0)^2+f^2(x+x_0)}{(1+x_0)(1+x)X^{\frac{3}{2}}}+\frac{4}{X^{\frac{3}{2}}}\left[\frac{(1+x_0x)(x-x_0)^2}{(1+x)^2(1+x_0)^2}+f^2\left(\frac{x}{(1+x)^2}+\frac{x_0}{(1+x_0)^2}\right)\right]\right\}, \quad (4)$$

$$X=(x-x_0)^2+2f^2(x+x_0)+f^4,$$

where f^2 is a quantity of order 10^{-4} for $E_0 \geqslant 100$. The function $f(x)=d\sigma_{\perp}-d\sigma_{\parallel}$ is plotted in Fig. 2. As seen from the graph, we can expect the integrated cross section for radiation polarized normally to the plane of emission to be larger than the cross section for radiation polarized parallel to that plane. The function $f(x)$ as well, of course, as $d\sigma_{\perp}+d\sigma_{\parallel}$ have sharp maxima at $x=x_0$ for which $q=q_{\min}$.

Let us define $\epsilon=k/E_0$, so that $E=(1-\epsilon)E_0$. Integrating (3) and (4) over x between 0 and any number large compared with 1 (actually $E_0{}^2\pi$) gives

$$d\sigma_{\perp}=2\frac{\bar{\phi}}{\pi}\frac{d\epsilon}{\epsilon}\frac{d\varphi_0dx_0}{(1+x_0)^2}\left\{\left[1-\epsilon+\frac{\epsilon^2}{2}\right]\log\frac{1+x_0}{f}-(1-\epsilon)-\frac{\epsilon^2}{4}\right\}, \quad (5)$$

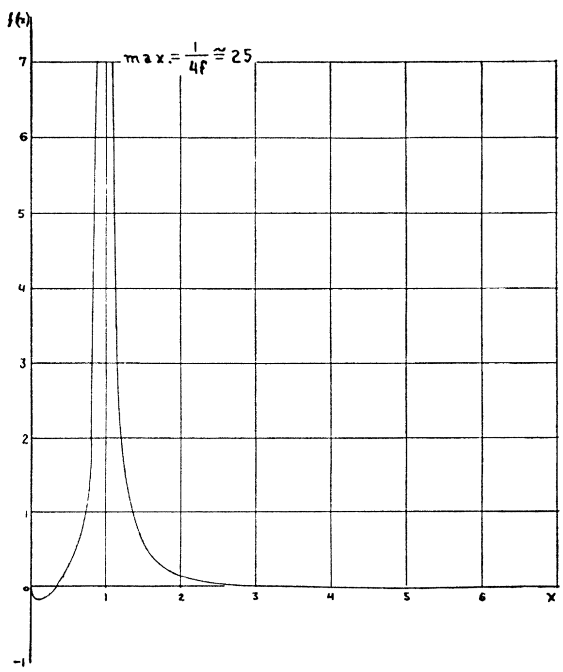

FIG. 2. $f(x)=\dfrac{d\sigma_{\perp}-d\sigma_{\parallel}}{\dfrac{\Phi}{2\pi}\dfrac{dk}{k}\dfrac{p}{p_0}}d\varphi_0dx\sqrt{x}$ for $x_0=1$.

$$d\sigma_{\parallel}=2\frac{\bar{\phi}}{\pi}\frac{d\epsilon}{\epsilon}\frac{d\varphi_0dx_0}{(1+x_0)^2}\left\{\left[1-\epsilon+\frac{\epsilon^2}{2}-4(1-\epsilon)\frac{x_0}{(1+x_0)^2}\right]\log\frac{1+x_0}{f}-\frac{\epsilon^2}{4}+(1-\epsilon)\left[1-2\left(\frac{1-x_0}{1+x_0}\right)^2\right]\right\}; \quad (6)$$

$$\begin{aligned}\therefore\ f(x_0,\epsilon)&=d\sigma_{\perp}-d\sigma_{\parallel}\\&=(8\bar{\phi}/\pi)[d\epsilon(1-\epsilon)/\epsilon][d\varphi_0x_0dx_0/(1+x_0)^4]\\&\quad\times[\log[(1+x_0)/f]-2] \quad (7)\\&>0.\end{aligned}$$

The equivalent formulas obtained by the method of

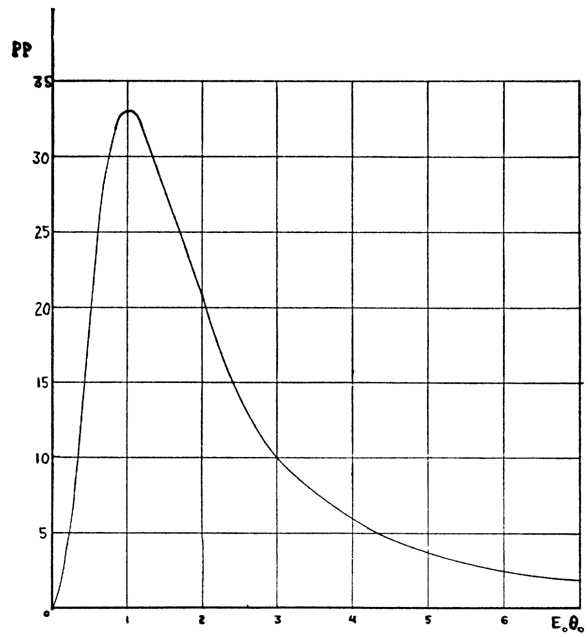

FIG. 3. $PP(x_0)$ for $F(\epsilon)=4$ ($\epsilon\cong 1/2$. $Z\cong 78$).

virtual quanta under the assumption of complete screening are the same as the terms containing $\log((1+x_0)/f)$ as a factor in formulas (5), (6), (7), except that the argument of the logarithm is $137Z^{-\frac{1}{3}}$ instead of $(1+x_0)108Z^{-\frac{1}{3}}$. Integrating over φ_0 and x_0 gives

$$f(\epsilon)=[2\bar{\phi}(1-\epsilon)d\epsilon/\epsilon][4\log^{\frac{1}{3}}(1/f)-14/9], \quad (8)$$

$$\sigma_{\text{tot}}(\epsilon)=(4\bar{\phi}d\epsilon/\epsilon)\{[4(1-\epsilon)/3+\epsilon^2](\tfrac{1}{2}+\log(1/f))+(1-\epsilon)/9\}. \quad (9)$$

The total cross section agrees with that given by Heitler[7] for the case of complete screening except that we obtain $\log 178Z^{-\frac{1}{3}}$ instead of $\log 183Z^{-\frac{1}{3}}$. The percent

[7] See reference 4, p. 170, Eq. (26).

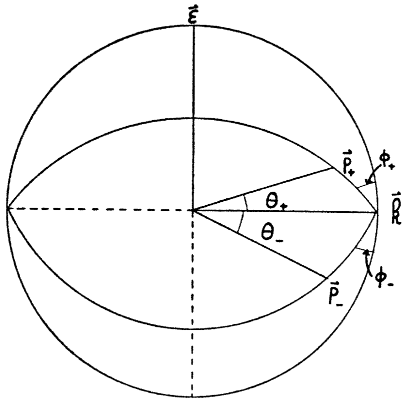

FIG. 4. Relationships between the angles involved in Eq. (11).

polarization, defined in the usual manner as

$$PP(x_0,\epsilon)=(d\sigma_\perp-d\sigma_{\parallel})/(d\sigma_\perp+d\sigma_{\parallel})=f(x_0,\epsilon)/d\sigma_{\mathrm{tot}}(x_0,\epsilon),$$

can be shown to decrease monotonically as ϵ increases from 0 to 1 for a given x_0. On the other hand, as a function of x_0, it increases from 0 in the forward direction to a maximum at $x_0=1$, then decreases as $1/x_0$ (see Fig. 3). If we take the logarithm to be nearly constant, we have

$$\begin{aligned}PP(x_0,\epsilon)&=4x_0[F(\epsilon)(1+x_0)^2-4x_0]^{-1},\\ F(\epsilon)&=\{[1+(1-\epsilon)^2]/(1-\epsilon)\}+(\log((1+x_0)/f)\\ &\quad-2)^{-1}[(2-\epsilon)^2+2\epsilon^2]/2(1-\epsilon)\\ &\cong\{[1+(1-\epsilon)^2]/(1-\epsilon)\}+(1/2)[(2-\epsilon)^2\\ &\quad+2\epsilon^2]/2(1-\epsilon)\quad\text{for Pt.}\end{aligned}\tag{10}$$

II. PAIR PRODUCTION

Under the same assumption for the screening as above, Eq. (2), the differential cross section for pair production by a polarized photon of momentum $\mathbf{k}$ in a nuclear field is[8]

$$\begin{aligned}d\sigma=\bar{\phi}\frac{p_+p_-dE_+}{4\pi^2k^3}\frac{d\Omega_+d\Omega_-}{(g^2+q^2)^2}\Bigg\{&-4\left[E_-\frac{\beta_+\sin\theta_+\cos\varphi_+}{1-\beta_+\cos\theta_+}+E_+\frac{\beta_-\sin\theta_-\cos\varphi_-}{1-\beta_-\cos\theta_-}\right]^2+q^2\left[\frac{\beta_+\sin\theta_+\cos\varphi_+}{1-\beta_+\cos\theta_+}-\frac{\beta_-\sin\theta_-\cos\varphi_-}{1-\beta_-\cos\theta_-}\right]^2\\ &+k^2\frac{p_+^2\sin^2\theta_++p_-^2\sin^2\theta_-+2p_+p_-\sin\theta_+\sin\theta_-\cos(\varphi_+-\varphi_-)}{E_+E_-(1-\beta_+\cos\theta_+)(1-\beta_-\cos\theta_-)}\Bigg\},\end{aligned}\tag{11}$$

where the subscripts + and − refer to the positive and negative electrons, so that

$$k=E_++E_-,\quad \mathbf{q}=\mathbf{k}-\mathbf{p}_+-\mathbf{p}_-,$$

and where the relationships between the angles are illustrated in Fig. 4. We shall define $d\sigma=d\sigma_{\parallel}$ when $\varphi_+=0$, i.e., when the positive electron is produced in the $\boldsymbol{\varepsilon}\mathbf{k}$ plane (plane of polarization); similarly, $d\sigma=d\sigma_\perp$ when $\varphi_+=\pi/2$, i.e., when the positive electron is produced in a plane normal to the plane of polarization. We then integrate $d\sigma_{\parallel}$ and $d\sigma_\perp$ over the angle φ_- which the plane containing the directions of the negative electron and of the photon makes with the plane of polarization. It must be noted, however, that the term proportional to q^2 in formula (11) may not be negligible even under our high energy, small angle approximation. This is because it contains a minus sign compared with the plus sign contained in the first term (this situation does not arise in connection with formula (1)). In particular, for $d\sigma=d\sigma_{\parallel}$ where $\cos\varphi_+=1$, the case $q^2\cong q^2_{\min}$ ($\therefore d\sigma_{\parallel}\cong d\sigma_{\parallel\,\max}$) implies that $\cos\varphi_-=-1$,[1] $\beta_+\sin\theta_+=\beta_-\sin\theta_-$. Under these conditions, both the first and the third terms in (11) become small. Defining $x_+=E_+^2\sin^2\theta_+$, $x_-=E_-^2\sin^2\theta_-$, we therefore distinguish between two cases.

Case A: $|x_+-x_-|\gg 0.01$ (taking $k\geqslant 100$). Then the term proportional to q^2 in (11) can be neglected and integration over φ_- yields

$$\begin{aligned}f(x_+,x_-)&=d\sigma_\perp-d\sigma_{\parallel}\\ &=\bar{\phi}\frac{p_+p_-dE_+}{2\pi k^3}d\varphi_+\frac{dx_+dx_-}{|x_+-x_-|}\\ &\quad\times\left\{-2\frac{x_++x_--|x_+-x_-|}{x_+(1+x_-)^2}+\frac{4(1+x_+x_-)}{(1+x_+)^2(1+x_-)^2}\right\}.\end{aligned}\tag{12}$$

This is precisely the negative of the corresponding expression for bremsstrahlung (which may be obtained from Eqs. (3) and (4) by taking $X=(x-x_0)^2$ and neglecting terms proportional to f^2). Therefore, the graph of Fig. 2 with the sign of $f(x)$ reversed will apply to the polarization of the plane of the pairs, except that its maximum has not yet been investigated.

Case B: $|x_+-x_-|=cf\cong 0.01$, c of order unity. Writing x for x_+ and integrating (11) without neglecting the term proportional to q^2 yields

$$f(x,c)=\bar{\phi}\frac{p_+p_-dE_+}{2\pi k^3}\frac{d\varphi_+dxdc}{(1+x)^2(c^2+4x)^{\frac{1}{2}}}\frac{1}{f}\times\left\{-\frac{2x[c^2+(1+x)^2]}{(c^2+4x)(1+x)^2}-\frac{k^2x}{E_+E_-}\right\}.\tag{13}$$

[8] See reference 4, p. 196. Heitler's φ_+ is our $\varphi_+-\varphi_-$.

In this expression, the first term is the negative of the corresponding expression for bremsstrahlung. The second term is new, but changes neither the sign nor the order of magnitude of our result.

It must be noted that if we put $\varphi_-=\pi$ in $d\sigma_{\|}$ and $\varphi_-=3\pi/2$ in $d\sigma_\perp$, it turns out that $d\sigma_\perp>d\sigma_{\|}$ unless $E_+\sin\theta_+=E_-\sin\theta_-$. This result has been investigated before[1,9] and means that, if the directions of the initial quantum and of the pair which it produces are in precisely the same plane, the cross section is larger if that plane is normal to the polarization vector than if it contains that vector. The integrated result, on the other hand, shows that the emission of either particle in the plane of polarization is favored over its emission in the normal plane.

Using the word polarization to denote preferred azimuth of a plane in general, it is interesting to note that the opposite signs displayed by the polarizations of the photon in bremsstrahlung and of either final particle in pair production cannot be accounted for by the obvious differences in sign between Eqs. (1) and (11). They must be ascribed to differences in the form of $\mathbf{q}$, the momentum given up to the nucleus, between the two processes. In the case of bremsstrahlung, as noted before, under the high energy, small angle approximation

$$q^2=q^2_{\min}+x_0+x-2(x_0x)^{\frac{1}{2}}\cos\varphi \qquad (14)$$

for both $d\sigma_{\|}$ and $d\sigma_\perp$. In the case of pair production, on the other hand,

$$\begin{aligned}&\text{for } d\sigma_{\|},\ q^2=q_{\|}^2=q^2_{\min}+x_++x_-+2(x_+x_-)^{\frac{1}{2}}\cos\varphi_-,\\ &\text{for } d\sigma_\perp,\ q^2=q_\perp^2=q^2_{\min}+x_++x_-+2(x_+x_-)^{\frac{1}{2}}\sin\varphi_-.\end{aligned} \qquad (15)$$

It is this difference between $q_{\|}^2$ and $q_\perp^2$ which accounts for the fact that $d\sigma_{\|}>d\sigma_\perp$ in pair production. For the anomalous Berlin-Madansky case mentioned above, where in general $d\sigma_{\|}>d\sigma_\perp$, we also have

$$q_{\|}^2=q_\perp^2=q^2_{\min}+(x_+^{\frac{1}{2}}-x_-^{\frac{1}{2}})^2. \qquad (16)$$

Therefore, the sign of the polarization is not a direct consequence of the sign of the energies belonging to the electron states involved, but rather stems from the formulation of the law of conservation of momentum and from the definitions of $d\sigma_{\|}$ and of $d\sigma_\perp$ for the two processes.

III. EFFECT OF MULTIPLE SCATTERING

Any measurement of the angular distribution of bremsstrahlung must be corrected for the deviation suffered by the electron in the course of multiple scattering prior to radiation. For the usual target of two mils thickness, this deviation is several times larger than the angle of maximum polarization. This may be seen from any of the several formulas[10] connecting the

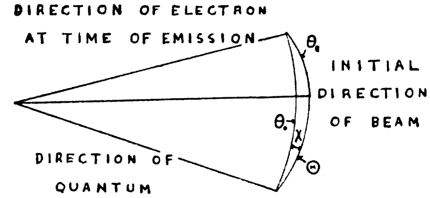

FIG. 5. Relationships between angles involved in Eq. (19).

target thickness t with the average angle of deviation in space $\bar{\theta}_e$. Since we are only interested in the small angle scattering, we shall use Williams' original[11] mean square angle in space computed for the gaussian distribution which holds closely for small angles:

$$\langle\theta_e^2\rangle_{Av}=2k\log(\phi_{\max}/\phi_{\min});\ k=(4\pi NtZ^2e^4)/(m^2p_0^2), \qquad (17)$$

where N = number of atoms per cc and the ϕ's are single scattering deviations projected on a fixed plane. $\phi_{\min}=Z^{\frac{1}{3}}/137E_0$ is determined by screening. To fit the actual gaussian distribution, we choose for $\phi_{\max}$ the angle such that the electron will suffer on the average one single scattering deviation through $\phi_{\max}$ or a larger angle while going through the target:

$$\int_{\phi_{\max}}^{\pi}P(\phi)d\phi=1.$$

Taking $P(\phi)d\phi=kd\phi/\phi^3$, the Rutherford distribution for small angles, gives $\phi_{\max}=(k/2)^{\frac{1}{2}}$. The finite size of the nucleus does not introduce an earlier cutoff for the thicknesses considered. For Pt, expressing t in radiation lengths,[12] Eq. (17) becomes

$$\langle\theta_e^2\rangle_{Av}=(229t/E_0)\log(243t^{\frac{1}{2}})\quad(p_0\cong E_0). \qquad (17')$$

At $t=10^{-4}$ (about the thinnest targets made),[13] the mean projected deviation due to multiple scattering is about equal to $\phi_{\max}$. That thickness may thus be considered the lower limit of the range of validity of the gaussian distribution in θ_e, $f(\theta_e, t)$.

We proceed to calculate the cross sections as functions of ϵ, t, and Θ, the angle which the emitted quantum makes with the initial direction of the electron beam by averaging over θ_e:[14]

$$d\sigma(\epsilon, t, \Theta)=\int d\Omega_e f(\theta_e, t)d\sigma(\epsilon, \theta_0). \qquad (18)$$

The relations between θ_0, θ_e, and Θ are shown in Fig. 5. Clearly,

$$d\Omega_e=\sin\theta_0 d\theta_0 d\chi,\quad \theta_e^2=\theta_0^2+\Theta^2-2\theta_0\Theta\cos\chi. \qquad (19)$$

[9] T. H. Berlin and L. Madansky, Phys. Rev. **78**, 623 (1950).

[10] E. J. Williams, Phys. Rev. **58**, 302 (1940), Eq. (3). S. Goudsmit and J. L. Saunderson, Phys. Rev. **58**, 39 (1940), Eq. (18).

[11] E. J. Williams, Proc. Roy. Soc. (London) **A167**, 545 (1939). Williams gives the projected mean square angle of deviation, or $\frac{1}{2}\bar{\theta}_e^2$.

[12] B. Rossi and K. Greisen, Revs. Modern Phys. **13**, 262 (1941).

[13] I am indebted to Professor W. K. H. Panofsky for this and other experimental information.

[14] I am indebted to Professor G. C. Wick for the bulk of this development.

TABLE I. Calculated values of the polarization.

v	β	Percent polarization [Eq. (25)] Formula	Value for $\epsilon=\frac{1}{2}$
1	$\frac{1}{2}$	$\frac{0.079}{1.4\varphi(\epsilon)-1}$	3%
1	1	$\frac{0.26}{1.4\varphi(\epsilon)-1}$	10%
1	5	$\frac{0.78}{1.1\varphi(\epsilon)-1}$	42%
2	$\frac{1}{2}$	$\frac{0.34}{1.3\varphi(\epsilon)-1}$	14%
2	1	$\frac{0.62}{1.3\varphi(\epsilon)-1}$	28%

In what follows, we neglect energy loss caused by ionization. At $E_0=300$, $Z=78$, $(dE/dx)_{\mathrm{rad}}/(dE/dx)_{\mathrm{ion}} \cong 150Z/800 \cong 15$ roughly.[15] We take for the distribution function[16]

$$f(\theta_e, t)=(\alpha/\pi)\exp(-\alpha\theta_e^2), \quad 1/\alpha=\bar{\theta}_e^2. \tag{20}$$

For the cross sections, we shall use only the log terms in (5) and (6), and we shall further consider the argument of the logarithm to be constant. This is equivalent to using the results obtained by the method of virtual quanta. Writing $x_0=E_0^2\theta_0^2$, we have in terms of the angle χ measured from the plane containing the initial electron beam and the emitted quantum

$$d\sigma_{\perp}=\frac{\bar{\phi}}{\pi}\frac{d\epsilon}{\epsilon}\frac{E_0^2\theta_0 d\theta_0 d\chi}{(1+E_0^2\theta_0^2)^2}\log\frac{137}{Z^{\frac{1}{3}}}\left\{1+(1-\epsilon)^2 -8(1-\epsilon)\frac{E_0^2\theta_0^2}{(1+E_0^2\theta_0^2)^2}\sin^2\chi\right\} \tag{21}$$

$$d\sigma_{\parallel}=\frac{\bar{\phi}}{\pi}\frac{d\epsilon}{\epsilon}\frac{E_0^2\theta_0 d\theta_0 d\chi}{(1+E_0^2\theta_0^2)^2}\log\frac{137}{Z^{\frac{1}{3}}}\left\{1+(1-\epsilon)^2 -8(1-\epsilon)\frac{E_0^2\theta_0^2}{(1+E_0^2\theta_0^2)^2}\cos^2\chi\right\}. \tag{22}$$

Introducing (19), (20), (21), (22) into (18) and integrating over χ gives

$$d\sigma_{\perp}=\frac{2\bar{\phi}}{\pi}\frac{d\epsilon}{\epsilon}\log\frac{137}{Z^{\frac{1}{3}}}\beta E_0^2 e^{-\beta v^2}du\frac{ue^{-\beta u^2}}{(1+u^2)^2}\times\left\{[1+(1-\epsilon)^2]I_0-\frac{8(1-\epsilon)u}{(1+u^2)^2}\frac{I_1}{2\beta vu}\right\}, \tag{23}$$

$$d\sigma_{\parallel}=\frac{2\bar{\phi}}{\pi}\frac{d\epsilon}{\epsilon}\log\frac{137}{Z^{\frac{1}{3}}}\beta E_0^2 e^{-\beta v^2}du\frac{ue^{-\beta u^2}}{(1+u^2)^2}\times\left\{[1+(1-\epsilon)^2]I_0-\frac{8(1-\epsilon)u}{(1+u^2)^2}\left(I_0-\frac{I_1}{2\beta vu}\right)\right\}, \tag{24}$$

$$PP=(d\sigma_{\perp}-d\sigma_{\parallel})/(d\sigma_{\perp}+d\sigma_{\parallel}) =\frac{\int_0^\infty du\frac{u^3}{(1+u^2)^4}e^{-\beta u^2}\left[I_0(2\beta vu)-\frac{I_1(2\beta vu)}{\beta vu}\right]}{\int_0^\infty du\left[\frac{\varphi(\epsilon)}{4}\frac{u}{(1+u^2)^2}-\frac{u^3}{(1+u^2)^4}\right]e^{-\beta u^2}I_0(2\beta vu)}, \tag{25}$$

$$\varphi(\epsilon)=[1+(1-\epsilon)^2]/(1-\epsilon),$$

where $u=E_0\theta_0$, $v=E_0\Theta$ and $\beta=\alpha/E_0^2=1/(E_0^2\bar{\theta}_e^2)$. The functions $I_0=I_0(2\beta vu)$ and $I_1=I_1(2\beta vu)$ are the Bessel functions of imaginary argument.[17]

It is the value of the parameter β which will determine the feasibility of measurement. If $\beta\ll 1$, the angular dependence of the polarization will be obliterated by the uncertainty in the calculated mean deviation due to multiple scattering. For $t=\frac{1}{2}\times 10^{-3}$, to the approximation of Eq. (17′), $\beta=5$. However, the thinner the targets, the more difficult it will be to separate the beam to be observed from the background radiation due to collimators, etc. . . . For this reason, it may also be useful to compute the polarization for smaller values of β. Thus, we have integrated expression (25) numerically for $\beta=\frac{1}{2}$, 1, and 5, corresponding to Pt targets 0.38, 0.22, and 0.058 mils thick, respectively. The results are shown in Table I. It may be noted that, since $e^{-Z}I_0(z)$ and $e^{-Z}I_1(z)$ are very slowly varying functions of their arguments, the cross sections decrease rapidly with increasing u once past a maximum of order unity.

I wish to express my sincere gratitude to Professor G. C. Wick, whose idea this investigation was, for a number of illuminating discussions. I should also like to thank Professor R. Serber, who read this paper and made several suggestions, and Professor W. K. H. Panofsky for discussions concerning possible experimental verification.

[15] E. Fermi, *Nuclear Physics* (University of Chicago Press, Chicago, 1950), p. 47.

[16] See reference 12, p. 267, Eq. (1.63). This formula is the distribution function for the projection of the angle of deviation on a fixed plane, say θ_x, and must be multiplied by a similar distribution function for θ_y to give (20).

[17] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Macmillan Company, New York, 1944), second edition, p. 79.