Wave Functions in Momentum Space*

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The integral equation, satisfied by the momentum-space wave function $\phi(p)$ for a nonrelativistic two-body problem with a phenomenological central interaction potential, is solved by means of an iteration method. A general prescription is given for finding suitable trial wave functions, which depend on some adjustable parameters. Reasonable values for these parameters are found by iteration of the wave function for particularly convenient values of the momentum. Successive iterations, giving better approximations $\phi_n(p)$ for $\phi(p)$, are carried out in a form suitable for numerical work. Besides $\phi_n(p)$, approximations are obtained for (a) the binding energy for certain bound states and (b) the phase shifts for scattering problems. For scattering at fairly low energies reasonable approximations are obtained with the same method both for weak and for fairly strong potentials.

Extensions of the method are discussed for (a) two-body problems including tensor forces, (b) simple three-body problems, and (c) a relativistic equation for the two-body problem.

I. INTRODUCTION

IN a number of current problems in nuclear physics the momentum distribution of nucleons in nuclei is of importance. For instance, the production cross sections of fast deuterons by the bombardment of nuclei by fast neutrons[1] depend critically on the momentum distribution of the protons in the nucleus. Similarly, the way in which the energy and momentum distribution of a π-meson and a nucleon produced by photon bombardment of hydrogen differs from that for bombardment of other nuclei again depends on the momentum distribution of the nucleons in such a nucleus. It therefore may be of some practical use to develop further the methods for finding wave functions in momentum space directly for a particle in a bound state in a fixed potential. For some problems involving the escape of slow nucleons from a nucleus it may also be of use to find the wave functions in momentum space for a particle of positive energy in the field of a fixed potential.

Approximation methods have been developed previously[2,3] for solving the wave equation in momentum space for a particle in a central potential. These methods are very powerful in cases where the necessary calculations can be carried out analytically, but are not very suitable in cases where numerical methods have to be used. The momentum-space wave function for a nonrelativistic particle in a fixed potential satisfies a well-known integral equation. It is the main aim of the present paper to show that a rapidly converging iteration method can also be found in many cases where the application of a variation method, like that used by Svartholm, is unsuitable. A general prescription for finding a reasonably good initial trial wave function for potentials with a finite range is given and the method is presented in a form suitable for numerical work.

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[1] G. F. Chew and M. L. Goldberger, Phys. Rev. **77**, 470 (1950).
[2] N. Svartholm, thesis, Lund (1945).
[3] M. Levy, Proc. Roy. Soc. (London) **A204**, 145 (1950).

II. GROUND STATE IN A CENTRAL POTENTIAL

Consider a bound state of energy E for a nonrelativistic particle of mass M in a fixed potential $U(\mathbf{r})$. We put

$$\hbar=c=1; \quad \gamma^2=-2ME. \tag{1}$$

The Schrödinger equation in ordinary space,

$$[(-\nabla^2/2M)+U(\mathbf{r})]\psi(\mathbf{r})=E\psi(\mathbf{r}), \tag{2}$$

transforms into an integral equation in momentum space:

$$(p^2+\gamma^2)\phi(\mathbf{p})=-2M\int d^{(3)}p' V(\mathbf{p}-\mathbf{p}')\phi(\mathbf{p}'). \tag{3}$$

There $[(2\pi)^{\frac{3}{2}}V(\mathbf{p})]$ and $\phi(\mathbf{p})$ are the p-space potential and wave function respectively; i.e., they are the three-dimensional fourier transforms of $U(\mathbf{r})$ and $\psi(\mathbf{r})$, respectively.

We restrict ourselves in this section to the special case of a spherically symmetric potential of finite "range" μ^{-1}. The p-space potential is then some spherically symmetric function $V(p;\mu)$ which, in most cases of interest here, is some monotonically decreasing function of p, decreasing fairly slowly for $p\lesssim\mu$ and decreasing rapidly for $p\gg\mu$ (usually a simple function of $[p^2+\gamma^2]$). Instead of calculating E for a fixed potential strength, we assume the binding energy E to be known (which is the case for the deuteron) and that the potential is

$$V(p)=\lambda V_0(p;\mu), \tag{4}$$

where $V_0(p;\mu)$ is a given function. We then have to solve Eq. (3) for the eigenvalue λ and for $\phi(\mathbf{p})$. We restrict ourselves further to the spherically symmetrical ground state wave function $\phi(p)$, which corresponds to a configuration-space wave function $\psi(r)$ with no nodes.

We wish to solve Eq. (3) by an iteration method; i.e., we assume some initial spherically symmetrical trial wave function $\phi_0(p)$; substitute this on the right-hand side of Eq. (3) and thus obtain a better approximation to the wave function, $\phi_1(p)$. On substituting $\phi_1(p)$ on the right-hand side of Eq. (3), a better approxi-

TABLE I. Trial wave functions $\psi_0(r)$ and $\phi_0(p)$ for some potential shapes $U_0(r)$ and $V_0(p)$, in configuration and momentum space, respectively, for a ground state of energy $(-\gamma^2/2M)$.

Shape	$U_0(r)$	$V_0(p)$	$\psi_0(r)$	$\phi_0(p)$
Yukawa	$e^{-\mu r}/\mu r$	$[2\pi^2\mu(\mu^2+p^2)]^{-1}$	$r^{-1}[e^{-\gamma r}-e^{-a\mu r}]$	$(2/\pi)^{\frac{1}{2}}(a^2\mu^2-\gamma^2)/(p^2+\gamma^2)(a^2\mu^2+p^2)$
Exponential	$e^{-\mu r}$	$\mu[\pi^2(\mu^2+p^2)^2]^{-1}$	$r^{-1}[e^{-\gamma r}-e^{-a\mu r}]-(a^2\mu^2-\gamma^2)e^{-a\mu r}/2a\mu$	$(2/\pi)^{\frac{1}{2}}(a^2\mu^2-\gamma^2)^2/(p^2+\gamma^2)(a^2\mu^2+p^2)^2$
Gaussian[a]	$e^{-\mu^2r^2}$	$[2\pi^{\frac{1}{2}}\mu]^{-3}e^{-p^2/4\mu^2}$	$f(r)r^{-1}\{e^{-\gamma r}-\exp[-\mu^2r^2-(2\mu^2+\gamma^2)^{\frac{1}{2}}r]\}$	$e^{-p^2/4a^2\mu^2}/(p^2+\gamma^2)$

[a] $f(r)$ is a slowly varying function of r. See reference 2. For large values of p the iterations for the Gaussian potential converge very slowly and the correct asymptotic behavior $\phi(p)$ is somewhat different from that of $\phi_0(p)$.

mation still, $\phi_2(p)$, is obtained, etc. It will be more convenient to use, instead of the $\phi_n(p)$ themselves,

$$\Phi_n(p)=\lambda^{-n}\phi(p),$$

which are then defined by the equation,

$$(p^2+\gamma^2)\Phi_{n+1}(\mathbf{p})=-2M\int d^{(3)}p'\,V_0(\mathbf{p}-\mathbf{p}';\mu)\Phi_n(\mathbf{p}'). \quad (5)$$

No general discussion of the convergence, and hence of the range of validity, of this iteration procedure will be given in this paper, but in each particular application the convergence can be investigated fairly easily.

The first step then is to find a general prescription for a good initial wave function $\phi_0(p)$. For the case of a monotonically decreasing potential $V_0(p,\mu)$ the correct wave function will be a monotonic function, decreasing with increasing values of (p/μ) more rapidly than $V_0(p;\mu)$. If we took as $\phi_0(p)$ a three-dimensional delta-function, we would obtain a $\Phi_1(p)$ which is proportional to $(p^2+\gamma^2)^{-1}V_0(p;\mu)$. Such arguments suggest that we try an initial wave function of similar form, but with an extra dimensionless parameter a which can be adjusted to make agreement even better. Our prescription for $\phi_0(p)$ is then

$$\phi_0(p)=(p^2+\gamma^2)^{-1}V_0(p/a;\mu). \quad (6)$$

In Table I we give the form of $\phi_0(p)$ for the special cases of a potential of Yukawa, exponential, and gaussian shape, respectively, as well as the form of the potential both in momentum space, $V_0(p)$, and in coordinate space, $U_0(r)$. The fourier transforms of $\phi_0(p)$ can be calculated in most of these cases and give a rough, but usually simple, approximation $\psi_0(r)$ to the wave function in coordinate space. For the Yukawa potential $\psi_0(r)$ has the form of the well known wave function for a Hulthén potential. For the exponential potential $\psi_0(r)$ is also a simple function and is given in Table I. For the gaussian potential $\psi_0(r)$ is a rather complicated function (see Svartholm),[2] but a simple function which approximates $\psi_0(r)$ is given in Table I.

In many cases it is impossible, or at least too tedious, to carry out the integral in Eq. (5), with n equal to zero, analytically for all values of p. In these cases the conventional variation method for finding the "best value" for the parameter a is not very suitable. Usually, however, the integral can be carried out much more easily (usually analytically and certainly numerically) for special values of p, like $p=0$, $p\ll\mu$, $p=a\mu$, $p\gg\mu$. A simple method for finding reasonable values for a is to find $\Phi_1(p)$ for some of these special values p_1, p_2, etc., and define a by means of the equation,

$$\Phi_1(p_1)/\Phi_1(p_2)=\phi_0(p_1)/\phi_0(p_2). \quad (7)$$

Since both ϕ_0 and Φ_1 are in general monotonically decreasing our criterion (7) means that we are using a value of a which makes the rapidity of the decrease of $\Phi_1(p)$ with p roughly the same as that of $\phi_0(p)$.

Having fixed the $\phi_0(p)$ to be used by means of Eqs. (6) and (7), we have to carry out the integrals (5) to obtain the successive approximations $\Phi_n(p)$. If no analytic formula can be found for $\Phi_n(p)$ for a particular value of n, it is most convenient to calculate $\Phi_n(p)$ numerically for a number of values of p between zero and infinity and from this to determine, for these values of p, a function $a_n(p)$ defined by the relation,

$$\phi_n(p)=(p^2+\gamma^2)^{-1}V_0[p/a_n(p);\mu]. \quad (8)$$

Whereas $\Phi_n(p)$ is a fairly rapidly varying function, tending towards zero for large p, $a_n(p)$ is a very slowly varying function and takes on finite values of equal orders of magnitude for both large and small values of p. This makes it possible to interpolate fairly accurately for $a_n(p)$ after it has been determined for only a few values of p. This interpolation furnishes a good approximation to $\Phi_n(p)$ to be substituted in Eq. (5), $\Phi_{n+1}(p)$ is then calculated for a few values of p, and so on.

As mentioned before, if $\Phi_n(p)$ is known, it is usually easier to calculate $\Phi_{n+1}(0)$ than Φ_{n+1} for other values of p. We therefore define an approximation to the eigenvalue λ of (3) by means of

$$\lambda_{n+1}=\Phi_n(0)/\Phi_{n+1}(0). \quad (9)$$

The approximations λ_n do not necessarily (unlike the approximations of Svartholm[2]) form a monotonically decreasing series, but they are much easier to calculate and approximate λ very closely for even quite low values of n.

As a very simple (and somewhat fortunate) example, consider the case of a Yukawa potential for which the binding energy of the ground state is known to be zero, $\gamma=0$. Equation (7) for p_1, p_2 equal to 0 and ∞, respectively, gives a as the solution of

$$a+1=a^2;\quad a\approx 1.62. \quad (10)$$

A similar calculation for $p_1=0$, $p_2\ll\mu$ gives a value for

a of 1.37. Taking a mean of these values, $a=1.50$, Eqs. (5) and (9) give then $\lambda_1 \approx 1.67$ after a very simple integration. In this case $\Phi_1(p)$ can be calculated analytically, and $a_1(p)$ varies from $a_1(0)=1.39$ to $a_1(\infty)=1.58$. Then λ_2 can be calculated, its value being 1.6823. The values of $a_1(p)$ are then accurate to within about 1 percent and λ_2 to within 1 in 10,000.

Approximations to $\phi(p)$ for the case of a Yukawa, exponential, and gaussian potential are being calculated by Mr. J. Goldstein for values of γ/μ corresponding to the experimental values for the triplet effective range for the n-p system and for the binding energy of the deuteron. These calculations are carried as far as $\Phi_2(p)$ and λ_3 and will be reported on in a later paper.

The integral equation (3) holds also for the states of higher angular momentum. For a central potential $V(p)$ the wave function $\phi(\mathbf{p})$ takes the general form,

$$\phi_{l,m}(\mathbf{p})=\phi_l(p)Y_{l,m}(\theta,\phi), \tag{11}$$

where $Y_{l,m}(\theta,\phi)$ is a spherical surface harmonic in p-space. Using the spherical harmonics addition theorem one can derive from Eq. (3) the equation,[4]

$$(p^2+\gamma^2)\phi_l(p)=-4\pi M\int_0^\infty dp'p'^2V_l(p;p')\phi_l(p'),$$
$$V_l(p;p')\equiv\int_{-1}^{1}dxV[(p^2+p'^2-2pp'x)^{\frac{1}{2}}]P_l(x). \tag{12}$$

For l bigger than zero $V_l(p;p')$ is no longer a monotonically decreasing function of p, $V_l(0;p')$ being zero, but will usually not change sign. For the lowest energy state, for a given value of l, $\phi_l(p)$ will have no nodes and an iteration method could be used to solve (12). A reasonable initial wave function $\phi_{l,0}(p)$ could be used of the form,

$$\phi_{l,0}(p)=(p^2+\gamma^2)^{-1}V_l(p/a;\mu). \tag{13}$$

Such an iteration method will probably still converge fairly rapidly even for moderately large values of l, but probably not as rapidly as for $l=0$. The method described in this section is not suitable without modification for the higher states for which the wave function has nodes.

III. SCATTERING IN A CENTRAL POTENTIAL

Consider the problem of the scattering of a nonrelativistic particle of definite energy E by a fixed central potential. If the wavelength of the particle is not too small compared with the range μ^{-1} of the potential, a partial wave analysis is a convenient method for solving the problem. Usually one attempts to solve the Schrödinger differential equation for $\psi_l(r)$, the radial wave function in configuration space for each particular angular momentum l, and the asymptotic phases η_l then determine the scattering cross section completely.

[4] Equation (12) is derived in Appendix 1.

A method of partial waves can also be applied in momentum space. The wave function $\phi_l(\mathbf{p})$ in momentum space for a particular value of l is of the form (11) with m equal to zero. For a scattering problem $\phi_l(p)$ satisfies an equation which is similar to Eq. (12); except that (a) γ^2 is replaced by $(-k^2)$, where

$$k=(2ME)^{\frac{1}{2}}$$

and (b) that the left-hand side contains an additional term proportional to $\delta(p-k)$ whose coefficient determines the phase η_l. For a monotonically decreasing potential $V(p)$ of sufficiently low strength, so that at most one bound state for a particular value of l is possible, and for sufficiently small energy ($k \lesssim \mu$), it seems likely that the methods described in the last section can be applied to obtain successive approximations to $\phi_l(p)$ and η_l, which converge fairly rapidly. This method might be particularly useful for some scattering problems for very low energy where S-scattering is predominant and where the momentum distribution for a fixed wave-number k (obtained automatically with this method) is of interest. We shall restrict ourselves in this section to the case of l equal to zero and shall omit the subscript l.

The asymptotic expression for the wave function in configuration space for $l=0$, $\psi(r)$, is then proportional to

$$\psi^{as}(r)=\sin(kr+\eta)/r\sin\eta, \tag{14}$$

where η is the S-phase which determines the S-wave scattering cross section. If $\psi(r)$ is normalized so that Eq. (14) is its asymptotic expression, then

$$\rho(E,E)=2\int_0^\infty drr^2\{|\psi^{as}(r)|^2-|\psi(r)|^2\} \tag{15}$$

is finite and is equal to the effective range as defined by Blatt and Jackson[5] and by Bethe.[6] The fourier transform of $\psi^{as}(r)$ is

$$\phi^{as}(p)=(2/\pi)^{\frac{1}{2}}\beta^{-1}\{\delta(p-k)+\beta/(p^2-k^2)\},$$
$$\beta\equiv(2/\pi)k\tan\eta, \tag{16}$$

where $\delta(p-k)$ is a Dirac delta-function.[7] Since $\rho(E,E)$ is merely a multiple of the difference of the normalization integrals of the two configuration space wave functions, it can also be written as

$$\rho(E,E)=2\int_0^\infty dpp^2\{|\phi^{as}(p)|^2-|\phi(p)|^2\} \tag{17}$$

if $\phi(p)$ is normalized to be the fourier transform of $\psi(r)$. For the integral (17) to be finite, $\phi(p)$ and $\phi^{as}(p)$ must be identically equal for $p\to k$.

The wave function $\phi(p)$ satisfies an integral equation,

[5] J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949).
[6] H. A. Bethe, Phys. Rev. **76**, 38 (1949).
[7] A derivation of Eq. (16) and some discussion is given in Appendix 2.

equivalent to Eq. (3),

$$(p^2-k^2)\{\phi(\mathbf{p})+c\delta(p-k)\}$$
$$=-2M\int d^{(3)}p' V(\mathbf{p}-\mathbf{p}')\phi(\mathbf{p}'), \quad (18)$$

where the constant c is as yet undetermined. We are interested only in the case $k\lesssim\mu$, for which the integral occurring on the right-hand side of Eq. (18) will be of fairly similar form to that occurring in Eq. (3), and an initial trial wave function analogous to Eq. (6) for $p\neq k$ should again be a good starting point. In the present case, however, we must choose $\phi(p)$ so that it is proportional to Eq. (16) for $p\to k$. We therefore take an initial wave function proportional to

$$\phi_0(p)=\{(p^2-k^2)^{-1}+\delta(p-k)/\beta_1\}V_0(p/a;\mu) \quad (19)$$

with β_1 left unspecified.

In the present problem both the total energy E and the strength of the potential are given. The unknown parameter to be determined by the iteration method is the constant β, of the same dimensions as the wave number k, which determines η and hence the S-wave scattering cross section. We define successive approximations to $\phi(p)$ by means of

$$\phi_n(\mathbf{p})=-\{(p^2-k^2)^{-1}+\delta(p-k)/\beta_{n+1}\}$$
$$\times\int d^{(3)}p' V(\mathbf{p}-\mathbf{p}')\phi_{n-1}(\mathbf{p}'), \quad (20)$$

starting with $n=1$. The integral in Eq. (20) is first evaluated with β_n in the expression for $\phi_{n-1}(p')$ left unspecified. Then β_n is determined by requiring that

$$\phi_n(p_0)=\phi_{n-1}(p_0) \quad (21)$$

for some suitably chosen fixed value, p_0, of $p\neq k$ (e.g., zero or a value differing from k by an infinitesimally small amount). The values β_n are then successive approximations to the parameter β which determines the phase shift. Before this iteration process can be carried out, a reasonable value for the parameter a must be found. This is again done by using Eq. (7). Equation (21) with $n=0$ and Eq. (7) both contain a and β_1 as unknowns, and these two equations have to be solved simultaneously. Since the exact choice of a is not very important, an approximate solution of these two simultaneous equations is sufficient.

One nice feature of this method is that it converges fairly rapidly both for a very weak potential and for one strong enough to allow one (but no more) bound state. For a potential, for instance, strong enough to introduce a phase-shift near $\pi/2$ (resonance) the values obtained for β_n will be very large compared with k; the term involving the delta-function in the integrand of Eq. (20) will be unimportant, and the iteration will proceed in close analogy to that for bound states.

For a very weak potential, on the other hand, Eqs. (20) and (21) with $n=1$ will yield a value of β_1 much smaller than k and the term involving the delta-function in the integrand of Eq. (21) will be much more important than the term involving $(p^2-k^2)^{-1}$. Complete omission of this term involving $(p^2-k^2)^{-1}$ in the expression for $\phi_0(p')$ in Eq. (21) would be equivalent to first-order Born approximation. With p_0 differing from k by an infinitesimally small amount, and making this omission, Eqs. (20) and (21) yield an approximation to β_1 which is quite independent of the value used for a:

$$\beta_1^{(B)}=-16\pi Mk^2\int_0^1 dyyV(2ky). \quad (22)$$

Since $\beta_1\ll k$, Eq. (16) gives

$$\eta^{(B)}\approx\tan\eta^{(B)}=-8\pi^2 Mk\int_0^1 dyyV(2ky). \quad (23)$$

A simple calculation shows that this expression is identical with the usual first-order Born approximation to η:

$$\eta^{(B)}=-2Mk^{-1}\int_0^\infty dr\sin^2(kr)U(r). \quad (24)$$

More generally, an expression independent of the choice of a and equivalent to the nth Born approximation could be derived from the nth iteration of the present method by the omission of certain terms. If, however, the iterations are carried out as described above without any omissions, then even β_1, obtained after one single iteration, contains (at least approximately) corrections proportional to higher powers of the potential, in addition to the linear term $\beta_1^{(B)}$. In fact, as discussed in the last paragraph, β_1 remains a fairly good approximation even for a fairly strong potential for which the Born approximation breaks down completely.

IV. OTHER APPLICATIONS

(A) Tensor Forces

The method discussed in the preceding two sections for the evaluation of momentum-space wave functions for a two-particle system can be extended to the case of an interaction potential which contains a tensor-force term in addition to the spherically symmetrical central-potential term. A potential in configuration-space of the form,

$$U_c(r)+gU_T(r)S_{12}(\theta_r) \quad (25)$$

corresponds to a momentum-space potential of form,

$$V_c(p)+gV_T(p)S_{12}(\theta_p), \quad (26)$$

where

$$S_{12}(\theta_r)=3[(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})/r^2]-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) \quad (27)$$

and $V_T(p)$ is proportional to

$$\int d^{(3)}rU_T(r)e^{i\mathbf{p}\cdot\mathbf{r}}P_2(\cos\theta_r). \quad (28)$$

For most of the commonly used simple forms of $U_T(r)$ the function $V_T(p)$ is usually fairly complicated. There are, however, a few types of tensor-potentials for which both $U_T(r)$ and $V_T(p)$ are fairly simple functions. For problems where calculations have to be carried out both in configuration and in momentum space it might be useful to employ such potentials. For instance,

$$V_T(p)=p^2/(p^2+\mu^2)^2 \tag{29}$$

corresponds to a $U_T(r)$ proportional to

$$(r^{-1}+\mu)e^{-\mu r}. \tag{30}$$

A consideration of momentum-space potentials leads to a more direct understanding of the well-known fact that, for very low energies, a tensor potential of form $U_T(r)$ is much less important than a central potential of the same form, $U_c(r)=U_T(r)$. Note, for instance, that $V_T(p)$ of the form (29) is proportional to p^2 for small momenta p, whereas the momentum-space potential $V_c(p)$ corresponding to a central potential $U_c(r)$ of form (30) is not.

The momentum-space wave function for the neutron-proton triplet system, for instance, with an interaction potential given by Eq. (26) is of the form,

$$\phi_c(p)\chi_m{}^3+\phi_T(p)\sum_{m'=-1}^{1}a_{m,m'}Y_{2,m'}(\theta,\phi)\chi_{-m'}{}^3 \tag{31}$$

where $\chi_m{}^3$ denotes one of the triplet spin-wave functions. The Schrödinger equation in momentum space can then be reduced to two coupled integral equations involving the two functions $\phi_c(p)$ and $\phi_T(p)$. For the case of the ground state of the deuteron, where the percentage of D-state is known to be small, an iteration method analogous to that described in Sec. II should be applicable. As initial trial wave functions one might take, in this case,

$$\begin{aligned}\phi_c(p)&=(p^2+\gamma^2)^{-1}V_c(p/a);\\ \phi_T(p)&=(p^2+\gamma^2)^{-1}GV_T(p/b),\end{aligned} \tag{32}$$

where there are now three parameters, a, b, and G, to be determined (instead of one, a, as in Sec. I).

(B) Three-Body Problem

The problem of the ground states of H^3 and He^3, using only central potentials, has already been treated by Svartholm[2] using a variation-iteration procedure in momentum-space. The momentum space wave function is here a function of two momentum variables $\mathbf{p}_1$ and $\mathbf{p}_2$. As Svartholm[2] has already pointed out it is convenient to employ as an initial wave function $\phi_0(\mathbf{p}_1,\mathbf{p}_2)$ a function of $(p_1{}^2+p_2{}^2+\mathbf{p}_1\cdot\mathbf{p}_2)$ only. As a starting point, either for a variation-iteration method as used by Svartholm or for an iteration method as described in this paper, more rapid convergence is obtained when an initial function of form,

$$\begin{aligned}\phi_0(\mathbf{p}_1,\mathbf{p}_2)=&(p_1{}^2+p_2{}^2+\mathbf{p}_1\cdot\mathbf{p}_2+\gamma^2)^{-1}\\ &\times V_0[(p_1{}^2+p_2{}^2+\mathbf{p}_1\cdot\mathbf{p}_2)/a;\mu]\end{aligned} \tag{33}$$

is used than with the functions previously used by Svartholm.[2]

(C) Relativistic Two-Body Equation

In the following paper[8] an integral equation is derived, from a relativistic treatment of the two-body problem, for a wave function $\psi(\mathbf{p},\epsilon)$, which is a function of a momentum variable p and an energy variable ϵ. This integral equation is of the form,

$$\begin{aligned}&F(p,\epsilon)\psi(\mathbf{p},\epsilon)\\ &=-\lambda\int\int d^{(3)}p'd\epsilon V_0(\mathbf{p}-\mathbf{p}',\epsilon-\epsilon';\mu)\psi(p',\epsilon').\end{aligned} \tag{34}$$

By analogy with Eqs. (3) and (6) an initial trial wave function of the form,

$$\psi_0(p,\epsilon)=F^{-1}(p,\epsilon)V_0(p/a,\epsilon/b;\mu) \tag{35}$$

can be used. Reasonable values for the two parameters a and b can then be found by means of the equation (analogous to Eq. (7)),

$$\psi_1(p_1,\epsilon_1)/\psi_1(p_2,\epsilon_2)=\psi_0(p_1,\epsilon_1)/\psi_0(p_2,\epsilon_2) \tag{36}$$

for suitably chosen combinations of the values p_1, ϵ_1 and p_2, ϵ_2.

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APPENDIX I. DERIVATION OF EQ. (12)

We shall first derive a general property of spherical harmonics, Eq. (A4). Let

$$Y_{l,m}(\Theta)=P_l{}^{|m|}(x)e^{im\phi},$$

where Θ denotes the unit vector (θ,ϕ) and $x=\cos\theta$. From the spherical harmonics addition theorem for $P_l(\cos|\Theta-\Theta_0|)$ and from the normalization integrals for the associated Legendre polynomials[9] it follows that

$$\begin{aligned}\int_{-1}^{1}dx'\int_0^{2\pi}d\phi'Y_{l,m}(\Theta')P_l(\cos|\Theta'-\Theta|)\\ =(4\pi/2l+1)Y_{l,m}(\Theta).\end{aligned} \tag{A1}$$

We can expand $Y_{l,m}(\Theta+\Theta_0)$ in terms of $Y_{l,m'}(\Theta)$,

$$Y_{l,m}(\Theta+\Theta_0)=\sum_{m'=-l}^{l}a_{m'}(\Theta_0)Y_{l,m'}(\Theta). \tag{A2}$$

[8] E. E. Salpeter and H. A. Bethe, Phys. Rev. 84, 1232 (1951).
[9] E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, New York, 1945), pp. 115 and 116.

Multiplying Eq. (A2) by $P_l(x)$ and integrating over x and ϕ, we obtain, after changing the variable of integration from Θ to $\Theta'\equiv(\Theta+\Theta_0)$ and using Eq. (A1),

$$a_0(\Theta_0)=Y_{l,m}(\Theta_0). \tag{A3}$$

If $f(\theta)$ is an arbitrary function of θ, not depending on ϕ, it follows from the above equations and from the orthogonality property of the spherical harmonics that

$$\int_{-1}^{1}dx\int_0^{2\pi}d\phi Y_{l,m}(\Theta+\Theta_0)f(\theta) =2\pi Y_{l,m}(\Theta_0)\int_{-1}^{1}dxP_l(x)f(\theta). \tag{A4}$$

Consider Eq. (3) for $\phi_{l,m}(\mathbf{p})$ of the form (11) with $\mathbf{p}$ parallel to the unit vector Θ_0. Let Θ now denote spherical polar coordinates with the vector Θ_0 taken as axis. Equation (3) then becomes

$$(p^2+\gamma^2)\phi_l(p)Y_{l,m}(\Theta_0) =-2M\int_0^{\infty}dp'p'^2\phi_l(p')\int_{-1}^{1}dx\int_0^{2\pi}d\phi \times V[(p^2+p'^2-2pp'x)^{\frac{1}{2}}]Y_{l,m}(\Theta+\Theta_0). \tag{A5}$$

Using Eq. (A4), we may reduce Eq. (A5) (after dropping the suffix zero) to Eq. (12) of the text.

APPENDIX II. DERIVATION OF EQ. (16)

We first note that, for an arbitrary analytic function $f(y)$,

$$\begin{aligned}&\operatorname*{Lim}_{\epsilon\to0}\int_{-\infty}^{\infty}dyf(y)\int_0^{\infty}dxe^{-\epsilon x}\cos(yx)=\pi f(0),\\ &\operatorname*{Lim}_{\epsilon\to0}\int_{-\infty}^{\infty}dyf(y)\int_0^{\infty}dxe^{-\epsilon x}\sin(yx) =\mathcal{P}\left\{\int_0^{\infty}dyf(y)/y\right\},\end{aligned} \tag{A6}$$

where $\bar{G}$ denotes the principal part. The Dirac delta-function $\delta(y)$ can be defined by means of its property

$$\int_{-\infty}^{\infty}dyf(y)\delta(y-a)=f(a). \tag{A7}$$

We therefore have the relations,

$$\begin{aligned}&\operatorname*{Lim}_{\epsilon\to0}\int_0^{\infty}dxe^{-\epsilon x}\cos(yx)=\pi\delta(y),\\ &\operatorname*{Lim}_{\epsilon\to0}\int_0^{\infty}dxe^{-\epsilon x}\sin(yx)=(1/y),\end{aligned} \tag{A8}$$

where it is to be understood that the principal part has to be taken of any integral in which this term $(1/y)$ occurs.

To obtain the fourier transform $\phi^{as}(\mathbf{p})$ of the wave function $\psi^{as}(\mathbf{r})$, Eq. (14), we first make the fourier integral definite by adding a convergence factor $e^{-\epsilon r}$ and then proceed to the limit of ϵ tending to zero.

$$\begin{aligned}\phi^{as}(p)&=(2\pi)^{-\frac{3}{2}}\operatorname*{Lim}_{\epsilon\to0}\int_0^{\infty}drr^2e^{-\epsilon r}\int_{-1}^{1}dx \times\int_0^{2\pi}d\phi e^{iprx}\sin(kr+\eta)/r\sin\eta\\ &=(2\pi)^{-\frac{1}{2}}(p\sin\eta)^{-1}\operatorname*{Lim}_{\epsilon\to0}\int_0^{\infty}dre^{-\epsilon r} \times\{\sin\eta[\sin(pr+kr)+\sin(pr-kr)]\\ &\quad+\cos\eta[\cos(pr-kr)-\cos(pr+kr)]\}.\end{aligned} \tag{A9}$$

Remembering that both p and k are positive, so that $\delta(p+k)$ can be replaced by zero, and using Eqs. (A8), we then have

$$\phi^{as}(\mathbf{p})=(\pi/2)^{\frac{1}{2}}\cot\eta\{p^{-1}\delta(p-k) +(2/\pi)\tan\eta/(p^2-k^2)\}. \tag{A10}$$

Since, from Eq. (A7), we have

$$(p-k)\delta(p-k)=0, \tag{A11}$$

the factor p^{-1} in Eq. (A10) can be replaced by k^{-1}, and Eq. (A10) then reduces to Eq. (16) of the text.

It also follows from Eq. (A11) that the coefficient c of the delta-function in the integral equation (18) is completely arbitrary. Hence, $\phi(k)$ cannot be determined from Eq. (18) alone, but is determined from the condition $\phi(k)=\phi^{as}(k)$. It should be noted that, throughout Sec. III of this paper, the principal part has to be taken of any integral over p in which the term $(p^2-k^2)^{-1}$ occurs.