The Solution of the Janossy G-Equation

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JANOSSY[1] in a recent paper gave the following equation describing the development of a nucleon cascade in homogeneous nuclear matter,

$$\Phi(\epsilon, n_1, n_2; x)=\int_0^x \exp[-(x-\theta)]d\theta$$
$$\times \sum_{\substack{n_1'+n_1''=n_1\\ n_2'+n_2''=n_2}} \int_0^\infty \int_0^\infty \Phi(\epsilon/\epsilon_1, n_1', n_2'; \theta)\Phi(\epsilon/\epsilon_2, n_1'', n_2''; \theta)$$
$$\times w(\epsilon_1, \epsilon_2)d\epsilon_1 d\epsilon_2, \quad (1)$$

where $\Phi(\epsilon, n_1, n_2; x)$ expresses the probability of finding n_1 nucleons with energies $>\epsilon E_0$ and n_2 nucleons with energies $<\epsilon E_0$ at a depth x in homogeneous nuclear matter as a result of a single primary nucleon of energy E_0. The cross section for nucleon-nucleon collisions is given by $w(\epsilon_1, \epsilon_2)d\epsilon_1 d\epsilon_2$. The difficulty of solving Eq. (1) directly led Janossy to introduce a generating function defined by

$$G(\epsilon, U; x)=\Sigma_N U^N \Phi(\epsilon, N; x), \quad (2)$$

in which U is written for u_1 and u_2; N for n_1; n_2 and U^N for $u_1^{n_1}$, $u_2^{n_2}$. Equation (1) was then transformed into

$$\frac{\partial}{\partial x}G(\epsilon, U; x)+G(\epsilon, U; x)$$
$$=\int_0^\infty \int_0^\infty G(\epsilon/\epsilon_1, U; x)G(\epsilon/\epsilon_2, U; x)w(\epsilon_1, \epsilon_2)d\epsilon_1 d\epsilon_2 \quad (3)$$

with the initial condition

$$G(\epsilon, u_1, u_2; 0)=\begin{cases} u_1 & \text{for} \quad \epsilon<1 \\ u_2 & \text{for} \quad \epsilon>1. \end{cases} \quad (4)$$

We have been able to develop methods whereby the solution of Eq. (1) may be obtained without resorting to the G equation. We find

$$\Phi(\epsilon, n_1, n_2; x)=2^{n-1}f_n(x)/n!(n-1)!$$
$$\times \underbrace{\int_0^\epsilon \cdots \int_0^\epsilon}_{n_2} \underbrace{\int_\epsilon^1 \cdots \int_\epsilon^1}_{n_1} L_n \Lambda_n(s_1, \cdots, s_n)d\epsilon_1 \cdots d\epsilon_n \quad (5)$$

with $n_1+n_2=n$ and

$$f_n(x)=e^{-x}(1-e^{-x})^{n-1}. \quad (6)$$

L_n is the n-fold inverse Mellin transform operator expressed by

$$L_n=\frac{1}{(2\pi i)^n}\int_{s_1'-i\infty}^{s_1'+i\infty} \cdots \int_{s_n'-i\infty}^{s_n'+i\infty} \epsilon_1^{-(s_1+1)} \cdots \epsilon^{-(s_n+1)}ds_1 \cdots ds_n \quad (7)$$

and

$$\Lambda_n(s_1, \cdots, s_n)= \sum_{a\neq b}^1 W(s_a, s_b) \sum_{a\neq b\neq c\neq d}^2 W(s_c, s_d)\cdots$$
$$\times \sum_{a\neq b\neq \cdots \neq y\neq z}^{n-1} W(s_y, s_z) \quad (8)$$

with

$$W(s_1, s_2)=\int_0^\infty \int_0^\infty \epsilon_1^{s_1}\epsilon_2^{s_2}w(\epsilon_1, \epsilon_2)d\epsilon_1 d\epsilon_2. \quad (9)$$

The superscripts beside each summation refer to the order in which the summations are carried out. The first summation yields W functions for all possible combinations of the n variables s_i, $i=1, \cdots, n$ taken two at a time. The second summation then gives W functions for all possible combinations of the $n-1$ variables s_a+s_b, s_1', $\cdots$, s_{n-2}' taken two at a time; similarly the third summation gives W functions for all possible combinations of the $n-2$ variables, s_c+s_d, s_1', $\cdots$, s_{n-3}' taken two at a time, and so on for the remainder of the summations.

The solution of the G equation (3), is now immediately given by Eqs. (2) and (5).

We have been successful in solving the complete fluctuation problem in nucleon cascade theory, and the results with details shall be presented in a subsequent publication. The methods developed are moreover directly applicable to the similar problem in electron-photon cascade theory. We shall present, shortly, solutions for the distribution functions and Janossy G-equations in this instance as well.

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Electrostatic Fields in Close-Packed Crystals

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THE splitting of free atomic or ionic energy levels by the electrostatic fields within a crystal is believed to play an important part in various phenomena (specific heat, magnetic susceptibility, etc.) in solids. In his original work on crystalline fields, Bethe[1] pointed out the usefulness of expanding the potentials in spherical harmonics. He also showed that for a crystal with cubic (point) symmetry, no spherical harmonics occur of degree less than 4. Bethe estimated the magnitude of the different terms for an ionic crystal by using Madelung's[2] method to expand the potential, and by calculating appropriate derivatives at the atomic positions.

We have been interested in electrostatic fields of metals in hexagonal close-packed and cubic-face centered crystals. In hexagonal crystals, spherical harmonics occur of degree 2 and 3, as well as higher degrees. In order to determine the magnitudes of spherical harmonics up to the 4th degree, we have approximated a metal as a collection of positive point charges of magnitude q at the ion sites, superimposed on a uniform negative charge of density ρ. For such a model, Madelung's method of calculating the potential cannot be used.

We choose the origin at the position of an ion. Then for a sufficiently large crystal the potential at small distances can be represented in spherical coordinates as

$$V=q/r-(2\pi/3)\rho r^2+\Sigma_{l,m}A_{lm}r^l Y_l^m(\vartheta, \varphi), \quad (1)$$

where the Y_l^m are surface spherical harmonics. The sum in Eq. (1) contains contributions from all the ions except that at the origin. It is possible to express the coefficients A_{lm} as

$$A_{lm}=[4\pi/(2l+1)]q\Sigma R^{-(l+1)}Y_l^m(\Theta, \Phi), \quad (2)$$

where R, Θ, Φ are coordinates of one of the ions, and the sum is taken over all ions (except the one at the origin).

Equation (2) provides a suitable method of calculating the coefficients for $l\geqslant 4$. The factor $R^{-(l+1)}$ makes the contributions of successive neighbors fall off in a sufficiently rapid way. For $l=2$ and 3, we used Ewald's[3] method of calculating the potential near the origin, and took appropriate derivatives.

For a hexagonal close-packed crystal with lattice parameters b perpendicular to the hexagonal axis, and c along the axis, we consider ions at $(x, y, z)=(0, 0, 0)$ and $(0, b/\sqrt{3}, c/2)$ in the unit cell. Writing the potential near the origin in the form,

$$V=q/r+(8\pi/3\sqrt{3})qr^2/b^2c+k_1(2z^2-x^2-y^2)+k_2(y^3-3x^2y)$$
$$+k_3[8z^4-24z^2(x^2+y^2)+3(x^2+y^2)^2]+\ldots, \quad (3)$$

we find for the "ideal" ratio of lattice parameters, $c/b=(8/3)^{\frac{1}{2}}$,

$$k_1=(0.001692\pm 0.000004)q/b^3,$$
$$k_2=(0.43780\pm 0.00002)q/b^4, \quad (4)$$
$$k_3=(0.1134\pm 0.0014)q/b^5.$$

Other spherical harmonics of degree 4 and less do not occur in Eq. (3) because of crystal symmetry.

We have also calculated derivatives of k_1 and k_2 with respect to the lattice parameter c. For $c/b=(8/3)^{\frac{1}{2}}$, we find

$$dk_1/dc=(-1.10721\pm0.00006)q/b^4, \qquad (5)$$
$$dk_2/dc=(-1.60363\pm0.00010)q/b^5.$$

For a face-centered cubic crystal of cubic lattice parameter a, the potential takes the form,

$$V=q/r+(8\pi/3)qr^2/a^3 +k_4[x^4+y^4+z^4-3(x^2y^2+x^2z^2+y^2z^2)]+\ldots, \qquad (6)$$

where coordinate axes are chosen parallel to cube edges, and

$$k_4=(-7.653\pm0.035)q/a^5. \qquad (7)$$

The arrangement of ions in a hexagonal close-packed crystal with ideal ratio of c/b is very similar to that in a face-centered cubic crystal. For the case $a=\sqrt{2}b$, the ions in each crystal have 12 nearest neighbors, with the same spacing, and almost the same arrangement. Nevertheless, not only does the hexagonal potential have extra terms, but even the 4th degree harmonics in the two cases differ in both form and magnitude.

In assuming a uniform distribution of conduction electrons, the model we have used neglects screening by these electrons. This screening is likely to be different for the different harmonics or multipoles.

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[2] M. Born and M. Goeppert-Mayer, *Handbuch der Physik* (Springer, Berlin, Germany, 1933), Vol. XXIV, Part 2, p. 709.

[3] M. Born and M. Goeppert-Mayer, *Handbuch der Physik* (Springer, Berlin, Germany, 1933), Vol. XXIV, Part 2, p. 710.

Photoconductivity of Trapped Electrons in KBr Crystals at Room Temperature

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THE room temperature photoconductivity of all the stable color center absorption bands attributed to electrons trapped at lattice defects has been reported briefly for KCl[1] and for KBr.[2] The quantitative data for KBr generally verified the initial findings with KCl and allowed a comparison of the product ηw—quantum yield times electron range—for the various bands. This note will discuss the KBr results in more detail. The crystals were colored by x-rays from a Machlett AEG-50A tube having a tungsten target and beryllium window and were polarized by 500 volts dc while continuous records of photocurrent *vs* wavelength were made.

Soft x-rays produce the *F*-absorption band and a much weaker *M*-absorption band, both of which are photoconductive. After a very short x-irradiation, when the *M*-band is too weak to be observed, a peak is found in the photoresponse (photocurrent/incident energy) at the *M*-band wavelength which is 500 times weaker than the *F*-peak. Further x-irradiation strengthens both absorption bands and their photoresponse until the condition shown in Fig. 1 is obtained. The splitting of the *F*-peak can be attributed to the inhomogeneous distribution of *F*-centers produced by soft x-rays. Since *F*-centers are good electron traps, the range being inversely proportional to *F*-center concentration, they make the photoresponse smaller at the center of the band where the monochromatic light is absorbed in a high concentration of *F*-centers near the entrance surface. Unstable centers (*F'*), which have a lifetime of the order of a second, form and decay during the measurement, raising the photoresponse nonuniformly. This is evident in the difference between the two curves obtained by sweeping in opposite directions through the spectral region.

Illuminating the crystal with *F*-light (*F*-bleach) weakens the *F*-band and also affects the *M*-band and forms the *R*-, *N*-, and *L*-bands (R_1 and R_2 are not usually resolved in KBr at room temperature; *L* designates the short wavelength companion of the *F*-band). It is found that *F* photoresponse is not proportional to absorptance during *F*-bleach. This is shown in Fig. 2(a) for a crystal x-rayed through $\frac{1}{16}$-inch aluminum to produce a more homogeneous distribution of *F*-centers. The rate of weakening during bleaching is much greater in *F* photoresponse than in *F* absorptance, and a decrease in the rate of weakening during bleaching is evident in both. This greater effect on photoresponse

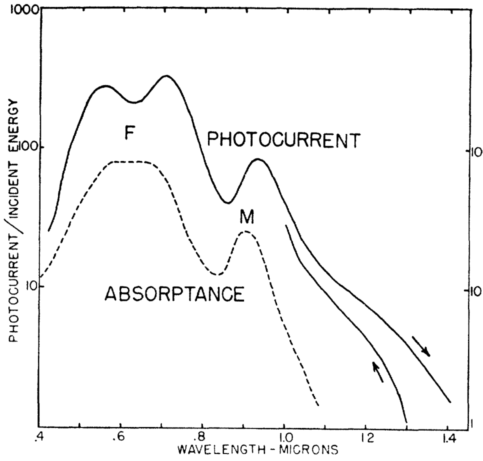

FIG. 1. Photoresponse and absorptance of *F*- and *M*-centers in KBr after long exposure to soft x-rays.

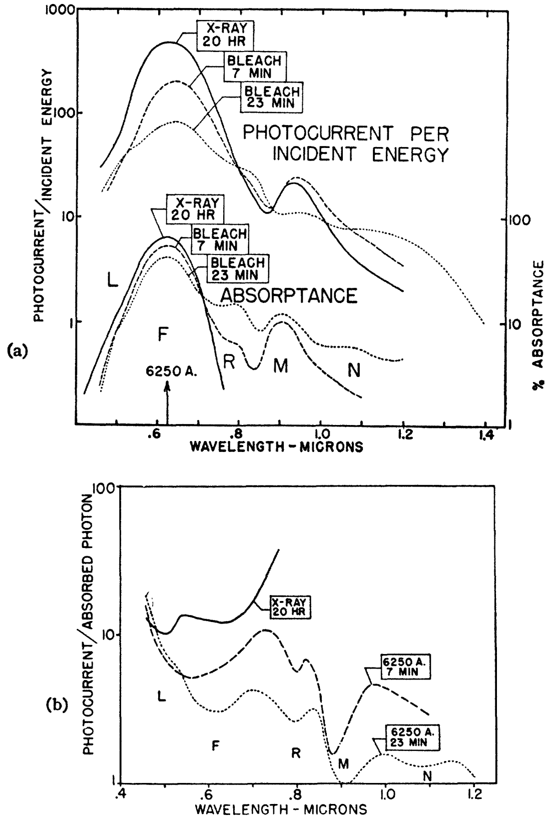

FIG. 2. Trapped electron centers in KBr after long exposure to hard x-rays and subsequent *F*-bleach: (a) photoresponse and absorptance; (b) product of quantum yield and electron range (ηw) in arbitrary units.