

**The Solution of the Janossy G-Equation**

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JANOSSY<sup>1</sup> in a recent paper gave the following equation describing the development of a nucleon cascade in homogeneous nuclear matter,

$$\Phi(\epsilon, n_1, n_2; x) = \int_0^x \exp[-(x-\theta)] d\theta \times \sum_{\substack{n_1'+n_1''=n_1 \\ n_2'+n_2''=n_2}} \int_0^\infty \int_0^\infty \Phi(\epsilon/\epsilon_1, n_1', n_2'; \theta) \Phi(\epsilon/\epsilon_2, n_1'', n_2''; \theta) \times w(\epsilon_1, \epsilon_2) d\epsilon_1 d\epsilon_2, \quad (1)$$

where  $\Phi(\epsilon, n_1, n_2; x)$  expresses the probability of finding  $n_1$  nucleons with energies  $> \epsilon E_0$  and  $n_2$  nucleons with energies  $< \epsilon E_0$  at a depth  $x$  in homogeneous nuclear matter as a result of a single primary nucleon of energy  $E_0$ . The cross section for nucleon-nucleon collisions is given by  $w(\epsilon_1, \epsilon_2) d\epsilon_1 d\epsilon_2$ . The difficulty of solving Eq. (1) directly led Janossy to introduce a generating function defined by

$$G(\epsilon, U; x) = \sum_N U^N \Phi(\epsilon, N; x), \quad (2)$$

in which  $U$  is written for  $u_1$  and  $u_2$ ;  $N$  for  $n_1; n_2$  and  $U^N$  for  $u_1^{n_1}, u_2^{n_2}$ . Equation (1) was then transformed into

$$\frac{\partial}{\partial x} G(\epsilon, U; x) + G(\epsilon, U; x) = \int_0^\infty \int_0^\infty G(\epsilon/\epsilon_1, U; x) G(\epsilon/\epsilon_2, U; x) w(\epsilon_1, \epsilon_2) d\epsilon_1 d\epsilon_2 \quad (3)$$

with the initial condition

$$G(\epsilon, u_1, u_2; 0) = \begin{cases} u_1 & \text{for } \epsilon < 1 \\ u_2 & \text{for } \epsilon > 1. \end{cases} \quad (4)$$

We have been able to develop methods whereby the solution of Eq. (1) may be obtained without resorting to the  $G$  equation. We find

$$\Phi(\epsilon, n_1, n_2; x) = 2^{n-1} f_n(x) / n!(n-1)! \times \int_0^\epsilon \dots \int_0^\epsilon \int_0^\epsilon \dots \int_0^\epsilon L_n \Lambda_n(s_1, \dots, s_n) d\epsilon_1 \dots d\epsilon_n \quad (5)$$

with  $n_1 + n_2 = n$  and

$$f_n(x) = e^{-x}(1-e^{-x})^{n-1}. \quad (6)$$

$L_n$  is the  $n$ -fold inverse Mellin transform operator expressed by

$$L_n = \frac{1}{(2\pi i)^n} \int_{s_1'-i\infty}^{s_1'+i\infty} \dots \int_{s_n'-i\infty}^{s_n'+i\infty} \epsilon_1^{-(s_1+1)} \dots \epsilon_n^{-(s_n+1)} d s_1 \dots d s_n \quad (7)$$

and

$$\Lambda_n(s_1, \dots, s_n) = \sum_{a \neq b}^1 W(s_a, s_b) \sum_{a \neq b \neq c \neq d}^2 W(s_c, s_d) \dots \times \sum_{a \neq b \neq \dots \neq y \neq z}^{n-1} W(s_y, s_z) \quad (8)$$

with

$$W(s_1, s_2) = \int_0^\infty \int_0^\infty \epsilon_1^{s_1} \epsilon_2^{s_2} w(\epsilon_1, \epsilon_2) d\epsilon_1 d\epsilon_2. \quad (9)$$

The superscripts beside each summation refer to the order in which the summations are carried out. The first summation yields  $W$  functions for all possible combinations of the  $n$  variables  $s_i, i=1, \dots, n$  taken two at a time. The second summation then gives  $W$  functions for all possible combinations of the  $n-1$  variables  $s_a + s_b, s_1', \dots, s_{n-2}'$  taken two at a time; similarly the third summation gives  $W$  functions for all possible combinations of the  $n-2$  variables,  $s_c + s_d, s_1', \dots, s_{n-3}'$  taken two at a time, and so on for the remainder of the summations.

The solution of the  $G$  equation (3), is now immediately given by Eqs. (2) and (5).

We have been successful in solving the complete fluctuation problem in nucleon cascade theory, and the results with details shall be presented in a subsequent publication. The methods developed are moreover directly applicable to the similar problem in electron-photon cascade theory. We shall present, shortly, solutions for the distribution functions and Janossy  $G$ -equations in this instance as well.

<sup>1</sup> L. Janossy, Proc. Phys. Soc. (London) **A63**, 241 (1949).

**Electrostatic Fields in Close-Packed Crystals**

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THE splitting of free atomic or ionic energy levels by the electrostatic fields within a crystal is believed to play an important part in various phenomena (specific heat, magnetic susceptibility, etc.) in solids. In his original work on crystalline fields, Bethe<sup>1</sup> pointed out the usefulness of expanding the potentials in spherical harmonics. He also showed that for a crystal with cubic (point) symmetry, no spherical harmonics occur of degree less than 4. Bethe estimated the magnitude of the different terms for an ionic crystal by using Madelung's<sup>2</sup> method to expand the potential, and by calculating appropriate derivatives at the atomic positions.

We have been interested in electrostatic fields of metals in hexagonal close-packed and cubic-face centered crystals. In hexagonal crystals, spherical harmonics occur of degree 2 and 3, as well as higher degrees. In order to determine the magnitudes of spherical harmonics up to the 4th degree, we have approximated a metal as a collection of positive point charges of magnitude  $q$  at the ion sites, superimposed on a uniform negative charge of density  $\rho$ . For such a model, Madelung's method of calculating the potential cannot be used.

We choose the origin at the position of an ion. Then for a sufficiently large crystal the potential at small distances can be represented in spherical coordinates as

$$V = q/r - (2\pi/3)\rho r^2 + \sum_l A_{lm} r^l Y_l^m(\vartheta, \varphi), \quad (1)$$

where the  $Y_l^m$  are surface spherical harmonics. The sum in Eq. (1) contains contributions from all the ions except that at the origin. It is possible to express the coefficients  $A_{lm}$  as

$$A_{lm} = [4\pi/(2l+1)]q \sum R^{-(l+1)} Y_l^m(\Theta, \Phi), \quad (2)$$

where  $R, \Theta, \Phi$  are coordinates of one of the ions, and the sum is taken over all ions (except the one at the origin).

Equation (2) provides a suitable method of calculating the coefficients for  $l \geq 4$ . The factor  $R^{-(l+1)}$  makes the contributions of successive neighbors fall off in a sufficiently rapid way. For  $l=2$  and 3, we used Ewald's<sup>3</sup> method of calculating the potential near the origin, and took appropriate derivatives.

For a hexagonal close-packed crystal with lattice parameters  $b$  perpendicular to the hexagonal axis, and  $c$  along the axis, we consider ions at  $(x, y, z) = (0, 0, 0)$  and  $(0, b/\sqrt{3}, c/2)$  in the unit cell. Writing the potential near the origin in the form,

$$V = q/r + (8\pi/3\sqrt{3})q\rho^2/b^2c + k_1(2z^2 - x^2 - y^2) + k_2(y^3 - 3x^2y) + k_3[8z^4 - 24z^2(x^2 + y^2) + 3(x^2 + y^2)^2] + \dots, \quad (3)$$

we find for the "ideal" ratio of lattice parameters,  $c/b = (8/3)^{1/2}$ ,

$$\begin{aligned} k_1 &= (0.001692 \pm 0.000004)q/b^3, \\ k_2 &= (0.43780 \pm 0.00002)q/b^4, \\ k_3 &= (0.1134 \pm 0.0014)q/b^5. \end{aligned} \quad (4)$$

Other spherical harmonics of degree 4 and less do not occur in Eq. (3) because of crystal symmetry.