The Radiation from an Electron Moving in a Uniform Magnetic Field

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The radiation from an electron moving in a uniform magnetic field is investigated quantum mechanically. Reasons are given for expecting deviations from the classical calculations at electron energies of about 100 Mev in the presence of a magnetic field of 10^4 gauss. The quantum-mechanical calculation is carried through and is compared with the classical calculation. Although the deviations are considerable, it is explained why the experiments of Elder, Langmuir, and Pollock of the G. E. synchrotron would not detect them.

I. INTRODUCTION

THE problem of the radiation of a relativistic electron moving in a uniform magnetic field has been treated classically.[1,2] The frequency spectrum of the radiation emitted contains only harmonics of the frequency of rotation of the electron. For a nonrelativistic electron, most of the radiation will be in the first harmonic, but for a relativistic particle it was pointed out by Schwinger that most of the radiation occurs in the higher harmonics.

For the sake of definiteness, we shall keep in mind during our calculations the specific case of a 200 mc^2 or 102-Mev electron moving in a magnetic field of 10^4 gauss. For these numbers the angular frequency of rotation of the electron is $\omega=8.8\times10^8$/sec. According to Schwinger, most of the radiation will occur near the harmonic $\lambda_c=(E/mc^2)^3$ and thus near the angular frequency $K_c=(E/mc^2)^3\omega$, where E is the energy of the electron and m its mass. In our case $\lambda_c=8\times10^6$ is the critical harmonic and $K_c=7.04\times10^{15}$/sec corresponding to a wavelength of 2680A.

The question now arises whether the classical calculation is correct for a high energy electron. The most evident criterion, and the one pointed out by Schwinger, is that the energy emitted of frequency K should be small compared to the energy of the electron, that is $\hbar K\ll E$. This may be interpreted that if $\hbar K\ll E$, then the particle-like or quantum properties of light may be disregarded and classical electrodynamics will succeed. If we put for K the critical frequency K_c, our criterion for the classical calculation to be correct over the major portion of the radiation spectrum becomes $E\ll mc^2(mc^2/\hbar\omega_0)$, $\omega_0=eH/mc$. For a magnetic field H of 10^4 gauss, this gives $E\ll2.34\times10^9$ Mev so that we seem quite safe at $E=200\ mc^2$.

However, a more stringent criterion arises from the following considerations. Classically, the initial state of the electron is taken to be a sharply defined orbit, and its motion is maintained by some external agent. Ignoring the reaction of the radiation on the electron is not important as the electron radiates very little of its energy; but quantum mechanically the initial space position of the electron cannot be so sharply defined. In the quantum-mechanical state in which the energy of the electron is well defined, this state being given by the eigenfunctions of the hamiltonian of this problem, we find that the orbit of the electron will be smeared out over a small distance $b=(\hbar/m\omega_0)^{\frac{1}{2}}$. This distance b depends only on the magnetic field H. For $H=10^4$ gauss, $b=2.58\times10^{-6}$ cm. If the wavelength of the emitted photon is large compared with b, then the smearing out of the orbit is not noticeable, and we should get the classical result; but if the wavelength of the radiation becomes small compared with b, then the uncertainty in the orbit will cause deviations from the classical result.

Our criterion for the correctness of the classical result is then that the wavelength of the radiation be large compared to the smearing out of the orbit, b; that is that $Kb/c\ll1$. If we put for K, the critical frequency K_c, then our criterion becomes $E\ll mc^2(mc^2/\hbar\omega_0)^{\frac{1}{2}}$, which we see is more severe than the previous criterion. For $H=10^4$ gauss, our criterion says $E\ll220$ Mev and indicates that deviations from the classical results should be obtained at $E=200\ mc^2$.

II. THE QUANTUM-MECHANICAL CALCULATION

We propose then, since we expect deviations from the classical results, to calculate the radiation from an electron moving in a uniform magnetic field H quantum mechanically. Our experimental situation is such that initially we have our electrons in a state in which the energy is well known, and the orbit is fairly well defined with a fairly well-defined radius. A little examination of the problem will show that we have only to follow the well-known pattern of the usual atomic radiation problem. We shall start out in an initial state which is an eigenfunction of our hamiltonian for a particle in a uniform magnetic field, which thus has a definite energy and which is associated (in a manner we will indicate later) with a definite orbit-radius. We shall then calculate the transition probability/time for the electron to emit a photon and to go to a final state of lower energy whose associated orbit-radius will be correspondingly smaller.

The initial and final states of the electron are eigenfunctions of the Dirac hamiltonian,

$$\mathcal{H}=\boldsymbol{\alpha}\cdot\boldsymbol{\Pi}+\beta m, \tag{1}$$

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[1] J. Schwinger, Phys. Rev. **75**, 1912 (1949).
[2] G. A. Schott, *Electromagnetic Radiation* (Cambridge University Press, Cambridge, England, 1912), pp. 109, 110.

where $\mathbf{\Pi}=\mathbf{p}+e\mathbf{A}$, $\mathbf{A}$ is the vector potential for a uniform magnetic field H in the z direction; $A_x=-\frac{1}{2}Hy$, $A_y=\frac{1}{2}Hx$, $A_z=0$. Also we have put $\hbar=c=1$.

These four-component Dirac eigenfunctions may be written down simply in terms of the eigenfunctions of the corresponding nonrelativistic Schroedinger equation. The nonrelativistic eigenfunctions of an electron in a uniform magnetic field H may be classified according to the following three properties;[3] the orbital energy of the electron given by the quantum number n, the z-component of the momentum p_z, and the position of the center of the orbit given by the quantum number l. We may write these eigenfunctions as[3,4]

$$\psi_{nlp_3}(\rho\varphi z)=u_{nl}(\rho,\ \varphi)e^{ip_3z}/L^{\frac{1}{2}} \tag{2}$$

and

$$u_{nl}(\rho,\ \varphi)=(l!/2\pi n!)^{\frac{1}{2}}(1/b)e^{-t/2}t^{\nu/2}L_l^{\nu}(t)\cdot e^{i\nu\varphi}, \tag{3}$$

where $t=\rho^2/2b^2$, $b^2=\hbar c/eH=\hbar/m\omega_0$, and $\nu=n-l$ gives the z-component of the angular momentum. $L_l^{\nu}(t)$ is the associated Laguerre polynomial[5] and $L^{-\frac{1}{2}}$ a box-normalization length.

As a nonrelativistic eigenfunction (2) represents an orbit with the energy $E_{np_3}=E_n+p_3^2/2m$, $E_n=\hbar\omega_0(n+\frac{1}{2})$, where $\omega_0=eH/mc$ is the nonrelativistic frequency of rotation whose radius R_n is given by the classical relationship $\frac{1}{2}\omega_0R_n^2=E_n$ or $R_n=b(2n+1)^{\frac{1}{2}}$. The l quantum number gives the location of the center of the orbit. Let x_0, y_0 be the coordinates of the orbit center, and let $\bar{R}=(x_0^2+y_0^2)^{\frac{1}{2}}$, then the center lies in the circle $\bar{R}_l=b(2l+1)^{\frac{1}{2}}$.

We can now write the Dirac eigenfunctions of Eq. (1) in terms of the foregoing nonrelativistic eigenfunction.[3] Let X_s be a spinor which is the simultaneous eigenfunction of σ_z and of β, $\sigma_zX_s=sX_s$, $s=\pm1$, and $\beta X_s=X_s$. Then the relativistic eigenfunctions for a state A, which is given by the quantum numbers nlp_3 and a spin index s, may be written

$$\phi_A=[2E_A(E_A+m)]^{-\frac{1}{2}}(E_A+\boldsymbol{\alpha}\cdot\mathbf{\Pi}+\beta m)\times X_s\psi_{nlp_3}(\rho,\ \varphi,\ z) \tag{4}$$

and corresponds to the energy

$$E_A=E_{np_3s}=\{p_3^2+m^2+m\omega_0(2n+s+1)\}^{\frac{1}{2}}. \tag{5}$$

Either sign of the root in Eq. (5) is possible.

In the initial state of our problem we have $p_3=0$; and if we neglect the spin term s and the mass of the electron M, the quantum number n of the state is given by $n=(E/m)^2(m/\omega_0)$. For a 200 mc^2 electron in a field of 10^4 gauss, we have $n=8.85\times10^{13}$.

The radius of the orbit is given by $R_n=b(2n+1)^{\frac{1}{2}}$ as determined by the radial dependence of the wave function, since the radius operator is no longer diagonal as in the relativistic case. One can also see that the classical relation for a highly relativistic particle $HeR_n=E_n$ is satisfied. The position of the orbit center is determined by l as before.

Our interaction with the radiation field is given by

$$\mathcal{H}'=e\boldsymbol{\alpha}\cdot\mathbf{A}_{\text{rad}}, \tag{6}$$

where $\mathbf{A}_{\text{rad}}$ is the vector potential of the radiation field which permits the creation and destruction of photons. For our problem we need to calculate the matrix element of H' from an initial state in which the electron has energy E_a, a certain radius R_n, $p_3=0$ and $l=0$ (so that the orbit-center is well located), to a final state with electron energy E_f, a smaller radius $R_{n'}$, the Z momentum $p_{3'}$, the orbit center given by l' and a photon emitted with momentum $\mathbf{K}$ and polarization vector $\boldsymbol{\varepsilon}_{\mathbf{K}\lambda}$. The direction of the photon makes an angle Θ with the Z-axis, and we will put the azimuth angle equal to zero for convenience. As pointed out in the introduction, the energy of the photon emitted is much smaller than the energy of the electron by a factor of about 10^7 in the major portion of the spectrum, so we will put $E_a=E_f$ in the following and regard the ratio K/E_a as very small. Our matrix element between the states ϕ_f and ϕ_a, is then

$$\mathcal{H}_{FA}'=e\int\phi_f^*\boldsymbol{\alpha}\cdot\mathbf{A}_{\mathbf{K}\lambda}\phi_a d\tau, \tag{7}$$

where

$$\mathbf{A}_{\mathbf{K}\lambda}=(2\pi/KL^3)^{\frac{1}{2}}e^{-i\mathbf{K}\cdot\mathbf{r}}\boldsymbol{\varepsilon}_{\mathbf{K}\lambda}$$

or, putting in Eq. (4)

$$\mathcal{H}_{FA}'=[e/2E_a(E_a+m)]\int d\tau\ \psi_{n'l'p_3'}X_{s'}^*\times(E_f+\boldsymbol{\alpha}\cdot\boldsymbol{\pi}+\beta m)\boldsymbol{\alpha}\cdot\mathbf{A}_{\mathbf{K}\lambda}(E_a+\boldsymbol{\alpha}\cdot\boldsymbol{\pi}+\beta m)X_s\psi_{n00}. \tag{8}$$

The spinor product can be greatly simplified, for

$$\begin{aligned}&X_{s'}^*(E_a+\boldsymbol{\alpha}\cdot\mathbf{\Pi}+\beta m)\boldsymbol{\alpha}\cdot\mathbf{A}_{\mathbf{K}\lambda}(E_a+\boldsymbol{\alpha}\cdot\mathbf{\Pi}+\beta m)X_s\\&\quad=X_{s'}^*(E_a+m)(\boldsymbol{\alpha}\cdot\mathbf{A}_{\mathbf{K}\lambda}\boldsymbol{\alpha}\cdot\mathbf{\Pi}+\boldsymbol{\alpha}\cdot\mathbf{\Pi}\boldsymbol{\alpha}\cdot\mathbf{A}_{\mathbf{K}\lambda})\\&\quad=(E_a+m)X_{s'}^*(2\mathbf{A}_{\mathbf{K}\lambda}\cdot\mathbf{\Pi}+\boldsymbol{\sigma}\cdot\text{curl}\mathbf{A}_{\mathbf{K}\lambda})X_s,\end{aligned} \tag{9}$$

using the results that $X_{s'}^*\boldsymbol{\alpha}X_s=0$, $\beta X_s=X_s$, and the rule $(\boldsymbol{\alpha}\cdot\mathbf{A})(\boldsymbol{\alpha}\cdot\mathbf{B})=\mathbf{A}\cdot\mathbf{B}+i\boldsymbol{\sigma}\cdot(\mathbf{A}\times\mathbf{B})$ where $\mathbf{A}$ and $\mathbf{B}$ commute with $\boldsymbol{\alpha}$.

Our matrix element reads now

$$\mathcal{H}_{FA}'=\frac{m}{E_a}\int d\tau\psi_{n'l'p_3'}^*\times\left\{\frac{e}{m}\mathbf{A}_{\mathbf{K}\lambda}\cdot\boldsymbol{\pi}+\frac{e}{2m}\boldsymbol{\sigma}\cdot\text{curl}\mathbf{A}_{\mathbf{K}\lambda}\right\}\psi_{n00}X_s. \tag{10}$$

This last result, apart from the factor m/E_a, is exactly what we would have written down if we had described the electron by means of the nonrelativistic Pauli spin theory. Indeed, both the Schroedinger and Dirac equations give identical results except that the frequency of rotation of the electron is $\omega_0=eH/mc$

[3] M. H. Johnson and B. A. Lippman, Phys. Rev. 77, 702 (1950).
[4] L. Page, Phys. Rev. 36, 444 (1930).
[5] Magnus and Oberhettinger, *Special Functions of Mathematical Physics* (Chelsea Publishing Company, New York, 1949), p. 84.

nonrelativistically, while it is $\omega=\omega_0(m/E_a)$ relativistically. This is true only when the energy carried off by the photon is small and is indicated by the classical calculation in which the motion of the electron is described neither relativistically nor nonrelativistically, but is simply maintained by an external agent.

To simplify further our matrix element, we make use of some properties of the $\mathbf{\Pi}$ operator. First, $\Pi_z\psi_{n00}=p_z$, $\psi_{n00}=0$, and $\Pi_\pm=\Pi_x\pm i\Pi_y$ have the properties of annihilation operators,

$$\Pi_+u_{nl}=im\omega_0 b2^{\frac{1}{2}} \quad (n+1)^{\frac{1}{2}} \quad u_{n+1,\,l}, \tag{11a}$$

$$\Pi_-u_{nl}=-im\omega_0 b2^{\frac{1}{2}} \quad n^{\frac{1}{2}} \quad u_{n-1,\,l}. \tag{11b}$$

So we write $\epsilon_{K\lambda}\cdot\mathbf{\Pi}=\frac{1}{2}\{\epsilon_{K\lambda}{}^-\Pi_+ + \epsilon_{K\lambda}{}^+\cdot\Pi_-\}$, where $\epsilon_{K\lambda}{}^\pm=\epsilon_{K\lambda}{}^x\pm i\epsilon_{K\lambda}{}^y$, and we can now write our matrix element as

$$\mathcal{H}_{FA}{}'=\frac{m}{E_a}\frac{e}{m}\left(\frac{\pi}{KL^3}\right)^{\frac{1}{2}} m\omega_0 b(n+1)^{\frac{1}{2}} \times\left\{\epsilon_{K\lambda}{}^- I(n',l'\,|\,n+1,0) +\left(\frac{n}{n+1}\right)^{\frac{1}{2}}\epsilon_{K\lambda}{}^+ I(n',l'\,|\,n-1,0)\right\}, \tag{12}$$

where

$$I(n',l'\,|\,nl)=\int u_{n'l'}{}^*\exp(-iK_\perp\rho\cos\varphi)u_{nl}\rho d\rho d\varphi, \tag{13}$$

$K_\perp=K\sin\Theta$ and we have dropped the spin term in Eq. (10), since it is smaller than the orbital term by the factor K/E_a.

Since the energy of the final state does not depend on l', to get the probability for the emission of a photon with a certain energy K, we will have to sum over all values of l'; that is, there are many final states with the same energy differing only in the location of the orbit centers. However, we will show in the Appendix that if the quantum numbers n and n' are very large, then as long as the momentum of the emitted photon is small compared with that of the electron, only the $l'=0$ transition is significant. If the photon carried away appreciable momentum, the recoil of the electron together with the action of the external magnetic field would cause the orbit center to shift and the transitions to higher l' would become important. Since we are interested in the low momentum photons, we will put $l'=0$ in Eq. (13) and write $I(n',0\,|\,n,0)=I_{n'n}$.

To calculate the transition probability per unit time for our process, we will need the density of final states $\rho(E_F)$. This is given simply by the density of the photon state, so that

$$\rho(E_F)dE_F=\rho(K)dK, \tag{14}$$

where

$$\rho(K)=(L/2\pi)^3K^2d\Omega,$$

$d\Omega$ is the solid angle in which the photon is emitted. Now as

$$E_F=\{p_3{}'^2+m^2+m\omega_0 2n'\}^{\frac{1}{2}}+K, \tag{15}$$

and $p_3'=K\cos\Theta$, then it follows that

$$(2E_F/2K)_\Theta=(K/E_f)\cos^2\Theta+1: \tag{16}$$

therefore, we have

$$\rho(E_F)=(L/2\pi)^3K^2d\Omega/[1+(\cos^2\Theta)K/E_f] \tag{17}$$

or

$$\rho(E_F)=(L/2\pi)^3K^2d\Omega,$$

since $K/E_f\ll 1$.

Let us now specify the polarization vector as being the unit vectors in the Θ- and Φ-directions. $\epsilon_{K1}=\epsilon_\Theta$, $\epsilon_{K2}=\epsilon_\Phi$. Then we have $\epsilon_\Theta{}^\pm=\cos\Theta i0$ and $\epsilon_\Phi{}^\pm=\pm i$.

Let $w_\Theta(\Theta,K)$ and $w_\Phi(\Theta,K)$ be the transition probabilities/sec for emitting the photon $\mathbf{K}$ with polarizations in the Θ- and Φ-directions, respectively. Then we find

$$w_\Theta(\Theta,K)=\frac{d\Omega}{4\pi}\cdot\frac{e^2}{\hbar c}\cdot\left[\frac{\omega b}{c}(n+1)^{\frac{1}{2}}\right]^2\frac{K}{\hbar\omega}\omega\cos^2\Theta \times\{I_{n',n+1}-I_{n',n-1}\}^2, \tag{18a}$$

$$w_\Phi(\Theta,K)=\frac{d\Omega}{4\pi}\cdot\frac{e^2}{\hbar c}\cdot\left[\frac{\omega b}{c}(n+1)^{\frac{1}{2}}\right]^2\frac{K}{\hbar\omega}\omega \times\{I_{n',n+1}+I_{n',n-1}\}^2, \tag{18b}$$

and the energy radiated/sec with Θ- or Φ-polarization is given by

$$P_\Theta(\Theta,K)=Kw_\Theta(\Theta,K) \tag{19a}$$

and

$$P_\Phi(\Theta,K)=Kw_\Phi(\Theta,K). \tag{19b}$$

It should be noted that only photons whose frequencies are harmonics of the frequency of rotation, ω, are emitted. By conservation of energy we have for the energy of the photon emitted

$$\begin{aligned}K&=E_a-E_f\\&=(m^2+2n\omega_0 m)^{\frac{1}{2}}-(m^2+2n'\omega_0 m+p_3{}'^2)^{\frac{1}{2}}\\&\cong(n-n')\hbar\omega-p_3'p_3'/E_a\\&\cong(n-n')\hbar\omega,\end{aligned} \tag{20}$$

where $\omega=\omega_0 m/E_a$ is the frequency of rotation of the electron and we have assumed that $K\ll E_a$ and thus $p_3'/E_a\ll 1$. So in the transition to the n' state, the harmonic emitted is $\lambda=n-n'$.

Our only task left now is to compute the integrals $I_{n',n+1}$ and $I_{n',n-1}$, which occur in Eq. (18). We will quote the result here and give the derivation in the Appendix;

$$I_{n',n}=(n'!/n!)^{\frac{1}{2}}\exp(-\tfrac{1}{2}\alpha^2)(-\alpha^2/2)^{\lambda/2}L_{n'}{}^\lambda(\alpha^2/2), \tag{21}$$

where $\alpha=Kb\sin\Theta=\lambda\omega b\sin\Theta$, $\lambda=n-n'$, is the harmonic radiated, and $L_n{}^\lambda$ is again an associated Laguerre

polynomial. The quantity α involves the ratio of the smearing out of the orbit, b, to the wavelength of the photon emitted, so that according to the argument presented in the introduction, the limit $\alpha \ll 1$ should give the classical results.

Now Eq. (21) involves a Laguerre polynomial of very high order and it can be expressed in terms of bessel functions. We have the following result for Laguerre polynomicals of very high order (proof in the Appendix).

$$L_n^{\lambda}(t)=\frac{(n+\lambda)!}{n!n^{\lambda}}\left(\frac{t}{n}\right)^{-\frac{1}{4}\lambda}\exp\left[-\lambda\left(\frac{t}{n}\right)^{\frac{1}{2}}+t\right]\times J_{\lambda}(2(nt)^{\frac{1}{2}}) \quad (22)$$

if both n and λ are very large and $(t/n)^{\frac{1}{2}}$ is sufficiently small. If we let $\delta=2(nt)^{\frac{1}{2}}/\lambda$ and $\gamma=(1-\delta^2)^{-\frac{1}{4}}$, then we should have $(t/n')^{\frac{1}{2}}\ll 1/\gamma^2$.

Substituting Eq. (22) into Eq. (21), we get

$$I_{n',n}=(-1)^{\lambda/2}\exp[-\lambda\alpha/(2n')^{\frac{1}{2}}]J_{\lambda}(\alpha(2n')^{\frac{1}{2}}), \quad (23)$$

where we used $[(n+\lambda)!/n!]^{\frac{1}{2}}\cong n^{\lambda/2}$, if n is large and $\lambda\ll n$.

We may eliminate n' from Eq. (23) in terms of more physical parameters. The radius of the orbit is given by $R=b(2n)^{\frac{1}{2}}$. So

$$\begin{aligned}\alpha(2n)^{\frac{1}{2}}&=\lambda\omega b\ (\sin\Theta)R/b\\&=\lambda\beta\sin\Theta,\end{aligned} \quad (24)$$

where β is the velocity of the electron. Also we have $\lambda/(2n)^{\frac{1}{2}}=\alpha/\beta\sin\Theta$. We put these results in Eq. (23) and get

$$I_{n',n}=(-1)^{\lambda/2}\exp(-\alpha^2/\beta\sin\Theta)J_{\lambda}(\lambda\beta\sin\Theta). \quad (25)$$

We put Eq. (25) in Eqs. (18) and (19) and get our final result for the energy radiated with the different polarizations:

$$P_{\Theta}(\Theta,K)=\frac{d\Omega}{2\pi}\frac{e^2}{k}\beta^3\lambda^2\omega\exp\left(-\frac{2\alpha^2}{\beta\sin\Theta}\right)\cos^2\Theta\times\left[\frac{J_{\lambda}(\lambda\beta\sin\Theta)}{\beta\sin\Theta}\right]^2, \quad (26a)$$

$$P_{\Phi}(\Theta,K)=\frac{d\Omega}{2\pi}\frac{e^2}{k}\beta^3\lambda^2\omega\exp\left(-\frac{2\alpha^2}{\beta\sin\Theta}\right)\times[J_{\lambda}'(\lambda\beta\sin\Theta)]^2, \quad (26b)$$

where $\alpha=Kb\sin\Theta=\lambda\omega b\sin\Theta/c$ and λ is the harmonic radiated.

If we compare Eq. (26) with the classical result,[1] we see that our quantum-mechanical result differs only in the factor $\exp[-2\alpha^2/\beta\sin\Theta]$. It is clear that for $\alpha\ll 1$, we get the classical result as we expect.

In the result (26), α may not become too large, as in formula (22) we had an upper limit on $(t/n)^{\frac{1}{2}}$. This imposes the limit on α, that

$$\alpha\ll(m/E)(m/\omega_0)^{\frac{1}{2}}, \quad (27)$$

or, in terms of the harmonic emitted, we should have

$$\lambda\ll m/\omega_0. \quad (28)$$

Since the important frequencies occur for $\lambda\sim(E/m)^3$, Eq. (28) yields the criterion for the correctness of our result that

$$E\ll m(m/\omega_0)^{\frac{1}{3}}. \quad (29)$$

For a field of 10^4 gauss, this means $E\ll 1.64\times 10^3$ Mev. This limitation is only a mathematical one owing to the use of the formula (22). In the Appendix the more exact result is given.

III. RESULTS

In Fig. 1 we have plotted the energy radiated per second per unit solid angle and per unit wavelength at $\Theta=\pi/2$, that is, in a direction which lies in the plane of the orbit against the wavelength of the radiation. It should be noted that the quantity plotted is $P_{\Phi}(\pi/2, K)$/wavelength, this being the quantity plotted in the experimental paper of Elder, Langmuir, and Pollock.[6]

We have plotted both the classical curve and the results of our quantum-mechanical calculation for an electron of 200 mc^2 energy moving in a field of 10^4 gauss. The deviations are evidently considerable.

From Fig. 1 we may see why no deviations were detected in the experiment. The wavelength range covered in the experiment was only the visible region from 4000A to 8000A and also only the relative intensity was measured. Figure 1 shows that in this region the classical and quantum-mechanical curves are similar in shape, though there is a considerable separation, and either curve could be made to match the experimental points.

APPENDIX

We will treat here the mathematical problem of calculating the matrix element

$$I(n'l'|n0)=\int u_{n'l'}^{*}\exp(-iK_{\perp}\rho\cos\varphi)u_{n0}\rho d\rho d\varphi \quad (30)$$

$$=\left(\frac{l'!}{n'!n!}\right)^{\frac{1}{2}}\frac{1}{2\pi}\int_0^{\infty}dt\int_0^{2\pi}d\varphi e^{-t}t^{\frac{1}{2}(n+n'-l')}L_{l'}^{n'-l'}(t)\times\exp[i\{(n-n'-l')\varphi-\alpha(2t)^{\frac{1}{2}}\cos\varphi\}], \quad (31)$$

where we substitute into Eq. (30) the eigenfunctions as given by Eq. (3) and $\alpha=Kb\sin\Theta$.

The integration over φ can be done and gives a bessel function, so that

$$I(n'l'|l0)=\left(\frac{l'!}{n'!n!}\right)^{\frac{1}{2}}i^{\lambda+l'}\int_0^{\infty}dte^{-t}t^{\frac{1}{2}(\lambda+l')+n'-l'}\times L_{l'}^{n'-l'}(t)J_{\lambda+l'}(\alpha(2t)^{\frac{1}{2}}), \quad (32)$$

where we put $\lambda=n-n'$.

Now we replace the Laguerre polynomial by its power series expansion

$$L_s^{r}(t)=(r+s)!\sum_{K=0}^{s}\frac{(-t)^K}{K!(s-K)!(r+K)!}, \quad (33)$$

[6] Elder, Langmuir, and Pollock, Phys. Rev. 74, 52 (1948).

and we get

$$I(n'l'|n0)=\left(\frac{l'!n'!}{n!}\right)^{\frac{1}{2}} i^{\lambda+l'} \sum_{K=0}^{l'} \left\{ \frac{(-1)^K}{K!(l'-K)!(K+n'-l')!} \times \int_0^\infty dt e^{-t} t^{\frac{1}{2}(\lambda+l')+n-l'+K} J_{\lambda+l'}(\alpha(2t)^{\frac{1}{2}}) \right\}. \quad (34)$$

For the integral over t we have the result[7]

$$\int_0^\infty dt e^{-t} t^{\frac{1}{2}(\lambda+l')+n'-l'+K} J_{\lambda+l'}(\alpha(2t)^{\frac{1}{2}}) = (n'-l'+K)!(\tfrac{1}{2}\alpha^2)^{\frac{1}{2}(\lambda+l')} \exp(-\tfrac{1}{2}\alpha^2) L_{n'-l'+K}{}^{\lambda+l'}(\alpha^2/2). \quad (35)$$

Thus, we have

$$I(n'l'|l0)=\left(\frac{l'!n'!}{n!}\right)^{\frac{1}{2}} (-\tfrac{1}{2}\alpha^2)^{\frac{1}{2}(\lambda+l')} \exp(-\tfrac{1}{2}\alpha^2) \times \sum_{K=0}^{l'} \frac{(-1)^K}{K!(l'-K)!} L_{n'-l'+K}{}^{\lambda+l'}\left(\frac{\alpha^2}{2}\right). \quad (36)$$

For the important case $l'=0$ we get

$$I_{n',n}=\left(\frac{n'!}{n!}\right)^{\frac{1}{2}} (-\tfrac{1}{2}\alpha^2)^{\frac{1}{2}\lambda} \exp(-\tfrac{1}{2}\alpha^2) L_{n'}{}^{\lambda}\left(\frac{\alpha^2}{2}\right), \quad (37)$$

which is the result quoted in Eq. (21).

For $l'>0$ we wish to show for n, n' very large that $I(n'l'|n0)$ will be smaller than $I_{n',n}$ by a factor of order $1/n$. In the sum over K occurring in Eq. (36), as long as $l'\ll n'$ and $n'=8.85\times10^{13}$ in our case, then one might expect that

$$L_{n'-l+K}{}^{\lambda+l'}=L_{n'}{}^{\lambda+l'}+0(1/n). \quad (38)$$

We will see in a later paragraph where we evaluate $L_{n'}{}^{\lambda}$ in terms of bessel function that this is actually the case. In Eq. (36) the sum over K gives

$$\sum_{K=0}^{l'} \frac{(-1)^K}{K!(l'-K)!} L_{n'-l'+K}{}^{\lambda+l'} = \sum_{K=0}^{l'} \frac{(-1)^K}{K!(l'-K)!} \times \left\{ L_{n'}{}^{\lambda+l'}+0\left(\frac{1}{n'}\right) \right\} = 0+0\left(\frac{1}{n'}\right). \quad (39)$$

And for very high l', $l'\sim n'$, the matrix element is small, since it will decrease as l' becomes larger. Thus, only the $l'=0$ transition is important.

We have now to establish the approximate relation, Eq. (22), used to evaluate the very high order Laguerre polynomial occurring in $I_{n',n}$. To this end we will expand $L_n{}^{\lambda}(t)$ in bessel function:[8]

$$L_n{}^{\lambda}(t)=\frac{(n+\lambda)!}{n!} \frac{e^t}{(nt)^{\lambda/2}} \sum_{m=0}^{\infty} C_m \frac{1}{m!} \left(-\lambda\left(\frac{t}{n}\right)^{\frac{1}{2}}\right)^m J_{\lambda+m}(2(tn)^{\frac{1}{2}}), \quad (40)$$

where, in general,

$$C_m=\left(\frac{-n}{\lambda}\right)^m \sum_{r=0}^{m} \frac{m!}{r!(m-r)!} \frac{(n+\lambda+r)!}{(n+\lambda)!} \left(-\frac{1}{n}\right)^r.$$

But in our case $\lambda\ll n$, and if we further assume that only $m\ll\lambda$ is important, since $\lambda\sim10^6$ in our problem, then $C_m=1$. For then we have $(n+\lambda+r)!/(n+\lambda)!\cong n^r(1+(\lambda/n))^r$, and the sum can be done by the binomial theorem.

In order to sum Eq. (40), we will make use of the fact that the order of the bessel functions involved is very large and thus $J_{m+\lambda}\simeq J_\lambda$ when $m\ll\lambda$. The more exact range of m for which we can factor out the bessel functions from the sum over m in Eq. (4) will be established later. For the moment let us assume this is permissible for $m<\bar{m}$. The sum in Eq. (40) now becomes the known

[7] See reference 5, p. 88.
[8] See reference 5, p. 95.

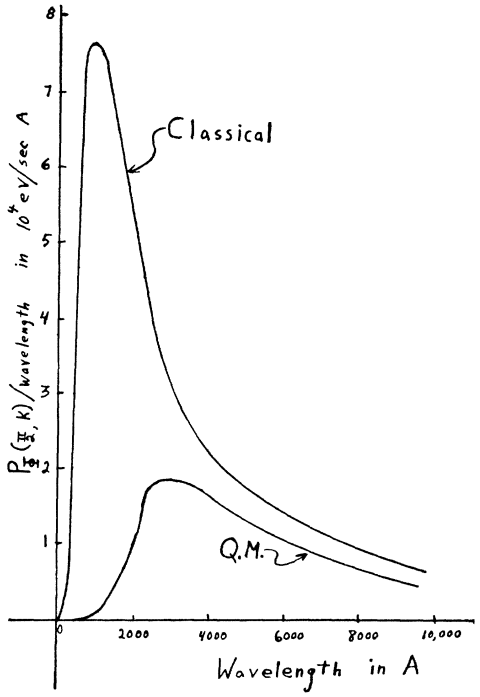

FIG. 1. Plot of the energy radiated per second per solid angle per unit wavelength in a direction which lies in the plane of the orbit against the wavelength of the radiation for an electron of 200 mc^2 energy moving in a magnetic field of 10^4 gauss.

exponential series, and we can write

$$L_n{}^{\lambda}(t)=\frac{(n+\lambda)!}{n!} \frac{1}{(nt)^{\lambda/2}} \exp\left[-\lambda\left(\frac{t}{n}\right)^{\frac{1}{2}}+t\right] J_\lambda(2(tn)^{\frac{1}{2}}), \quad (41)$$

provided we can neglect those terms from which $m>\bar{m}$; and this is so if we have the restriction

$$\lambda(t/n)^{\frac{1}{2}}\ll\bar{m}. \quad (42)$$

Equation (41) is the result we desire except that we have yet to find the restriction under which it holds; that is, we must determine $\bar{m}$ for which we can say $J_{\lambda+m}=J_\lambda$, $\lambda\gg\bar{m}$.

For large order bessel functions we may write $J_\lambda(2(tn)^{\frac{1}{2}})=J_\lambda(\lambda\delta)$, where in our case δ is smaller than, but very close to, one, and[9]

$$J_\lambda(2(tn)^{\frac{1}{2}})=\frac{\epsilon}{\pi 3^{\frac{1}{2}}} K_{1/3}(\tfrac{1}{3}\lambda\epsilon^3), \quad (43)$$

where $\epsilon=(1-\delta^2)^{\frac{1}{2}}\ll1$ in our case. Also

$$J_{\lambda+m}=\frac{\epsilon_m}{\pi 3^{\frac{1}{2}}} K_{1/3}(\tfrac{1}{3}(\lambda+m)\epsilon_m^3), \quad (44)$$

where

$$\epsilon_m=\{1-\delta^2(\lambda/(\lambda+m))^2\}^{\frac{1}{2}}\cong[1-\delta^2(1-2m/\lambda)]^{\frac{1}{2}}.$$

It is easy to see now that if we restrict m so that $m\ll\lambda\epsilon^2$, then we find $\epsilon_m-\epsilon=\epsilon(m/\lambda)\ll\epsilon$. So we put $\bar{m}=\lambda\epsilon^2$, and for $m\ll\bar{m}$, ϵ_m will differ only slightly from ϵ and $J_{\lambda+m}\cong J_\lambda$.

Combining this value for $\bar{m}$ with Eq. (42) we get the restriction for Eq. (41) to be correct:

$$(t/n)^{\frac{1}{2}}\ll\epsilon^2=1/\gamma^2. \quad (43)$$

[9] G. W. Watson, *Bessel Functions* (Macmillan Company, New York, 1945), p. 248.