# On a Phenomenological Neutron-Proton Interaction*†

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This paper describes the calculations and results obtained in fitting a phenomenological interaction to the properties of the deuteron and related low energy phenomena. This interaction is a linear combination of a central and tensor potential, where both potentials are of the Yukawa shape, but with different ranges. The calculations were based on the variation-iteration method which has the advantage of providing systematically improved trial wave functions, together with limits of error for the well depth eigenvalue at every stage of the process. The nonpositive definite character of the tensor potential complicates the nature of the convergence to the correct eigenvalue. Special methods are described to overcome this difficulty. The successive iterations were performed numerically on the Harvard Mark I calculator. The accuracy obtained for the well depth parameter was limited in practice by the finite intervals employed in the numerical integration, and is estimated to be one in $10^4$. The quadrupole moment, as a nonstationary quantity, does not exhibit the same degree of convergence; the accuracy obtained was here estimated as one in $10^3$. The results are presented in tabular form as values of the well depth parameter and of the quadrupole moment for four central potential ranges, five tensor ranges, and three values of the tensor strength. These tables also include values of the fractional amount of $D$ state and of the effective triplet range. The experimental magnitudes of the latter quantities serve to delimit the permissible values of the tensor range. The photoelectric cross section of the deuteron is shown to involve only the triplet effective range in addition to the familiar zero-range formula. The cross section for photomagnetic capture contains the singlet and triplet phenomenological parameters, and in addition, a mixed effective range which also includes the effect of an exchange magnetic moment. The value of the mixed range, as inferred from the experimental capture cross section, agrees with the average of the singlet and triplet effective ranges, within the rather large experimental uncertainties. For the energy domain in which it is appreciable, the photomagnetic cross section is almost uniquely fixed by the capture cross section. The comparison of these cross sections with experiment is satisfactory.

## I. INTRODUCTION

IN the phenomenological theory of nuclear forces, the shape of the nuclear potential first enters in a sensitive fashion in the theory of $H^3$ and in high energy nucleon-nucleon scattering. However, in order to treat these phenomena, it is necessary to take into account the low energy two-body data which serve to relate and limit the ranges and depths of the nuclear potentials but which are not strongly dependent upon well shape. In the present paper, we are particularly interested in the neutron-proton interaction in the triplet state. Calculations have been performed for various potentials to be described subsequently in which the well depths required to yield the experimental binding energy and quadrupole moment of the deuteron have been evaluated for a variety of ranges. The limitation on these ranges provided by other low energy phenomena such as neutron-proton scattering and the photodisintegration of the deuteron is also considered.

The general form of the interaction potential in the triplet state as a function of $\mathbf{r}$, the relative position vector for the neutron-proton system, is

$$V(\mathbf{r})=-[V_c f(r/r_c)+V_t g(r/r_t)S_{12}]. \quad (1)$$

Here $-V_c f(r/r_c)$ is the potential for the central force characterized by a depth $V_c$, a range $r_c$ and shape $f(x)$. The expression $-V_t g(r/r_t)$ is the analogous quantity for the tensor force containing the dependence of the interaction upon the internucleon distance, while

$$S_{12}=[3(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})/r^2]-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 \quad (2)$$

contains the dependence upon the direction of the vector $\mathbf{r}$ relative to the spin vectors $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ of the two particles.

Since our principal interest is in the deuteron, we have not specified the exchange properties of $V(\mathbf{r})$. It should be emphasized that the form of the interaction contained in Eq. (1) omits possible velocity-dependent forces.

In the present paper, the Yukawa potential is employed for both $f$ and $g$:

$$f(x)=g(x)=e^{-x}/x, \quad (3)$$

where $r_c$ and $r_t$ are varied over a considerable range. The Yukawa well is chosen for it appears to be compatible with a symmetrical nuclear hamiltonian.[1] In addition preliminary calculations[2] have indicated that a satisfactory theory of $H^3$ may be obtained.

Similar potentials have been adopted by a number of authors. Rarita and Schwinger[3] placed $f=g$, and used a

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square well with a range of $2.8\times10^{-13}$ cm. Guindon[4] employed the square well shape for both $f$ and $g$ but permitted slightly different values for $r_c$ and $r_t$. Biedenharn,[5] Padfield,[6] and Christian and Hart[7] have also used square well potentials, but have considered wider variations in $r_c$ and $r_t$ in order to fit recent neutron-proton scattering data. Several authors have employed Yukawa wells, including Chew and Goldberger,[8] Hu and Massey,[9] Rarita,[10] and Christian and Hart. Generally, $r_c$ was placed equal to $r_t$ except for one case in which Christian and Hart considered a number of values of $r_t$ for $r_c=1.185\times10^{-13}$ cm. The exponential well with $r_c=r_t$ has been treated by Rarita, and by Hu and Massey. The latter have also employed the gaussian potential. Jauch and Hu,[11] and Wu and Foley,[12] employ the $V(r)$ predicted by a meson mixture theory.

## II. DETERMINATION OF THE FORCE CONSTANTS

The triplet neutron-proton potential (1) contains four constants $V_c$, $V_t$, $r_c$, and $r_t$. Two of these are determined by

$\epsilon=$binding energy of the deuteron[13] $2.23\pm0.007$ Mev,
$Q=$quadrupole moment of the deuteron[14,15] $(2.766\pm0.02)\times10^{-27}$ cm$^2$.

This leaves two free parameters, which we have chosen to be $r_c$ and $r_t$. The central force ranges employed in the present calculations were varied about that given by proton-proton scattering, while $r_t$ is kept larger than $r_c$ in most instances, as is indicated by the theory[2] of $H^3$. Calculations were made for a sufficiently wide range of values of $r_c$ and $r_t$, and the ratio $(V_t/V_c)$, so as to include any possible changes in the experimental data. Computations were performed for four values of $r_c$, five of $r_t$ for each $r_c$, and three values of $(V_t/V_c)$ for each $(r_c, r_t)$ pair, making sixty in all.

We employ the variational-iterational method.[16] This method is particularly valuable in the present problem because it provides (1) a method for estimating the errors in the eigenvalue by giving upper and lower bounds to them, (2) a method for systematically improving the wave functions by iteration, and (3) a method for extrapolation. Since the quadrupole moment is a determining quantity, which depends rather critically upon the wave functions, it was essential to employ a systematic method with the properties already outlined.

The wave function for the deuteron may be written as a linear combination of an $S$ and $D$ state

$$\psi=(1/r)[u(r)+(1/8^{\frac{1}{2}})S_{12}w(r)]\chi_m, \tag{4}$$

where $\chi_m$ is the spin function with magnetic quantum number $m$. The differential equations determining $u$ and $w$ are to be given in terms of the independent variable

$$x=r/r_c. \tag{5}$$

Then, if

$$\eta=(M\epsilon/\hbar^2)^{\frac{1}{2}}r_c, \quad \lambda=MV_cr_c^2/\hbar^2, \quad \tau=r_c/r_t, \quad \gamma=V_t/V_c, \quad \lambda/\eta^2=V_c/\epsilon, \quad \eta/\tau=(M\epsilon/\hbar^2)^{\frac{1}{2}}r_t, \tag{5a}$$

we obtain

$$(d^2u/dx^2)+[-\eta^2+\lambda f(x)]u=-2^{\frac{3}{2}}\gamma\lambda g(\tau x)w, \tag{6a}$$

$$(d^2w/dx^2)-(6w/x^2)+\{-\eta^2+\lambda[f(x)-2\gamma g(\tau x)]\}w=-2^{\frac{3}{2}}\gamma\lambda g(\tau x)u. \tag{6b}$$

The quadrupole moment is

$$Q=(2^{\frac{1}{2}}/10)r_c^2\int_0^\infty x^2[uw-(1/8^{\frac{1}{2}})w^2]dx. \tag{7}$$

To facilitate the solution by iteration, it is convenient to replace Eqs. (6) by equivalent integral equations:

$$u=\lambda\int_0^\infty G_\eta^{(0)}(x,x')[f(x')u(x')+2^{\frac{3}{2}}\gamma g(\tau x')w(x')]dx', \tag{8a}$$

$$w=\lambda\int_0^\infty G_\eta^{(2)}(x,x')\{2^{\frac{3}{2}}\gamma g(\tau x')u(x')+[f(x')-2\gamma g(\tau x')]w(x')\}dx', \tag{8b}$$

where

$$d^2G_\eta^{(l)}(x,x')/dx^2-[(l)(l+1)/x^2]G_\eta^{(l)}(x,x')-\eta^2G_\eta^{(l)}(x,x')=-\delta(x-x'),$$
$$G_\eta^{(l)}(0,x')=0, \quad G_\eta^{(l)}(\infty,x')=0.$$

Hence, we obtain

$$G_\eta^{(0)}(x,x')=(1/\eta)\sinh\eta x_<e^{-\eta x_>},$$

$$G_\eta^{(2)}(x,x')=\left(\frac{1}{-\eta}\right)\left[\left(1+\frac{3}{(\eta x_<)^2}\right)\sinh\eta x_<-\frac{3}{\eta x_<}\cosh\eta x_<\right]\left[\left(1+\frac{3}{\eta x_>}+\frac{3}{(\eta x_>)^2}\right)e^{-\eta x_>}\right]. \tag{9}$$

Here the smaller of $x$ and $x'$ is substituted for $x_<$ and the larger for $x_>$.

At this point it becomes possible to describe the iteration part of the calculation. Starting from an initial

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$Q=(2.738\pm0.016)\times10^{-27}$ cm$^2$.
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TABLE I. Result of several iterations, and comparison with extrapolated $Q$.

| Iteration | $\lambda$ | $Q$ ($10^{-27}$ cm$^2$) |
|---|---|---|
| $\lambda_0^{(0)}$ | 9.5728 | |
| $\lambda_0^{(\frac{1}{2})}$ | 9.5726 | 2.708 |
| $\lambda_0^{(1)}$ | 9.5727 | 2.680 |
| Extrapolated | $9.5726_6$ | 2.689 |
| $\lambda_0^{(\frac{3}{2})}$ | $9.5726_5$ | 2.695 |

pair of trial functions $u_0$ and $w_0$ whose choice will be discussed subsequently, we may obtain a first iterate $u_1$ and $w_1$ by introducing $u_0$ and $w_0$ in the right-hand side of Eq. (8). The factor $\lambda$ is a common proportionality factor, so that it may be absorbed into the definition of $u_1$ and $w_1$. To obtain the second iterate, introduce $u_1$ and $w_1$ in the right-hand side of Eq. (8), and so on for the higher iterates.

The wave functions developed in this manner are then employed as trial functions in a variational principle for $\lambda$. Equations (8) are a pair of integral equations with eigenvalue $\lambda$; i.e., the depth of the potential $V_0$ required to yield the experimental binding energy, as included in the parameter $\eta$, is now the solution of an eigenvalue problem. This characteristic feature of the iterational scheme corresponds to the present state of knowledge in the theory of the deuteron, for the binding energy of the deuteron is known from experiment and one is looking for a corresponding value for the potential depth. The necessary variational principles may be obtained from the original Schrödinger equation. However, it is somewhat simpler to work directly from Eqs. (6) and (8). Let us define a matrix

$$\phi=\begin{pmatrix} u \\ w \end{pmatrix} \tag{10}$$

and matrix operators

$$\mathbf{A}=\begin{pmatrix} -(d^2/dx^2)+\eta^2 & 0 \\ 0 & -(d^2/dx^2)+(6/x^2)+\eta^2 \end{pmatrix}, \tag{11}$$

$$\mathbf{B}=\begin{pmatrix} f(x) & 2^{\frac{3}{2}}\gamma g(\tau x) \\ 2^{\frac{3}{2}}\gamma g(\tau x) & f(x)-2\gamma g(\tau x) \end{pmatrix}. \tag{12}$$

Then Eqs. (6) and (8) may be written

$$\mathbf{A}\phi=\lambda\mathbf{B}\phi, \tag{13}$$

$$\phi=\lambda\mathbf{A}^{-1}\mathbf{B}\phi. \tag{14}$$

TABLE II. Result of several iterations in a case of near degeneracy.

| Iteration | $\lambda$ | $Q$ ($10^{-27}$ cm$^2$) |
|---|---|---|
| $\lambda_0^{(0)}$ | 4.52261 | |
| $\lambda_0^{(\frac{1}{2})}$ | 4.52118 | 2.587 |
| $\lambda_0^{(1)}$ | 4.52257 | 2.665 |
| Extrapolated values | 4.52188 | 2.626 |
| $\lambda_0^{(0)}$ | 4.52186 | |
| $\lambda_0^{(\frac{1}{2})}$ | 4.52186 | 2.625 |
| $\lambda_0^{(1)}$ | 4.52186 | 2.626 |

From Eqs. (13) and (14) two variational principles may be obtained. We shall be interested in situations in which the eigenvalues need not be of a single sign. Let $\lambda_0$ be the eigenvalue of least absolute value. We adjust operator $\mathbf{B}$ so that it is positive. Then, we have

$$\lambda_0=\text{extremum}[(\phi, \mathbf{A}\phi)/(\phi, \mathbf{B}\phi)] \tag{15a}$$

and

$$\lambda_0=\text{extremum}[(\phi, \mathbf{B}\phi)/(\phi, \mathbf{B}\mathbf{A}^{-1}\mathbf{B}\phi)], \tag{15b}$$

where the symbol $(\phi, \psi)$ signifies the scalar product of $\phi$ and $\psi$ integrated over $x$ from 0 to infinity. Upon introducing the successive iterates $\phi_n=\begin{pmatrix} u_n \\ w_n \end{pmatrix}$ into Eq. (15), we obtain a set of values $\lambda_0^{(n)}$, which are approximations to $\lambda_0$:

$$\begin{aligned} \lambda_0^{(n)}&=(n, n-1)/(n, n), \\ \lambda_0^{(n+\frac{1}{2})}&=(n, n)/(n, n+1), \end{aligned} \tag{16}$$

where

$$(n, m)=(\phi_n, \mathbf{B}\phi_m)$$

or, in terms of $u_n$ and $w_n$,

$$(n, m)=\int_0^\infty [fu_nu_m+2^{\frac{3}{2}}\gamma g(u_nw_m+w_nu_m) +(f-2\gamma g)w_nw_m]dx. \tag{17}$$

It may be shown[16] that the sequences $\lambda_0^{(n)}$, $\lambda_0^{(n+\frac{1}{2})}$ converge to $\lambda_0$. The manner in which this occurs is complicated in this case by the nonpositive definite character of $\mathbf{B}$. A sufficient condition that $\mathbf{B}$ be definite is that the determinant

$$\begin{vmatrix} f(x) & 2^{\frac{3}{2}}\gamma g(\tau x) \\ 2^{\frac{3}{2}}\gamma g(\tau x) & f(x)-2\gamma g(\tau x) \end{vmatrix}$$

be of a given sign. This is certainly the case for $\gamma=0$, but the sign changes when $(\gamma g/f)$ exceeds $(\frac{1}{4})$ or $(-\gamma g/f)$ exceeds $(\frac{1}{2})$, where it has been assumed that both $f$ and $g$ are positive. The nonpositive definite character of $\mathbf{B}$ has the consequence that the eigenvalues $\lambda$ may be both positive and negative extending to $+\infty$ and $-\infty$. One sequence of eigenvalues corresponds to those states for which the $S$ state is the principal component; i.e., it gives the values of the potential $V_c$ for which a state, mostly $S$ in character, has the binding energy $\epsilon$. The smallest positive $\lambda$, $\lambda_0$, is that potential for which $\epsilon$ is the binding energy of the ground state. The other sequence of eigenvalues corresponds to those states for which the $D$ state is the principal component.

Two consequences of the above are of importance in the calculation. The inequalities satisfied by $\{\lambda_0^{(n)}\}$ and $\{\lambda_0^{(n+\frac{1}{2})}\}$ are considerably less specific when $\mathbf{B}$ is not positive definite compared with the positive definite situation. Thus, in the latter case, these successive approximations to $\lambda_0$ form a monotonic sequence approaching $\lambda_0$ from above:

$$\lambda_0^{(n)}\geqslant\lambda_0^{(n+\frac{1}{2})}\geqslant\lambda_0^{(n+1)}\geqslant\cdots\geqslant\lambda_0.$$

For nonpositive definite $\mathbf{B}$ the $\{\lambda_0^{(n)}\}$ satisfy a weaker

set of conditions:

$$|\lambda_0^{(n)}| \geqslant \lambda_0,$$
$$|\lambda_0^{(n)}| \geqslant |\lambda_0^{(n+\frac{1}{2})}|, \tag{18}$$
$$\lambda_0^{(n)}\lambda_0^{(n+\frac{1}{2})} \geqslant \lambda_0^{(n+1)}\lambda_0^{(n+\frac{1}{2})} \geqslant \cdots \geqslant \lambda_0^2.$$

Let $\lambda_1$ be the eigenvalue whose absolute value is next in size to $\lambda_0$. If, after some number of iterations, convergence to $\lambda_0$ has proceeded to the stage that

$$\lambda_1^2 \geqslant \lambda_0^{(n+\frac{1}{2})}\lambda_0^{(n+1)}, \quad n \geqslant N,$$

then we have

$$\lambda_0^{(n)} \geqslant \lambda_0^{(n+1)} \geqslant \cdots \geqslant \lambda_0, \quad n \geqslant N. \tag{19}$$

We thus see that the set $\{\lambda_0^{(n)}\}$, $n$ integral, will, for sufficiently large $n$, approach $\lambda_0$ monotonically from above. The values of $\lambda_0^{(n+\frac{1}{2})}$, $n$ integral, can be either above or below $\lambda_0$. Eventually, however, the sign of $(\lambda_0^{(n+\frac{1}{2})}-\lambda_0)$ will become independent of $n$, being positive or negative in accordance with the sign of $(\lambda_0/\lambda_1)$. Thus, when $\lambda_0/\lambda_1<0$, the set $\lambda_0^{(n+\frac{1}{2})}$ is asymptotic to $\lambda_0$ from below.

Other upper and lower bounds to $\lambda_0$ may be obtained. We give several of these:

$$\lambda_0 \geqslant \lambda_0^{(n+\frac{1}{2})}\left[1-\frac{\lambda_0^{(n)}-\lambda_0^{(n+\frac{1}{2})}}{|\lambda_1|-\lambda_0^{(n+\frac{1}{2})}}\right], \tag{20a}$$

$$\lambda_0^2 \geqslant \lambda_0^{(n+\frac{1}{2})}\lambda_0^{(n+1)}\left[1-\frac{\lambda_0^{(n)}-\lambda_0^{(n+1)}}{|\lambda_1|-\lambda_0^{(n+1)}}\right], \quad \lambda_0^{(n+1)}>0. \tag{20b}$$

If $\lambda_1$ is known to be negative, then we have

$$\lambda_0 \geqslant \lambda_0^{(n+\frac{1}{2})}\left[1-\frac{\lambda_0^{(n)}-\lambda_0^{(n+\frac{1}{2})}}{|\lambda_2|-\lambda_0^{(n+\frac{1}{2})}}\right], \tag{20c}$$

a considerable improvement over Eq. (20b). Here, $\lambda_2$ is the third in the sequence of absolute eigenvalues. Because of the possibility that $\lambda_0^{(n+\frac{1}{2})}$ may be less than $\lambda_0$, another upper bound which is useful may be obtained:

$$\lambda_0 \leqslant \lambda_0^{(n+\frac{1}{2})}\left[1+\frac{\lambda_0^{(n)}-\lambda_0^{(n+\frac{1}{2})}}{|\lambda_1|+\lambda_0^{(n+1)}}\right]. \tag{20d}$$

These inequalities are valid if the denominators are positive. To apply these inequalities, some method for obtaining a lower bound for $|\lambda_1|$ is necessary. We have employed the relation,

$$\lambda_1^2 \geqslant [\text{spur}(\mathbf{A}^{-1}\mathbf{B})^2-(\lambda_0^{(n)})^{-2}], \tag{21}$$

where

$$\text{spur}(\mathbf{A}^{-1}\mathbf{B})^2=\eta^2\int dx'\int dx''\{G_\eta^{(0)}(x',x'')f(x'')$$
$$\times G_\eta^{(0)}(x'',x')f(x')+16\gamma^2 G_\eta^{(0)}(x',x'')g(\tau x'')$$
$$\times G_\eta^{(2)}(x'',x')g(\tau x')+G_\eta^{(2)}(x',x'')[f(x'')$$
$$-2\gamma g(\tau x'')]G_\eta^{(2)}(x'',x')[f(x')-2\gamma g(\tau x')]\}.$$

TABLE III. Values of the parameter $\eta$ corresponding to various values of $\mu$; $\mu$ is given in terms of the electron mass.

| $\eta$ | 0.313 | 0.275 | 0.256 | 0.224 |
|---|---|---|---|---|
| $\mu$ | 286 | 326 | 350 | 400 |

A similar method[16] is available for determination of $\lambda_2$ in (20c).

Another remark should be made with regard to the product sequence in (18) or sequence (19). The rate of convergence is best when the ratio $|(\lambda_0/\lambda_1)|$ is very small compared with 1. For some values of the parameter $\gamma$, this ratio is very close to one, corresponding to an approximate degeneracy in the eigenvalues of the iterated operator in Eq. (15), $(\mathbf{A}^{-1}\mathbf{B})^2$, which has eigenvalues $\lambda_0^{-2}$, $\lambda_1^{-2}$, $\cdots$. This is particularly serious in the computation of such nonstationary quantities as the amount of $D$ state and the quadrupole moment. Near-degeneracy may produce sizable fluctuations in these quantities with successive iterations. In practice, these fluctuations are in the relative amplitude of the $S$ and $D$ state rather than in the shape of the radial functions. Accordingly, the relative amplitude of the $S$ and $D$ state may, after a sufficient number of iterations, be introduced as a variational parameter. A simpler procedure is given by the following extrapolation technique.

Let $F^{(0)}$, $F^{(1)}$, $F^{(2)}$, $\cdots$ be the values of a quantity $F$ obtained by successive iterations. The quantity $F$ may be $\lambda$, or $Q$, or the value of $u$ or $w$ at a given point. If the iterations at stage $F^{(0)}$ have proceeded far enough, it may be assumed that

$$F^{(0)}=F+f, \quad F^{(1)}=F+\mathcal{E}f, \quad F^{(2)}=F+\mathcal{E}^2 f,$$

where $f$ is the error at stage $F^{(0)}$, while $\mathcal{E}$, the parameter giving the rate of convergence, is $\lambda_0/\lambda_1$. We have essentially assumed that all the error comes from the admixture of the eigenfunction of $\lambda_1$ to that of $\lambda_0$. From these three equations we may determine $F$:

$$F=F^{(0)}-[F^{(0)}-F^{(1)}]^2/[F^{(0)}-2F^{(1)}+F^{(2)}]. \tag{22}$$

If two sets of numbers $F$ and $G$ depend on $\mathcal{E}$ in the same way (e.g., $\lambda$ and $Q$), then it is possible to obtain $\mathcal{E}$ from one sequence and use it in the other so that

$$F=F^{(0)}-(F^{(0)}-F^{(1)})[G^{(0)}-G^{(1)}]/$$
$$[G^{(0)}-2G^{(1)}+G^{(2)}]. \tag{23}$$

The extrapolation method was checked by comparison with successive iterations in some of the worst cases which occurred and proved to be very accurate.

## Computational Details

The first step in the calculation involves the determination of a suitable initial trial function $\phi_0$. We

TABLE IV. Values of $(\eta/\tau)$ and the corresponding equivalent meson masses.

| $(\eta/\tau)$ | 0.640 | 0.493 | 0.355$_5$ | 0.320 | 0.275 |
|---|---|---|---|---|---|
| $\mu$ | 140 | 182 | 252 | 280 | 326 |

TABLE V. Values of $\lambda=MV_c r_c^2/\hbar^2$, $\gamma=V_t/V_c$, $Q$=quadrupole moment, $p_D$=amount of $D$ state, $\rho_1$=effective triplet range for various values of $\eta=(M\epsilon/\hbar^2)^{\frac{1}{2}}r_c$, $\eta/\tau=(M\epsilon/\hbar^2)r_t$.

| $\eta/\tau$ | $\gamma$ | $\lambda$ | $Q$ ($10^{-27}$ cm$^2$) | $p_D$ | $\rho_1$ ($10^{-13}$ cm) |
|---|---|---|---|---|---|
| | | $\eta=0.313$ | | | |
| | 0.1140 | 2.1520 | 2.52 | 0.016 | 1.92 |
| 0.640 | 0.1300 | 2.1016 | 2.78 | 0.019 | 1.96 |
| | 0.1350 | 2.0855 | 2.86 | 0.020 | 1.96 |
| | 0.2400 | 2.0129 | 2.53 | 0.021 | 1.88 |
| 0.492 | 0.2700 | 1.9496 | 2.74 | 0.024 | 1.93 |
| | 0.3000 | 1.8886 | 2.89 | 0.026 | 1.93 |
| | 0.8550 | 1.5429 | 2.62 | 0.032 | 1.79 |
| $0.355_5$ | 0.9300 | 1.4784 | 2.72 | 0.035 | 1.79 |
| | 1.0000 | 1.4220 | 2.79 | 0.036 | 1.79 |
| | 1.5600 | 1.2227 | 2.67 | 0.042 | 1.71 |
| 0.320 | 1.7000 | 1.1575 | 2.74 | 0.042 | 1.71 |
| | 1.8400 | 1.0983 | 2.78 | 0.042 | 1.71 |
| | 6.340 | 0.5214 | 2.65 | 0.054 | 1.52 |
| 0.275 | 9.200 | 0.3757 | 2.72 | 0.054 | 1.52 |
| | 13.000 | 0.2737 | 2.77 | 0.054 | 1.52 |
| | | $\eta=0.275$ | | | |
| | 0.1000 | 2.0498 | 2.64 | 0.018 | 1.85 |
| 0.640 | 0.1040 | 2.0347 | 2.71 | $0.019_5$ | 1.85 |
| | 0.1080 | 2.0195 | 2.78 | $0.020_5$ | 1.85 |
| | 0.2239 | 1.8676 | 2.72 | 0.025 | 1.79 |
| 0.492 | 0.2400 | 1.8277 | 2.83 | 0.027 | 1.82 |
| | 0.3000 | 1.6860 | 3.18 | 0.034 | 1.85 |
| | 0.7107 | 1.4631 | 2.63 | 0.034 | 1.71 |
| $0.355_5$ | 0.7660 | 1.4070 | 2.71 | 0.036 | 1.71 |
| | 0.7820 | 1.3915 | 2.72 | 0.037 | 1.71 |
| | 1.3392 | 1.1285 | 2.69 | 0.042 | 1.66 |
| 0.320 | 1.4280 | 1.0818 | 2.73 | 0.041 | 1.66 |
| | 1.5600 | 1.0190 | 2.79 | 0.046 | 1.66 |
| | 4.7300 | 0.5331 | 2.63 | 0.051 | 1.52 |
| 0.275 | 5.8000 | 0.4470 | 2.67 | 0.053 | 1.52 |
| | 9.2000 | 0.2952 | 2.75 | 0.056 | 1.49 |
| | | $\eta=0.256$ | | | |
| | 0.0920 | 2.0031 | 2.67 | 0.020 | 1.79 |
| 0.640 | 0.0978 | 1.9789 | 2.80 | 0.021 | 1.79 |
| | 0.1140 | 1.9100 | 3.08 | 0.025 | 1.84 |
| | 0.1900 | 1.8616 | 2.64 | $0.024_5$ | 1.74 |
| 0.492 | 0.2100 | 1.8064 | 2.79 | 0.027 | 1.77 |
| | 0.2400 | 1.7260 | 3.00 | 0.030 | 1.79 |
| | 0.7300 | 1.3297 | 2.76 | 0.039 | 1.68 |
| $0.355_5$ | 0.8100 | 1.2524 | 2.85 | 0.040 | 1.68 |
| | 0.8550 | 1.2125 | 2.89 | 0.040 | 1.71 |
| | 1.2000 | 1.1009 | 2.69 | 0.043 | 1.63 |
| 0.320 | 1.3392 | 1.0231 | 2.76 | 0.045 | 1.63 |
| | 1.5600 | 0.9192 | 2.85 | 0.046 | 1.63 |
| | 6.2400 | 0.3692 | 2.70 | 0.055 | 1.49 |
| 0.275 | 10.5000 | 0.2288 | 2.77 | 0.054 | 1.49 |
| | 13.0000 | 0.1871 | 2.79 | 0.052 | 1.49 |
| | | $\eta=0.224$ | | | |
| | 0.0740 | 1.9429 | 2.62 | 0.020 | 1.68 |
| 0.640 | 0.0770 | 1.9279 | 2.71 | 0.021 | 1.68 |
| | 0.0850 | 1.8871 | 2.97 | $0.023_5$ | 1.74 |
| | 0.1550 | 1.7961 | 2.61 | 0.025 | 1.66 |
| 0.492 | 0.1700 | 1.7461 | 2.77 | 0.028 | 1.68 |
| | 0.1900 | 1.6808 | 2.94 | 0.030 | 1.73 |
| | 0.5800 | 1.2899 | 2.72 | 0.039 | 1.60 |
| $0.355_5$ | 0.6240 | 1.2370 | 2.81 | 0.040 | 1.63 |
| | 0.7107 | 1.1434 | 2.91 | 0.044 | 1.63 |
| | 0.9500 | 1.0687 | 2.60 | 0.044 | 1.57 |
| 0.320 | 1.1000 | 0.9681 | 2.75 | 0.046 | 1.57 |
| | 1.2000 | 0.9105 | 2.81 | 0.045 | 1.60 |
| | 4.7300 | 0.3713 | 2.69 | 0.055 | 1.46 |
| 0.275 | 5.8000 | 0.3087 | 2.72 | $0.054_5$ | 1.49 |
| | 10.5000 | 0.1773 | 2.78 | 0.059 | 1.49 |

employed the integral equation variational principle (15b), in conjunction with simple one-parameter functions for $u$ and for $w$, which satisfied the boundary conditions. The values of $\lambda_0$, and therefore $V_0$, obtained were in error by a few percent. The error in $Q$ was considerably larger, and it must be concluded that discussions based on the variational method employing such simple functions without further improvement are not reliable. The function $\phi_0$ was then introduced into the right-hand side of Eq. (8) to obtain $\phi_1$, etc.; the resulting approximations for $\lambda_0$ and $Q$ were obtained from Eqs. (16) and (7), respectively.

These calculations were performed on the Mark I calculator of the Harvard Computation Laboratory, using Simpson's trapezoidal rule over most of the range. The mesh was chosen so as to provide an over-all accuracy for the final values of $\lambda_0$ of at least one part in $10^4$. There was some difficulty in the region near $x=0$, since integration (8b) involves a rapidly varying function of $x$ behaving as $1/x^2$ as $x\to 0$. A special integration formula for this region was devised in which the known properties of $u$ were utilized. In the region of the origin, $u\sim x(A_0+A_1x+A_2x^2+\cdots)$. The coefficients $A_1$ may be expressed in terms of the tabulated values of $u$. The integration for each power of $x$ may then be performed and the final integral expressed in terms of tabulated values of $u$ multiplied by known numerical coefficients. The convergence obtained from successive $\lambda_0^{(n)}$ was very good. It was generally necessary to perform only two iterations, which is sufficient to permit the utilization of extrapolation formulas (22) and (23). The convergence to $Q$ was not nearly as good, but with the extrapolation method it was possible to obtain $Q$'s which were correct to the third significant figure. A case which was carried beyond the second iteration is given in Table I, together with a comparison between the extrapolated $Q$ and the next iterate.

A second case in which there is a near degeneracy is illustrated in the Table II. The values of $\lambda_0^{(n)}$ obtained from the first two iterations $\lambda_0^{(0)}$, $\lambda_0^{(\frac{1}{2})}$, $\lambda_0^{(1)}$, are seen to oscillate, and the second significant figure in $Q$ is uncertain. At this point the extrapolation formula (23)

was applied to $\phi$ and then two further iterations performed. It is seen that the $\lambda_0^{(n)}$-sequence for this new initial trial function is constant to six significant figures, and the first three figures of $Q$ are now determined. It is interesting to examine just how accurate the extrapolation technique applied directly to the first group of $\lambda_0^{(n)}$ would be in this particularly difficult case. It is seen that the extrapolated $\lambda$ agrees with the second, and correct group, to 1 part in 400,000, while the extrapolated $Q$ is in agreement to 1 part in 2600.

In any numerical method there is an unavoidable error following from the finiteness of the mesh. As a consequence an auxiliary problem has been solved; the exact problem is obtained only when the size of the mesh approaches zero. We may consider the difference between the two problems as a perturbation in $\mathbf{B}$; and in this way we obtain $\delta\lambda_0$, the error in $\lambda_0$, from the variational principle (15):

$$\delta\lambda_0/\lambda_0 = -(\phi, \delta\mathbf{B}\phi)/(\phi, \mathbf{B}\phi). \quad (24)$$

Assuming Simpson's rule is employed from $x_0$ to infinity, the numerator may be approximately evaluated as

$$(\phi, \delta\mathbf{B}\phi) \simeq (\Delta^3/90)(\partial^3/\partial x_0^3)(\phi, \mathbf{B}\phi)'_{x=x_0}, \quad (25)$$

where the prime signifies that only the matrix product is to be taken and $\Delta$ is the mesh size. An evaluation of Eq. (24) for several cases gives $(\delta\lambda_0)/\lambda_0 \sim 10^{-4}$. The error in $Q$ from this source may be expected to be of the same order of magnitude.

## III. RESULTS

A total of sixty cases were computed, a $\lambda$ and $Q$ being obtained for each. The values of the parameter $\eta$, proportional to $r_c$, the range of the central force, were centered about the range suggested by the scattering data.[17] The various ranges employed may be related to an equivalent meson mass by the relation $r_c = (\hbar/\mu c)$. The four values of $\mu$, in terms of the electron mass and the corresponding values of $\eta$ are given in Table III. For each value of $r_c$, five values of the tensor range $r_t$ were employed. The values of $(\eta/\tau)$ (which is proportional to $r_t$) and the corresponding equivalent meson masses are given in Table IV. It will be noticed that for the most part $r_t$ was chosen so that $r_t \geqslant r_c$. For each pair $(r_c, r_t)$ three values of $\gamma$ were used. These were chosen so as to center the consequent values of $Q$ about the experimental value of $2.766\times10^{-27}$ cm$^2$ and so that accurate interpolation in the event of changes in $Q$ and $\epsilon$ would be possible.

The values of $\lambda$ and $Q$ for each value of $r_c$, $r_t$, and $\gamma$ are given in Table V. The values of $\gamma$ and $\lambda$ needed to give $Q = 2.766\times10^{-27}$ cm$^2$ tabulated in Table VI were obtained by interpolation from Table V.

Some qualitative features of these results are as follows:

1. For a given $r_c$ and $r_t$, $\lambda$ decreases with increasing $\lambda$ or more precisely

$$(1+\gamma/\tau)\lambda \sim \text{constant}.$$

2. For a given $r_c$ and $r_t$, $Q$ and $p_D$ increase with increasing tensor interaction

$$\gamma\lambda/Q \sim \text{constant}, \quad p_D/Q \sim \text{constant}.$$

3. For a given $r_c$ and $Q$, $\gamma\lambda$ increases rapidly with decreasing $r_t$:

$$\gamma\lambda/\tau^3 \sim \text{constant} \quad \text{and} \quad (1+\gamma/\tau)\lambda \sim \text{constant}.$$

4. For a given $r_t$ and $\gamma$, $\lambda$ increases with increasing $r_c$:

$$\lambda/\eta \sim \text{constant}.$$

5. For a given $r_t$ and $Q$, $\gamma$ increases with increasing $r_c$:

$$\gamma/\eta \sim \text{constant}.$$

The relations in groups 1 and 2 hold rather well, while those in 3, 4, and 5 are much more approximate. For example, relation 1 holds generally to better than 1 percent, relation 2 to somewhat less than 1 percent, while relation 3 holds for the most part to better than 10 percent. These relations are empirical results and should not be extended beyond the range of the parameters considered in these calculations.

## IV. APPLICATIONS

In this section, we shall investigate the limits set on the possible values of $r_c$ and $r_t$ as determined by other available data on the neutron-proton system. We shall consider the magnetic moment of the deuteron, low energy neutron-proton scattering, the photodisintegration of the deuteron, and neutron capture in hydrogen.

TABLE VI. Interpolated values of $\gamma$, $\lambda$, $p_D$, $\rho_1$ for $Q = 2.766\times10^{-27}$ cm$^2$ for various values of $\eta$ and $(\eta/\tau)$.

| $\eta$ | $\eta/\tau$ | $\gamma$ | $\lambda$ | $p_D$ | $\rho_1$ ($10^{-13}$ cm) |
|---|---|---|---|---|---|
| 0.313 | 0.640 | 0.1291 | 2.1044 | 0.019 | 1.96 |
| | 0.492 | 0.2747 | 1.9400 | 0.024 | 1.93 |
| | $0.355_5$ | 0.9744 | 1.4424 | 0.036 | 1.79 |
| | 0.320 | 1.7860 | 1.1209 | 0.042 | 1.71 |
| | 0.275 | 12.6421 | 0.2819 | 0.054 | 1.52 |
| 0.275 | 0.640 | 0.1072 | 2.0226 | 0.020 | 1.85 |
| | 0.492 | 0.2305 | 1.8512 | 0.026 | 1.79 |
| | $0.355_5$ | 0.8359 | 1.3406 | 0.038 | 1.71 |
| | 0.320 | 1.5074 | 1.0431 | 0.043 | 1.66 |
| | 0.275 | 10.0816 | 0.2681 | 0.056 | 1.49 |
| 0.256 | 0.640 | 0.0962 | 1.9857 | 0.021 | 1.79 |
| | 0.492 | 0.2067 | 1.8153 | 0.027 | 1.77 |
| | $0.355_5$ | 0.7344 | 1.3251 | 0.039 | 1.68 |
| | 0.320 | 1.3525 | 1.0164 | 0.045 | 1.63 |
| | 0.275 | 10.0684 | 0.2371 | 0.054 | 1.49 |
| 0.224 | 0.640 | 0.0788 | 1.9188 | 0.021 | 1.68 |
| | 0.492 | 0.1696 | 1.7475 | 0.028 | 1.68 |
| | $0.355_5$ | 0.5985 | 1.2666 | 0.039 | 1.63 |
| | 0.320 | 1.1244 | 0.9537 | 0.046 | 1.57 |
| | 0.275 | 9.0980 | 0.2087 | 0.057 | 1.49 |

[17] J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949); J. D. Jackson and J. M. Blatt, Revs. Modern Phys. **22**, 77 (1950); H. A. Bethe, Phys. Rev. **76**, 38 (1949); G. Breit and W. G. Bouricious, Phys. Rev. **75**, 1029 (1949).

### a. Magnetic Moment of the Deuteron

The admixture of the $D$ state contributes to the magnetic moment of the deuteron as follows:

$$\mu_D=\mu_n+\mu_p-(\tfrac{3}{2})[\mu_n+\mu_p-(\tfrac{1}{2})]p_D. \tag{26}$$

Here, $\mu_D$, $\mu_n$, $\mu_p$ are the magnetic moments in nuclear magnetons of the deuteron, neutron, and proton, respectively, and $p_D$ is the fractional amount of $D$ state as defined by

$$p_D=\int_0^\infty w^2dx \Big/ \int_0^\infty (u^2+w^2)dx. \tag{27}$$

However, Eq. (26) is incomplete because of other contributions to the magnetic moment of the deuteron arising from relativistic[18] effects. The latter is difficult to estimate from the nonrelativistic potential, since the specific relativistic transformation properties of the interactions are involved. An investigation of the various simple models indicates that the correction to Eq. (26) is uncertain by $\pm(T/2Mc^2)\sim\pm 0.008$, where $T$ is the average internal kinetic energy of the deuteron. The consequent uncertainty in $p_D$ is $\pm 0.016$. In Table V the value of $p_D$ for each value of $\gamma$ and is tabulated. It is seen that $p_D$ depends principally upon $\tau$, not being sensitive to variations in $\gamma$ for a given $\tau$. Most of the values of $p_D$ for $0.275<\eta/\tau<0.640(\eta/\tau=(M\epsilon/\hbar^2)^{\frac{1}{2}}r_t)$ fall within the admissible range for $p_D=0.04\pm 0.016$. It is clear, however, that the regions $\eta/\tau>0.640$ and $\eta/\tau<0.275$ are excluded.

### b. Neutron-Proton Scattering

The phase shift $\delta_1$ for the $^3S_1$ scattering of neutrons by protons may be determined from the expansion:

$$\begin{gathered} k\cot\delta_1=-\alpha+\tfrac{1}{2}(k^2+\alpha^2)\rho_1+0[(k^2+\alpha^2)^2\rho_1^3], \\ \alpha^2=M\epsilon/\hbar^2=(\eta/r_c)^2, \end{gathered} \tag{28}$$

where $\rho_1$ is the effective triplet range. The parameter $\rho_1$ may be expressed in terms of the deuteron wave function by the expression,

$$\rho_1=2\int_0^\infty [e^{-2\alpha r}-(u^2+w^2)]dr. \tag{29}$$

Here, $(u^2+w^2)$ has been normalized so that it approaches $e^{-2\alpha r}$ as $r\to\infty$. The values of the effective triplet range $\rho_1$ computed from Eq. (29) are given in Table V. The recent experimental data for the coherent[19] neutron-proton scattering amplitude, the neutron-proton[20] cross section for epithermal neutrons, together with the binding energy of the deuteron, can be used to determine the triplet scattering length $a_1$ (the limit of $(\delta_1/k)$ as $k$ approaches zero) and the triplet range $\rho_1$:

$$a_1=-(5.39\pm 0.03)\times 10^{-13}\text{ cm},$$

$$\rho_1=(1.73\pm 0.04)\times 10^{-13}\text{ cm}.^{21}$$

Many of the values given for $\rho_1$ in Tables V and VI are incompatible with the measurements, so that the possible values of the range of the tensor force for a given central force range are sharply restricted. In particular, the admissible tensor force ranges are significantly greater than the central force range. It should be noted that this conclusion is independent of the exact value of the quadrupole moment for, as may be seen from Table V, the value of $\rho_1$ is not sensitive to the value of $Q$.

### c. Photodisintegration of the Deuteron

We develop a phenomenological description of both the photoelectric and photomagnetic effect. In the former case, the empirical absence of interaction in the final state makes it possible to express the cross section in terms of parameters describing the triplet $n-p$ system which are already known experimentally. On the other hand, for photomagnetic effects which involves transitions between triplet and singlet states, one needs, in addition to the singlet scattering parameters, a new phenomenological constant which also contains the possible effects of exchange currents.

*Photoelectric Disintegration*

The cross section for this process is proportional to the quantity

$$P=\left|\mathbf{e}\cdot\int e^{-i\mathbf{k}\cdot\mathbf{r}}\mathbf{r}(u/r)d\tau\right|^2 \Big/ \int (u^2+w^2)dr. \tag{30}$$

Here, $\mathbf{e}$ is the polarization vector for the incident photon, and $\hbar\mathbf{k}$ is the relative momentum in the center-of-mass system for the final $P$ state in which the interactions between neutron and proton have been neglected. The transitions from the $^3D$ part of the deuteron ground state wave function may also be omitted. Equation (30) is based on the interaction $\frac{1}{2}e\mathbf{E}\cdot\mathbf{r}$ in place of $-(e/Mc)\mathbf{p}\cdot\mathbf{A}$, since the former includes the effects of exchange currents.

We compare $P$ with its value in the zero-range approximation:

$$P_0=\left|\mathbf{e}\cdot\int e^{-i\mathbf{k}\cdot\mathbf{r}}\mathbf{r}(e^{-\alpha r}/r)d\tau\right|^2 \Big/ \int e^{-2\alpha r}dr. \tag{31}$$

Adopting the normalization

$$(u^2+w^2)\underset{r\to\infty}{\longrightarrow}e^{-2\alpha r},$$

the ratio of the denominators in $P$ and $P_0$ may be

[18] H. Primakoff, Phys. Rev. **72**, 118 (1947); G. Briet and I. Bloch, Phys. Rev. **72**, 135 (1947).
[19] Hughes, Burgy, and Ringo, Phys. Rev. **79**, 227 (1950).
[20] E. Milkonian, Phys. Rev. **76**, 1744 (1949).

[21] Inclusion of the effect of the next term in expansion (28) as determined by J. M. Blatt and L. C. Biedenharn (private communication) reduces the value of $\rho_1$ by 1.6 percent to $1.70\times 10^{-13}$ cm which is within the quoted experimental error.

expressed in terms of $\rho_1$:

$$\int (u^2+w^2)dr \Big/ \int e^{-2\alpha r}dr = (1-\alpha\rho_1). \tag{32}$$

The integral in the numerator of $P$ is

$$\mathbf{N}=\int e^{-i\mathbf{k}\cdot\mathbf{r}}\mathbf{r}(u/r)d\tau$$

$$=i\nabla_k\int e^{-i\mathbf{k}\cdot\mathbf{r}}(u/r)d\tau$$

$$=4\pi i\nabla_k\left[(1/k)\int_0^\infty \sin krudr\right], \tag{33}$$

where $\nabla_k$ is the gradient operator in $k$ space. The asymptotic behavior of $u$ is given by $v=(1-\zeta_0^2)^{\frac{1}{2}}e^{-\alpha r}$, where

$$(\zeta_0/(1-\zeta_0^2)^{\frac{1}{2}})=\lim_{r\to\infty}(w/u).$$

Since $\zeta_0^2\sim 10^{-3}$, we shall ignore the factor $(1-\zeta_0^2)^2$ in the discussion below. We may write

$$\int_0^\infty u\sin kr\,dr=\int_0^\infty v\sin kr\,dr-\int_0^\infty (v-u)\sin kr\,dr.$$

Since $v-u$ vanishes outside the range of nuclear forces, it is permissible for the energies being considered here to insert a power series expansion for $\sin kr$ in the second integral. $\mathbf{N}$ then becomes

$$\mathbf{N}=-[8\pi i\mathbf{k}/(k^2+\alpha^2)^2]\left[1-\tfrac{1}{6}(k^2+\alpha^2)^2\int_0^\infty r^3(v-u)dr\right]. \tag{34}$$

The first term in $\mathbf{N}$ gives the zero-range approximation. The second range-dependent term is proportional to $(k^2+\alpha^2)^2\rho_1^4$, the proportionality constant being a rather small number. A closer estimate of the correction may be obtained through the use of the approximate wave function

$$u=e^{-\alpha r}-e^{-\gamma r},$$

$\gamma\simeq(3/\rho_1)(1+\cdots)$, which is suitable for the meson, exponential, or Hulthen potential wells. Then the range dependence of $\mathbf{N}$ is measured by

$$(k^2+\alpha^2)^2\rho_1^4/81\sim 3\times 10^{-4}(1+k^2/\alpha^2)^2.$$

This is very small compared with one even at relatively high photon energies. We shall therefore omit the correction, so that the quantity $P$ is given by

$$P=P_0/(1-\alpha\rho_1).$$

Hence, the photoelectric cross section[22] is

$$\sigma_e=\sigma_e^{(0)}/(1-\alpha\rho_1), \tag{35}$$

where $\sigma_e^{(0)}$ is the cross section for zero-range nuclear force:

$$\sigma_e^{(0)}=\frac{8\pi}{3}\left(\frac{e^2}{\hbar c}\right)\left(\frac{1}{\alpha^2}\right)\frac{(\epsilon E)^{\frac{3}{2}}}{(E+\epsilon)^3}=1.15\times 10^{-26}\ \text{cm}^2\frac{(\epsilon E)^{\frac{3}{2}}}{(E+\epsilon)^3},$$

in which $E+\epsilon$ is the energy of the incident photon.

Thus, the energy dependence of the photoelectric cross section is given by the zero-range formula. Introducing a finite range changes only the magnitude of the cross section, so that a measurement of the absolute cross section will give a determination of $\rho_1$, the triplet scattering range. We shall compare this result with experiment after photomagnetic effects have been considered.

*Photomagnetic Disintegration*

The cross section for photomagnetic disintegration in which the very weak transitions to the $D$ state have been omitted is

$$\sigma_m=\frac{\pi}{3}\left(\frac{e^2}{\hbar c}\right)(\mu_n-\mu_p)^2\frac{E+\epsilon}{Mc^2}\frac{\sin^2\delta_0}{k}\times\frac{\left[\int uu_k(1+g(r))dr\right]^2}{\int (u^2+w^2)dr}, \tag{36}$$

where $u_k$ is the singlet $S$ radial wave function multiplied by $r$, normalized so as to approach $\sin(kr+\delta_0)/\sin\delta_0$ for large $r$, and we again normalize $(u^2+w^2)$ to approach $e^{-2\alpha r}$. The function $g(r)$ has been introduced to represent the effect of the exchange magnetic moment as it appears in the triplet-singlet transitions.[23] The spin operator character of this moment need not be specified, since the transition to the singlet state is from a unique triplet spin state. In addition, it may be expected that $g(r)$ is appreciable only within the range of nuclear forces.

The cross section for the inverse of this process, the radiative capture of slow neutrons is

$$\sigma_c=\pi\left(\frac{\epsilon}{2E_n}\right)^{\frac{1}{2}}\left(\frac{e^2}{\hbar c}\right)(\mu_n-\mu_p)^2\left(\frac{\epsilon}{Mc^2}\right)^2\alpha a_0^2\times\frac{\left[\int uu_0(1+g(r))dr\right]^2}{\int (u^2+w^2)dr}, \tag{37}$$

[22] This formula was first given by H. A. Bethe and C. Longmire, Phys. Rev. 77, 647 (1950), and J. M. Blatt (unpublished).

[23] The exchange magnetic moment can be related phenomenologically to the exchange tensor interaction. A calculation of this contribution to the mixed effective range and of the magnetic moments of $H^3$ and $He^3$ is in progress.

where

$$u_0 \underset{r\to\infty}{\longrightarrow} 1+\left(\frac{r}{a_0}\right),$$

and $a_0$, the singlet scattering amplitude, is the low energy limit of $(\delta_0/k)$, and $E_n$ is the neutron kinetic energy in the laboratory system. The experimental value of $a_0$ is $(23.7\pm0.1)\times10^{-13}$ cm.

The integral in the numerator of Eq. (36) may be rewritten as follows:

$$M_k=\int_0^\infty u[1+g(r)]u_k dr=\int_0^\infty vv_k dr - \int (vv_k-uu_k)dr+\int_0^\infty ug(r)u_k dr,$$

where $v=[\sin(kr+\delta_0)/\sin\delta_0]$, $v=e^{-\alpha r}$ give the asymptotic behavior of $u_k$ and $u$, respectively. Hence, we have

$$M_k=[(k\cot\delta_0+\alpha)/(\alpha^2+k^2)]-\tfrac{1}{2}\bar\rho,$$
$$(\tfrac{1}{2}\bar\rho)=\int_0^\infty (vv_k-uu_k)dr-\int_0^\infty uu_k g(r)dr. \quad (38)$$

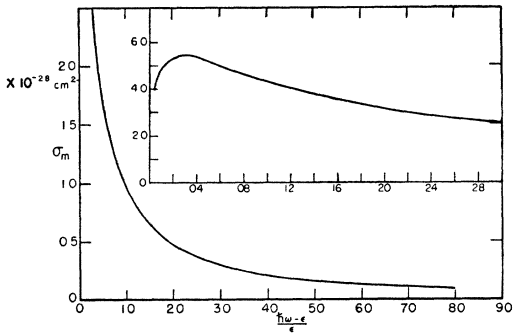


FIG. 1. The cross section for the photomagnetic disintegration of the deuteron, where $\hbar\omega$ is the incident gamma-ray energy and $\epsilon$ is the deuteron binding energy.

Since both contributions in Eq. (38) involve only the behavior of the wave functions within the range of nuclear forces, $\bar\rho$, a mixed effective range, will be insensitive to the kinetic energy of the emergent particles. Therefore, $M_0$, in particular, will be given by

$$M_0=(1/\alpha)[1+(1/\alpha a_0)]-\tfrac{1}{2}\bar\rho$$

with the same $\bar\rho$ as in Eq. (38). In terms of the zero energy triplet scattering length $a_1$,

$$a_1=-(1/\alpha)[1/(1-\tfrac{1}{2}\alpha\rho_1)],$$

$M_0$ becomes

$$M_0=(1/\alpha a_0)[a_0(1-\tfrac{1}{2}\alpha\bar\rho)-a_1(1-\tfrac{1}{2}\alpha\rho_1)]$$
$$=[(a_0-a_1)/\alpha a_0][1-\tfrac{1}{2}\alpha(a_0\bar\rho-a_1\rho_1)/(a_0-a_1)].$$

Introducing this result into Eq. (37), we obtain

$$\sigma_c=(2\epsilon/E_n)^{\frac{1}{2}}(\epsilon/Mc^2)\bar\sigma, \quad (39)$$

where

$$\bar\sigma=\pi\left(\frac{e^2}{\hbar c}\right)(\mu_n-\mu_p)^2\left(\frac{\epsilon}{Mc^2}\right)(a_0-a_1)^2 \times\frac{[1-\frac{1}{2}\alpha(a_0\bar\rho-a_1\rho_1)/(a_0-a_1)]^2}{1-\alpha\rho_1}$$
$$=(1.02\pm0.01)\times10^{-26}\text{ cm}^2 \times\frac{[1-\frac{1}{2}\alpha(a_0\bar\rho-a_1\rho_1)/(a_0-a_1)]^2}{1-\alpha\rho_1}. \quad (40)$$

The cross section for photomagnetic disintegration may be expressed in terms of $\bar\sigma$:

$$\sigma_m=\tfrac{2}{3}\bar\sigma\frac{k\alpha}{(\alpha^2+k^2)} \times\frac{[1+(k^2/2\alpha^2)\alpha(\rho_0-\bar\rho)/(1-\frac{1}{2}\alpha\bar\rho+1/\alpha a_0)]^2}{1+k^2a_0(a_0+\rho_0)+(\frac{1}{4})(k^2a_0\rho_0)^2}. \quad (41)$$

With the exception of $\bar\rho$, all of the quantities appearing

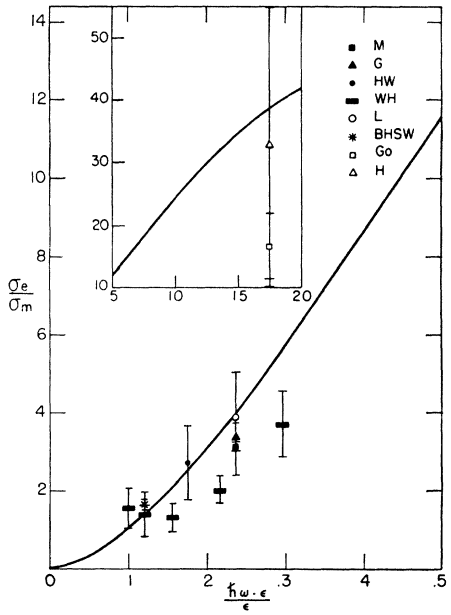


FIG. 2. The ratio of the cross section for photoelectric disintegration and the cross section for the photomagnetic disintegration of the deuteron, where $\hbar\omega$ is the incident gamma-ray energy and $\epsilon$ is the deuteron binding energy. The references to experiment are:

L=N. O. Lassen, Phys. Rev. **74**, 1533 (1948); **75**, 1099 (1949).
G=F. Genevese, Phys. Rev. **76**, 1288 (1949).
M=E. P. Meiners, Jr., Phys. Rev. **76**, 259 (1949).
WH=W. M. Woodward and J. Halpern, Phys. Rev. **76**, 107 (1949).
HW=B. Hamermesh and A. Wattenberg, Phys. Rev. **76**, 1408 (1949).
BHSW=Bishop, Halban, Shaw, and Wilson, Phys. Rev. **81**, 220 (1951).
Go=G. Goldhaber, Phys. Rev. **81**, 930 (1951).
H=P. V. C. Hough, thesis.

in the expression for the capture cross section are accurately known experimentally. In the cross section for photodisintegration we encounter $\rho_0$, the singlet scattering range which has not been accurately determined from scattering data. It is noted that the energy dependence of $\sigma_m$ is more sensitive to the value of $\rho_0$ than the corresponding singlet $n-p$ scattering cross section. Unfortunately, this additional energy dependence will first become significant for $k^2 \sim \alpha^2$, where the magnetic cross section is a small fraction of the total. Thus, for the low energy domain where the photomagnetic cross section is appreciable, the latter is essentially determined by $\bar{\sigma}$ which is however better fixed by the more accurate experimental capture cross section.

The experimental value of $0.310 \pm 0.02$ barn[24] for the capture cross section at a neutron energy of (1/40) ev implies a value of $(0.976 \pm 0.07) \times 10^{-26}$ cm$^2$ for $\bar{\sigma}$. From this $\bar{\rho}$ may now be obtained:

$$\bar{\rho} = (2.18 \pm 0.3) \times 10^{-13} \text{ cm}.$$

It may be noted that, within the large experimental uncertainty, the value of $\bar{\rho}$ agrees with the average of $\rho_1 = (1.73 \pm 0.04) \times 10^{-13}$ cm and $\rho_0 = (2.5 \pm 0.5) \times 10^{-13}$ cm the latter being inferred from neutron-proton scattering data, this average being $(2.12 \pm 0.25) \times 10^{-13}$ cm. This is what would be expected if the exchange moment were not appreciable and if the shape of the singlet and triplet $S$ wave functions were similar within the region of nuclear interaction. For example, if we employ the approximate wave functions $u = e^{-\alpha r} - e^{-\gamma r}$, $u_0 = 1 - e^{-\xi r}$ where $\gamma \sim 3/\rho_1$ and $\xi \sim 3/\rho_0$ we obtain, with neglect of the exchange moment contribution,

$$\begin{aligned} \bar{\rho} &= 2\int (vv_0 - uu_0)dr \\ &= \tfrac{2}{3}[\rho_0 + \rho_1 - \rho_1\rho_0/(\rho_0+\rho_1)] \\ &= \tfrac{1}{2}(\rho_0+\rho_1)\{1+\tfrac{1}{3}[(\rho_0-\rho_1)/(\rho_0+\rho_1)]^2\}, \end{aligned}$$

which differs negligibly from the average[25] $(\rho_0+\rho_1)/2$ for the values of $\rho_0$ and $\rho_1$ of interest.

A curve for the theoretical photomagnetic disintegration cross section is given in Fig. 1. The experimental quantity which involves this cross section most sensitively is the ratio $\sigma_e/\sigma_m$, as inferred from the angular distribution of the disintegration products. The not very precise experimental determinations are compared with theory in Fig. 2 with no evident inconsistency.

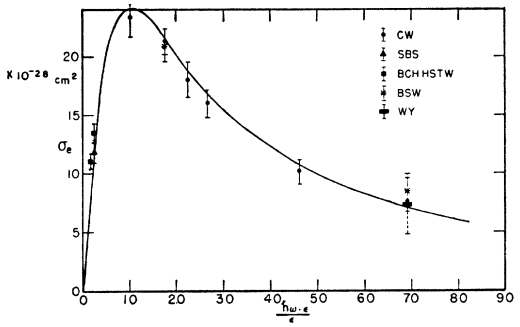


FIG. 3: The cross section for the photoelectric disintegration of the deuteron, where $\hbar\omega$ is the incident gamma-ray energy and $\epsilon$ is the deuteron binding energy. The references to experiment are:

CW = J. H. Carver and D. H. Williamson, Nature **167**, 154 (1951).
SBS = Snell, Barker, and Sternberg, Phys. Rev. **75**, 1290 (1949).
BCHHSTW = Bishop, *et al.*, reference 24.
BSW = Barnes, Stafford, and Wilkinson, Nature **165**, 70 (1950).
WY = H. Wäffler and S. Younis, Helv. Phys. Acta **22**, 414 (1949).

If the theoretical values of $\sigma_m$ are subtracted from the experimentally determined total cross section, one obtains photoelectric cross sections which are compared with theory in Fig. 3. The satisfactory agreement indicates the consistency of the values of $\rho_1$ obtained from the photoelectric cross section and from scattering and binding energy data.

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[24] W. J. Whitehouse and G. A. R. Graham, Can. J. Research **A25**, 261 (1947). See comment in Bishop, Collie, Halban, Hedgran, Siegbahn, Du Toit, and Wilson, Phys. Rev. **80**, 211 (1950).

[25] This is the simple form assumed in reference 21.

[26] Feshbach, Eisenstein, and Schwinger, Phys. Rev. **74**, 1223 (1948).