Range Energy Relation for 340-Mev Protons

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The angle of emission of the Čerenkov radiation is used to find the velocity of a beam of protons. Their range is also measured and we obtain points of the range energy relation for energies near 340 Mev for Be, C, Al, Cu, Sn, and Pb. The data are used to evaluate the average excitation energy I for these substances.

THE range-energy relations for protons is interesting for two types of reasons: its study has considerable intrinsic importance as a problem of physics; in addition, the numerical results are extensively used by experimenters in determining energies.

For high energies (300 Mev) many of the serious difficulties besetting the very low energy part of the curve become negligible and the formula of Bethe[1]

$$-\frac{dE}{dx}=\frac{4\pi e^4}{mv^2}NZ\left(\log\frac{2mv^2}{I(1-\beta^2)}-\beta^2-\frac{C_k}{Z}\right) \quad (1)$$

can be used. Bloch has shown, using the Fermi-Thomas model of the atom, that I is proportional to the atomic number of the stopping substance.[2] The "constant" B,

$$I=BZ, \quad (2)$$

has been determined for several substances by Bakker and Segrè,[3] using the two values of I for Al and Be which have been determined by Wilson,[4] and Madsen and Venkateswarlu.[5] The quantities C_k and a corresponding C_L represent relatively small corrections required by the fact that the velocity of the proton is not extremely large compared to the velocity of the electron in the K and L shell of the stopping substance.

It is clearly desirable to extend the experiment to an absolute measurement, eliminating the necessity of using the results of Wilson, and Madsen and Venkateswarlu, which are obtained with light substances for which the statistical model is not well applicable. To do this a knowledge of the initial $\beta=v/c$ of the proton is necessary. This can be approximately obtained from the characteristics of the cyclotron accelerating the protons, but because of the precession of the orbits and other reasons this method is not very precise. Deflection of the beam in a known magnetic field would also give a way of measuring its energy, but although our deflected beam is very monoenergetic, as we shall see later, our deflecting magnet is not calibrated to give a precise absolute measurement of the energy.

Recently Mather[6] has developed an apparatus to measure the angle of emission of the Čerenkov radiation produced by the beam in a piece of flint glass and has perfected this method to such an extent that it gives very accurate values of β. This technique affords an opportunity to measure the energy of the beam on an absolute scale and hence to determine the range of particles of known energy. Integration of (1) gives

$$R=\int_0^E(-dE/dx)^{-1}dE. \quad (3)$$

If the range R and the limit of the integral E are known, Eq. (3) is an equation with I as the only unknown, if the corrections C_k, etc., have been separately calculated. Unfortunately this has not been done yet, although the extensive computational work necessary is in progress.[7] In this paper C_k has been taken into account crudely; C_L and C_M have been neglected. The final values of $B=I/Z$ are expected to be slightly lower (up to 1 ev) than those given in Table II.

For the practical problem of determining the energy given the "range" of a particle, we have to examine a little more carefully what we mean by range. The range given by Eq. (3) is the mean range: half of the particles travel in the material for a length larger than R and half for a length smaller than R. The length considered is the rectified trajectory, and due to multiple scattering this is not the same as the distance from the entrance point in the material of a plane perpendicular to the initial direction of the beam through which half of the particles pass. Clearly the mean range given by Eq. (3) is larger than the latter "range" measured as indicated above, which we shall call R^*. We can obtain a crude estimate of the importance of this effect by the following consideration which gives $(R-R^*)/R$. Divide the range R into small lengths l, and call θ_i the angle between l_i and the direction of the incoming beam. We have

$$R-R^*=\sum l_i(1-\cos\theta_i)\approx\sum \tfrac{1}{2}l_i\theta_i^2, \quad (4)$$

if θ_i is small. Now the average value of θ_i^2 is

$$\Theta_i^2=(Zm/M)\log(E_0/E_i) \quad (5)$$

(m mass of electron, M mass of proton). This formula is a crude approximation obtained from Williams' formula and Eq. (1).

[1] M. S. Livingston and H. A. Bethe, Revs. Modern Phys. **9**, 263 (1937).
[2] F. Bloch, Z. Physik **81**, 363 (1938).
[3] C. J. Bakker and E. Segrè, Phys. Rev. **81**, 489 (1951).
[4] R. R. Wilson, Phys. Rev. **60**, 749 (1941).
[5] C. Madsen and P. Venkateswarlu, Phys. Rev. **74**, 648 (1948).
[6] R. L. Mather, Phys. Rev. **84**, 181 (1951).
[7] H. A. Bethe and M. C. Walske, private communication.

TABLE I. The experimental proton range R^* for various absorbers at proton energies near 340 Mev, the rectified experimental range R, the theoretical and experimental standard deviation of the range distribution σ, and the resulting standard deviation of the proton energy distribution ΔE.

Energy Mev	Absorber	R^* g/cm²	R g/cm²	σ_{th} g/cm²	σ_{exp} g/cm²	ΔE Mev
339.7	$_4$Be	76.68	76.73	0.65	0.91	1.75
339.7	$_6$C	69.97	70.03	0.62	0.88	1.83
339.7	$_{13}$Al	79.26	79.42	0.75	1.04	1.84
338.5	$_{13}$Al	78.47	78.63	0.75	0.92	1.40
337.9	$_{29}$Cu	91.43	91.84	0.92	1.12	1.44
338.5	$_{29}$Cu	91.36	91.77	0.92	1.25	1.89
339.7	$_{29}$Cu	92.27	92.69	0.92	1.24	1.88
339.7	$_{50}$Sn	106.58	107.41	—	1.50	—
339.7	$_{82}$Pb	122.80	124.37	1.35	1.90	2.32
338.5	$_{82}$Pb	121.21	122.76	1.35	1.84	2.25

We replace in Eq. (4) the sum by an integral and use (5) to obtain

$$R-R^*=\frac{Zm}{2M}\int_0^R\left(\log\frac{E_0}{E(x)}\right)dx. \tag{6}$$

If we assume $R\approx E^{1.75}$, which is a good approximation of the range-energy relation, we have

$$R-R^*=ZmR_0/2M(1.75), \tag{7}$$

or

$$(R-R^*)/R=Z/6400. \tag{8}$$

The values of R in Table I are obtained from the values of R^*, directly observed, with the help of Eq. (8).

Our experimental arrangement is practically the same as the one used by Bakker and Segrè in their investigation mentioned above. The deflected beam of the 184-inch cyclotron is collimated to 1-inch diameter, passes through the Čerenkov radiation apparatus and enters an ionization chamber full of argon at atmospheric pressure.[8] The chamber is closed by foils of copper-beryllium alloy, 2 mils thick, and its interior walls are of aluminum 7 mg/cm² thick. The depth of the part used is 5 cm and the diameter is 10 cm. After passing through this chamber the beam goes through a variable copper absorber carried by a wheel. This absorber can be varied from 0 to 8.62 gram/cm² of copper in 12 equal steps. After having passed the variable absorber the beam goes through a stack of plates of the material under investigation and then passes through an ionization chamber identical to the one described above. The ratio of the ionization current in the two chambers is plotted as a function of the absorber between the two.

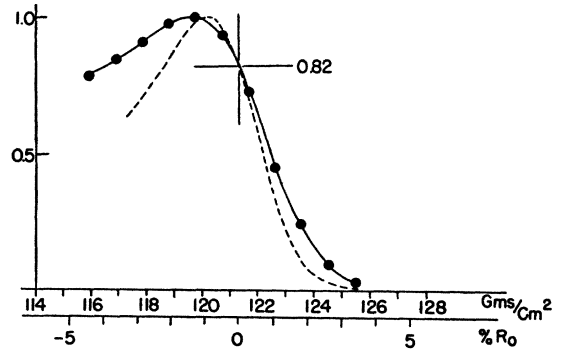

FIG. 1. Relative ionization *vs* range; solid curve—experimental data; dotted curve—calculated curve using $\sigma=\sigma_{theor}$.

[8] A sketch of the ionization chamber will be given in Chamberlain, Segrè, and Wiegand, Phys. Rev. **83**, 923 (1951).

We compute all thicknesses of the wheel absorbers, windows, etc., in g cm^{-2} of equivalent stopping power as if they were composed of the same substance as the main aborber, using the results of reference 3. The thickness of these absorbers is in any event a small fraction of the total thickness.

As an example of the curves obtained the case of lead is shown in Fig. 1.

We must now obtain from these data the mean range. If we call $i(t)$ the ionization per cm of argon in the ionization chamber produced by a single particle at distance t from the end of its range in the absorbing material, and assume for the distribution of ranges due to straggling the gaussian form of probability

$$P(R)=(2\pi)^{-\frac{1}{2}}\sigma^{-1}\exp[-(R-R^*)^2/2\sigma^2] \tag{9}$$

we have for the ionization measured in our chamber

$$I(R)=k(2\pi)^{-\frac{1}{2}}\sigma^{-1}\int_R^\infty\exp[-(x-R^*)^2/2\sigma^2]i(x-R)dx. \tag{10}$$

Assuming a new variable $(x-R)/\sigma=u$ and calling $(R-R^*)/\sigma=v$, formula 10 becomes

$$I=K\int_0^\infty\exp[-\tfrac{1}{2}(u+v)^2]i(\sigma u)du, \tag{11}$$

where K is a constant. i is represented accurately enough by

$$i=(\text{const})t^{-0.46}, \tag{12}$$

and we compute numerically the integral

$$f(x)=\int_0^\infty\exp[-\tfrac{1}{2}(x+t)^2]t^{-0.46}dt. \tag{13}$$

This is given in Fig. 2. $f(x)$ uses as a unit of length the standard deviation of the gaussian. It will be noticed that if we normalize the ordinates in such a way as to call the maximum 1, then $f(0)=0.82$. This means that, no matter what the value of the standard deviation, the center of the gaussian occurs at that value of the thickness for which $f(x)$ is equal to 0.82 times its maximum. This is R^*.

The experimental standard deviation of the range distribution, σ_{exp}, is obtained by comparing the experimental curves with Fig. 2. We normalize them by multiplying the ordinates by such factors as to make the

maxima of the curves equal. We then multiply the abscissas of each experimental curve by such a factor that the theoretical and experimental curve may be superimposed upon each other. The thickness of material in the experimental curve corresponding to $x=1$ in the theoretical curve is the experimental standard deviation.

Theoretically the straggling can be calculated with the formula of Bohr[9]

$$\sigma^2=4\pi NZe^4\int_0^E(-dE/dx)^{-3}dE. \tag{14}$$

The values of σ_{theor} of Table I are computed by numerical integration from Eq. (14) and the values of dE/dx given in the tables of Aron *et al.*[10] It will be noticed that they are about 0.75 times the experimental value. If we try to attribute the difference to inhomogeneity of the energy of the beam ΔE, we obtain

$$(\sigma_{\text{exp}}^2-\sigma_{\text{theor}}^2)^{\frac{1}{2}}=(-dE/dx)^{-1}\Delta E. \tag{15}$$

Numerically ΔE is given in column 7 of Table I. It is clear that $\Delta E/E$ is about 0.5×10^{-2}, a very good definition of the beam energy.

Unfortunately there is a disagreement between these computations and experiment which is not entirely clear to us. If we examine Fig. 1, the experimental (solid curve) and theoretical results (dotted curve) agree for the region of the curve past the maximum, but not for the region preceding it. More protons have suffered a larger loss of energy than we expected. There are several possible reasons for this, the most probable being the effect of nuclear collisions, but we have been unable to account for this effect quantitatively. We do not think however that it affects the determination of R. We estimate the standard deviations of these measurements to be approximately 1 Mev for the energy and 0.2 g cm^{-2} for the ranges. Since dE/dx is of the order of 2 Mev/g cm^{-2} at 340 Mev, an error of 1 Mev corresponds to an error of 0.5 g cm^{-2} in the range and hence most of the uncertainty comes from the energy measurements. The uncertainty in energy ΔE (column 7 of Table I), if present, is too small to produce an appreciable broadening of the Čerenkov line and is not detectable in this way. From figures analyzed as indicated above we have the results shown in Table I.

TABLE II. The rectified measured proton range R, the predicted range from reference 10 R_{Aron}, and the difference between them R_A-R for various absorbers at proton energies near 340 Mev. The values of the average ionization potential I and Block's "constant" I/Z are those which, if they had been used in the calculations of reference 10, would have made R_A equal to R.

Energy Mev	Absorber	R g/cm²	R_{Aron} g/cm²	R_A-R g/cm²	I ev	I/Z ev
339.7	$_4$Be	76.73	74.57	−2.16	59.0	14.75
339.7	$_6$C	70.03	69.40	−0.63	74.4	12.91
339.7	$_{13}$Al	79.42	79.40	−0.02	150.3	11.56
338.5	$_{13}$Al	78.63	78.95	0.32	145.5	11.19
337.9	$_{29}$Cu	91.84	92.72	0.88	312.3	10.77
338.5	$_{29}$Cu	91.77	93.01	1.24	304.0	10.48
339.7	$_{29}$Cu	92.69	93.53	0.84	313.4	10.81
339.7	$_{50}$Sn	107.41	—			
339.7	$_{82}$Pb	124.37	127.15	2.78	828.7	10.11
338.5	$_{82}$Pb	122.76	126.45	3.69	792.6	9.67

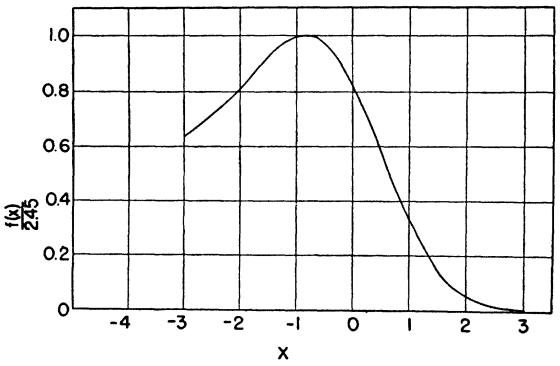

FIG. 2. Calculated shape of the ionization *vs* range curve from Eq. (13).

With regard to the chemical purity of the samples used, we have these data: Beryllium: 99.9 percent. Carbon: 99$^+$ percent. Aluminum: 99.2 percent, impurities Fe, Cu. Copper: 99.9 percent, impurities O, P. Tin: 99.8 percent, impurities Pb, Sb, As. Lead: 99.85 percent. Bi 0.15 percent.

In order to analyze our data further we have reported in Table II the energy, material, rectified experimental range and the range calculated by Aron *et al.* for the substances studied. As shown by column 5 ($R_{\text{Aron}}-R$) is a small quantity, showing that Aron's tables are quite accurate. It is however possible to improve them by changing the value of I used in their calculation in such a way as to bring them in exact agreement with the experimental results. This has been performed by Aron, and the values of I thus obtained are given in column 6 of Table II. Column 7 gives I/Z for the same substances.

It will be noted that I_{Al} is practically identical with the value of 150 ev found by Wilson in 1940, and I_{Be} is also in excellent agreement with the measurements of Madsen and Venkateswarlu. For the other substances our results agree quite well with the less direct measurements of Bakker and Segrè.

We wish to thank Mr. W. Aron for his generous and valuable cooperation in this work and Professors Bethe and Wick for interesting discussions.

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[9] N. Bohr, *The Penetration of Atomic Particles through Matter*, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **18**, 8 (1948), [Eq. (5.2.6)].

[10] Aron, Hoffman, and Williams, "Range Energy Curves," AECU-663, unpublished (UCRL-121), unpublished. The range-energy curves are being recalculated by Mr. Aron using the information supplied in this paper and the paper of Bakker and Segrè, see reference 3. Unfortunately the calculations of Mr. Walske, see reference 7, will not be available for this revision.