On the Theory of Multiple Scattering, Particularly of Charged Particles*

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The general theory of the elastic multiple scattering of particles with a strongly anisotropic scattering function is investigated without making the small-angle approximation. The rigorous transport equation is used and approximations are introduced at a later stage. The paper consists of four parts. In the first part the general formulation of the problem is given. The approximations involved in the existing theories of small-angle forward scattering are discussed in some detail. In the second part the spherical harmonic method is formulated in a manner so as to permit an explicit expression for the general nth approximation. There is an ambiguity both in (a) the way of defining successive approximations and in (b) the way of introducing approximate boundary conditions. We have chosen (b) to give the best approximation to the exact solution of the Schwarzschild-Milne problem. In the third part it is shown that our choice of (a) for the spherical harmonic method leads to the same final formulas as the gaussian quadrature method. The relation of these two methods is discussed in detail. In the fourth part the problem of anisotropic multiple scattering is reduced to a quasi-isotropic one by using a generalized Goudsmit-Saunderson type distribution function (defined also for back scattering) as a first approximation. Three different methods are given for forward scattering (including large angles). The first method is a perturbation treatment. The second method is based on the approximate delta-function character of the scattering function and employs a Fokker-Planck type development for the peaked part of the scattering function. The third method is a Liouville-Neumann type of iteration applied directly to the transport equation. For back scattering the second and third of these methods also apply. In addition a special method is developed, based on the smallness of the back single scattering cross section. The generalized Goudsmit-Saunderson distribution function is developed in powers of the thickness of the scatterer and it is shown that all three methods lead to the same single scattering tail. The three methods can be applied for any single scattering law. The screened Born-Rutherford law is introduced in some cases as an example. The relation of the present work to previous theories is discussed.

I. INTRODUCTION

THE theory of multiple scattering has been developed in the past primarily for three groups of problems, namely, (1) radiative transfer, (2) multiple scattering of neutrons, and (3) multiple scattering of charged particles, in particular of electrons. In any problem of multiple scattering we have two steps to consider (a) the law of single scattering, (b) the statistical problem of obtaining the spatial and angular distribution of the multiply scattered light or particles, which is properly governed by a Boltzmann integro-differential equation.

We shall be concerned with the case of axial symmetry in the single scattering law and multiple scattering in a plane-parallel stratified medium. Then the distribution function will depend only on two variables: a cartesian and a polar coordinate describing the spatial and angular behavior respectively. The difficulties inherent in the statistical problem of multiple scattering are well illustrated by the fact that a closed expression for the angular and spatial distribution was obtained only for the simplest problem in multiple scattering.

A method of expanding the distribution function into Legendre polynomials[1] (spherical harmonic method, called SH in the sequel) has been applied by Gratton[2] and Chandrasekhar[3] to the isotropic problem, unfortunately using an inconvenient way of defining successive approximations. This prohibited the general formulation of the method, and made it look inferior to the method of gaussian quadrature (called GQ in the sequel). Wick[4] replaced the integral in the Boltzmann equation by a sum employing gaussian quadrature.[5] Guided by the nth approximation in the GQ method, Chandrasekhar[6] obtained exact expressions for the angular distribution for semi-infinite and finite media.

The SH method was used by several authors, notably Marshak,[7] Mark, Glauber, and Rarita from 1944 on for a variety of neutron diffusion problems. To the authors' knowledge, however, no general expressions were obtained for the nth approximation for the general anisotropic problem, nor was the relation to the GQ method treated in a general manner. The general form of the SH method as developed in this paper will be the principal method we employ. There is an ambiguity in the way of defining (a) successive approximations and (b) approximate boundary conditions in both the SH and the GQ method. This ambiguity does not seem to have been recognized in the literature. The usual procedure for (a) and (b) in the GQ method was chosen for its analytical simplicity. We shall show that for the SH method (a) can be chosen in a natural way so that the GQ and the SH methods give exactly the same

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[1] Such methods have, of course, been applied to a great extent to even more complicated Boltzmann equations; see for instance, E. Guth and J. Mayerhofer, Phys. Rev. **57**, 908 (1940).

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[3] S. Chandrasekhar, Astrophys. J. **99**, 180 (1944).

[4] G. C. Wick, Z. Physik **121**, 702 (1943).

[5] The GQ method has been applied to integral equations earlier by Nystrom, Acta Math. **54**, 185 (1930).

[6] S. Chandrasekhar, *Radiative Transfer* (Clarendon Press, Oxford, 1950).

[7] R. E. Marshak, Phys. Rev. **71**, 443 (1947).

solution. Thus GQ and SH methods differ only in analytical details, which do not affect the final form of the solution. (This is not the case when, for the SH method, Gratton or Chandrasekhar's (a) is chosen.) For the SH method our (b) differs from that used in the GQ method. Our (b) was chosen among other reasonable choices to give the best approximation to the exact solution for the isotropic case. This approximation is better than the one obtained by (b) of the GQ method. In the limit $n\rightarrow\infty$, both the GQ (b), and our (b), lead to the same exact solution. Although the GQ method and our formulation of the SH method lead to the same final result, there is some analytical advantage to the use of GQ for isotropic or quasi-isotropic problems, while SH is simpler to apply to anisotropic cases. For strongly anisotropic problems exact solutions involving Chandrasekhar's H and X, Y-functions are not practicable, since very many of these functions would have to be tabulated. For quasi-isotropic problems, on the other hand, one needs only a few of these functions.

Placzek[8] has shown that a simple iteration procedure using integral properties of the known exact solution leads to a more accurate solution than polynomial approximations using approximate boundary conditions. For the anisotropic problem, unfortunately, Placzek's procedure cannot be used, because the exact solution is not known.

The simpler features of the multiple scattering of charged particles have been described in a very instructive manner in a recent paper by Bohr.[9] For this reason we shall restrict ourselves to a brief summary of the attempts toward a more rigorous theory. Bothe[10] started with the correct Boltzmann integro-differential equation, but did not state the exact boundary conditions. From the Boltzmann equation he derived a Fokker-Planck type of differential equation. Though the transition "from Boltzmann to Fokker-Planck" assumes the "small-angle approximation," Bothe retained in the latter a factor ($\cos\theta$) which goes to unity in that approximation. He then tried to solve this Fokker-Planck type equation with an inexact boundary condition. Bethe, Rose, and Smith[11] have tried to obtain a solution of the same differential equation with the exact boundary conditions. We shall show later [see Sec. II(C)], however, that it seems doubtful whether such a solution does exist. Bethe, Rose, and Smith also derived a diffusion equation for back scattering from thick foils by a procedure similar to the "age" theory of neutron diffusion. Williams[12] developed a consistent theory in the small-angle approximation based on the Fokker-Planck equation, putting the factor mentioned ($\cos\theta$) equal to unity and using an approximate boundary condition neglecting the "back scattering." In this approximation the problem is very similar to rotatory Brownian motion, i.e., diffusion on a sphere. Goudsmit and Saunderson[13] developed a more accurate theory, which can be formulated as follows: one applies the small-angle approximation (putting $\cos\theta=1$) to the Boltzmann equation (instead of the Fokker-Planck equation of Williams), and uses the same approximate boundary condition as Williams does neglecting "back scattering." Goudsmit and Saunderson point out that a parameter in their theory can be considered either as the thickness of the medium or as path length, and that their solution is an exact one for the somewhat unrealistic problem of equal path lengths. Molière[14] uses an older theory of Wentzel[15] to derive an expression which is just that of Goudsmit and Saunderson if one replaces a series by an integral, and evaluates the integral. Snyder and Scott[16] derive essentially the same integral form from equations equivalent to the approximate Boltzmann equation which lead to the Goudsmit-Saunderson theory. These authors give the most extensive numerical evaluation of their formulas. For thin scatterers both Moliere and Snyder-Scott show how the multiple scattering approaches the single scattering tail. A different approach by Butler[17] leads for thin scatterers also to a separation of the gaussian multiple scattering and the single scattering tail in the small angle approximation. Butler's procedure was generalized to larger angles, still neglecting back scattering, by Teichmann.[18] Lewis[19] treats multiple scattering in an infinite medium using the path-length as a variable. He re-derives the Goudsmit-Saunderson solution and shows the transition to the Snyder-Scott integral form. One may solve the problem by getting a functional relation between path length and actual thickness of the foil, but, of course, this is just as complicated as the original problem. Rose,[20] in a note, gives an approximate expression for the path length-thickness ratio.

As is seen from this brief summary, no treatment of the exact Boltzmann equation with the exact boundary condition seems to exist in the literature. Since all consistent approximative treatments neglect backscattering, no reliable theory of this phenomenon is available in the literature either. (Bothe[21] has recently given elementary considerations on backscattering.)

It would be very cumbersome to apply in a straightforward manner the SH (or GQ) method to the case of

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[17] S. T. Butler, Proc. Phys. Soc. (London) **A63**, 599 (1950).
[18] T. Teichmann, *Multiple Scattering of High Energy Charged Particles in Thin Foils*, Palmer Physical Laboratory, Princeton University, Princeton, New Jersey, unpublished.
[19] H. W. Lewis, Phys. Rev. **78**, 526 (1950).
[20] M. E. Rose, Phys. Rev. **58**, 90 (1940).
[21] W. Bothe, Ann. Physik **6**, 44 (1949).

a strongly anisotropic single scattering, like Rutherford scattering. We shall show, however, how the anisotropic problem can be reduced to a quasi-isotropic one. Most of our solutions use the Goudsmit-Saunderson solution as a first approximation. However, we always start from the exact Boltzmann equation with the exact boundary conditions and make approximations at a later stage.

Section II introduces (A) the single scattering function, (B) the exact Boltzmann equation with the exact boundary condition, and (C) the different kinds of approximation for forward scattering. Section III brings the development of the SH method for the isotropic problem without incident current. Section IV contains the application of the SH method (A) to the approximate Boltzmann equation neglecting back scattering (equivalent to Goudsmit-Saunderson theory), and (B) to the exact Boltzmann equation of II(B). Section V makes a comparison of the SH and GQ methods for the case of (A) isotropic scattering and (B) anisotropic scattering, in which we also give the GQ procedure of solving the problem in IV(A). The identity of our SH and the customary GQ method is shown explicitly, apart from the boundary conditions. For the case of the isotropic problem of III, a numerical comparison of the exact, SH and GQ methods is presented. Section VI presents our theory of multiple scattering of charged particles, particularly of electrons, by reducing the anisotropic problem to a quasi-isotropic one. The sub-section (A) on forward scattering consists of: (1) a perturbation treatment using the Goudsmit-Saunderson solution as a first approximation, (2) a simple improvement of the Goudsmit-Saunderson solution based on the δ-function character of the single Rutherford scattering, and (3) an iteration procedure starting again with the Goudsmit-Saunderson solution. The subsection (B) on back scattering consists of (1) a procedure of solving the exact Boltzmann equation based on the smallness of the "backward" single scattering cross section, (2) an approximate solution similar to (A2), and (3) an iteration method similar to (A3) In subsection (C) it is shown that as the thickness of the scatterer approaches zero all three methods lead to the same single scattering tail. This is exhibited both for forward and for back scattering, using a straightforward expansion of the distribution function in powers of the thickness.

II. GENERAL FORMULATION

(A) Law of Elastic Single Scattering for Axially Asymmetric Scattering Potential

Let α be the angle between the incident velocity and the scattered velocity, and $\Phi(\cos\alpha)$ be the single scattering law. The total scattering cross section σ is given by

$$\sigma=\int_0^{2\pi} d\phi \int_0^{\pi} \Phi(\cos\alpha)\,\sin\alpha d\alpha,$$

where ϕ is the azimuth angle. Then we define our scattering function by

$$p(\cos\alpha)=(4\pi/\sigma)\Phi(\cos\alpha), \tag{1}$$

so that when we develop $p(\cos\alpha)$ in Legendre polynomials

$$p(\cos\alpha)=\sum_{r=0}^{\infty}\omega_r P_r(\cos\alpha), \tag{2}$$

we have $\omega_0=1$. $\Phi(\cos\alpha)$ is related to the amplitude of the scattered wave $f(\cos\alpha)$ by

$$\begin{aligned}\Phi(\cos\alpha)&=|f(\cos\alpha)|^2\\ &=|(1/2ik)\textstyle\sum_r(2r+1)(e^{2i\delta_r}-1)P_r(\cos\alpha)|^2,\end{aligned} \tag{3}$$

where the δ_r's are the phases. The relation between the ω_r's and the δ_r's can be obtained by using the formula for the integration of the product of three Legendre polynomials given by Rose *et al.*[22]

Now with reference to an arbitrary spherical coordinate system, the directions of the incident velocity and the scattered velocity can be specified by the angles θ, ϕ and θ', ϕ' respectively. These angles are related to the angle α by the equation

$$\cos\alpha=\mu\mu'+[(1-\mu^2)(1-\mu'^2)]^{\frac{1}{2}}\cos(\phi-\phi') \tag{4}$$

where $\mu=\cos\theta$ and $\mu'=\cos\theta'$. Equation (2) then becomes

$$p(\mu,\phi,\mu',\phi')=\sum_{r=0}^{\infty}\omega_r P_r\langle\mu\mu'+[(1-\mu^2)(1-\mu'^2)]^{\frac{1}{2}}\cos(\phi-\phi')\rangle.$$

Expanding the Legendre polynomials by the addition theorem and integrating over the variable ϕ, we get

$$p(\mu,\mu')=\frac{1}{2\pi}\int_0^{2\pi}p(\mu,\phi,\mu',\phi')d\phi=\sum_{r=0}^{\infty}\omega_r P_r(\mu)P_r(\mu'). \tag{5}$$

Practically, one does not have to take the whole infinite series (2). There is always a maximum term at which one can terminate the series with no appreciable error.

(B) Boltzmann Equation

An axially symmetric beam of particles is incident on a plane slab of scattering material with two of its dimensions infinite. Let the axis parallel to the finite dimension be the x-axis, which is also the axis of symmetry of the incident beam. The beam is incident on the surface $x=0$, and the other surface is $x=a>0$. The steady-state distribution function of the scattered

[22] Rose, Goertzel, Spinrad, Harr, and Strong, Phys. Rev. 83, 79 (1951), Eq. (11a).

particles obeys the well-known Boltzmann equation

$$\mu(\partial f/\partial\tau)+f=\tfrac{1}{2}\int_{-1}^{1}f(\tau,\mu')p(\mu,\mu')d\mu', \tag{6}$$

where μ is the cosine of the angle between the velocity of the particle and the x axis, $\tau=N\sigma x$, and N is the number of scattering centers per unit volume. σ and $p(\mu,\mu')$ are defined above. The problem is to solve Eq. (6) with the boundary conditions

$$f(0,\mu)=\pi F\delta(\mu-\mu_0),\quad \mu>0, \tag{7a}$$

$$f(t,\mu)=0,\qquad \mu<0, \tag{7b}$$

where $t=N\sigma a$, and $\mu_0>0$. For simplicity here we have used a δ-function as the incident beam. Since the integral equation is linear, the solution for any arbitrary axially symmetric incident beam is just a superposition of such fundamental solutions. The integral of the δ-function over the whole solid angle is normalized to unity, while the strength of the incident beam is governed by the constant F.

If we define the integral in Eq. (6) by the function $J(\tau,\mu)$, i.e.,

$$J(\tau,\mu)=\tfrac{1}{2}\int_{-1}^{1}f(\tau,\mu')p(\mu,\mu')d\mu', \tag{8}$$

Eq. (6) becomes

$$\mu(\partial f/\partial\tau)+f=J(\tau,\mu). \tag{9}$$

The formal solution of Eq. (9), satisfying the boundary conditions (7), is clearly

$$f(\tau,+\mu)=e^{-\tau/\mu}\int_0^{\tau}J(\tau,\mu)e^{\tau/\mu}(d\tau/\mu)+\pi \mathrm{F}e^{-\tau/\mu}\delta(\mu-\mu_0), \tag{10a}$$

$$f(\tau,-\mu)=e^{\tau/\mu}\int_{\tau}^{t}J(\tau,-\mu)e^{-\tau/\mu}d\tau/\mu. \tag{10b}$$

Here μ is in absolute value. The expressions (10a) and (10b) correspond to the range $\mu>0$ and $\mu<0$ respectively. If one substitutes Eq. (10) into Eq. (8), one gets an integral equation for $J(\tau,\mu)$. That is

$$J(\tau,\mu)=\tfrac{1}{4}Fp(\mu,\mu_0)e^{-\tau/\mu_0}+\tfrac{1}{2}\left[\int_0^{\tau}d\tau'\int_0^1 d\mu'-\int_{\tau}^{t}d\tau'\int_{-1}^{0}d\mu'\right]HJ(\tau',\mu'),$$

where $H=(1/\mu')p(\mu,\mu')e^{(\tau'-\tau)/\mu'}$. The function $J(\tau,\mu)$ determines the solution completely by Eqs. (10). Thus one may obtain the solution of the problem by solving the integral equation for J instead of the original integro-differential equation. For the simple case of semi-infinite, plane-parallel, isotropically scattering material without incident current, the last integral equation for $J(\tau,\mu)$ reduces to

$$J(\tau)=\int_0^{\infty}H(|\tau-\tau'|)J(\tau')d\tau',$$

with

$$H(x)=\tfrac{1}{2}\int_1^{\infty}e^{-sx}ds/s.$$

This is the Schwarzschild-Milne integral equation.

(C) Approximations for Forward Scattering

If the single scattering is mainly in the forward direction, a reasonable approximation is to replace the factor μ in the first term of Eq. (6) by μ_0 and neglect back scattering in the boundary condition. If we take $\mu_0=1$ for simplicity, the problem now becomes

$$(\partial f/\partial\tau)+f=\tfrac{1}{2}\int_{-1}^{1}f(\tau,\mu')p(\mu,\mu')d\mu', \tag{11}$$

with the boundary condition

$$f(0,\mu)=\pi F\delta(\mu-1),\quad -1\leq\mu\leq 1. \tag{12}$$

The exact solution of this approximate problem is

$$f(\tau,\mu)=\frac{F}{4}\sum_{r=0}^{\infty}(2r+1)P_r(\mu)e^{-k_r\tau}, \tag{13}$$

where $k_r=[1-\omega_r/(2r+1)]$, with ω_r given by Eq. (2). This series was first derived by Goudsmit and Saunderson without explicit use of the Boltzmann equation. Their assumptions are thus equivalent to Eqs. (11) and (12).

Another kind of small-angle approximation is to replace the exact integral equation (6) by a differential equation of Fokker-Planck type. Using the definition of the function $p(\mu,\mu')$, we can rewrite Eq. (6) into the form

$$\mu\frac{\partial f}{\partial\tau}=\frac{1}{4\pi}\int_{-1}^{1}d\mu'\int_0^{2\pi}d\phi p(\cos\alpha)[f(\tau,\mu')-f(\tau,\mu)]. \tag{14}$$

The three angles α, $\theta=\cos^{-1}\mu$, and $\theta'=\cos^{-1}\mu'$ form a spherical triangle as shown by Eq. (4). Now let us use the edge opposite the angle θ' as the polar axis, then we have a relation similar to (4),

$$\cos\theta'=\cos\theta\cos\alpha+\sin\theta\sin\alpha\cos\beta, \tag{15}$$

where β is the azimuth angle in this case. Then one develops $f(\tau,\mu')$ into a Taylor series around μ, keeping terms up to α^2, and gets

$$f(\tau,\mu')=f(\tau,\mu)+\frac{\partial f}{\partial\mu}\left[\{(1-\mu^2)\}^{\frac{1}{2}}\alpha\cos\beta-\frac{\mu\alpha^2}{2}\right]+\frac{1}{2}\frac{\partial^2 f}{\partial\mu^2}(1-\mu^2)\alpha^2\cos^2\beta\cdots. \tag{16}$$

Putting the last expression for $f(\tau, \mu')$ into Eq. (14) and integrating over the angles α and β instead of μ' and ϕ, we get

$$\mu\frac{\partial f}{\partial\tau}=\frac{1}{\lambda}\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial f}{\partial\mu}, \qquad (17)$$

where $1/\lambda=\frac{1}{8}\int_0^\pi p(\cos\alpha)\alpha^2\sin\alpha d\alpha$, a constant. Bethe, Rose, and Smith have tried to obtain a solution of (17) with the exact boundary conditions

$$\begin{aligned} f(0,\mu)&=\pi F\delta(\mu-1), && \mu>0,\\ f(t,\mu)&=0, && \mu<0. \end{aligned} \qquad (18)$$

It seems doubtful whether such a solution does exist. In fact, we shall show that the Fokker-Planck Eq. (17) and the exact boundary conditions (18) are most likely not compatible with each other.

One can easily show that the two-dimensional analog of Eq. (17) is

$$\cos\theta\partial f/\partial\tau=(1/\lambda)\partial^2 f/\partial\theta^2 \qquad (19)$$

and the boundary conditions are

$$\begin{aligned} f(0,\theta)&=\pi F\delta(\theta-\theta_0), && 0\leq\theta<\pi/2,\\ f(t,\theta)&=0, && \pi/2<\theta\leq\pi \end{aligned} \qquad (20)$$

If we put $\cos\theta=1$ in the left-hand member of (19), it becomes precisely the differential equation for heat conduction in one dimension, with f playing the role of the temperature, τ the time, and θ the distance along the one-dimensional bar. Then the boundary condition (20) corresponds to the following heat conduction problem.

A bar of length π is insulated all over the surface. If the initial temperature distribution along one-half of the bar is given and if it is required that the temperature along the other half of the bar should be zero at a given later time, what is the temperature distribution along the whole bar at any time in between? It is a well-known fact in the theory of heat conduction that the temperature distribution along the bar at any later time is always analytic no matter whether the initial temperature distribution is analytic or not. Therefore, it is impossible to have the temperature zero along part of the bar at a certain later time. Now the original Eq. (19) and its corresponding three-dimensional Eq. (17) belong to the same class of parabolic equations as the heat equation, so we should expect that such general analyticity theorem holds true for all of them. How could then Bethe, Rose, and Smith obtain a solution to this seemingly insoluble problem? The answer lies, of course, in the approximations which these authors made in addition to the small-angle development. Their approximate solution implies that backscattering results mostly from such numerous small-angle single deflexions, for which Eq. (16) is a valid approximation.

The above demonstration[23] of the incompatibility of the approximate Fokker-Planck equation and the exact boundary condition indicates that if we approximate the exact integro-differential Eq. (6) by Eq. (17), we must also approximate, to be self consistent, the exact conditions (7). A self-consistent system results if we apply the same "forward" approximations in connection with the Fokker-Planck equation (17) as we did before, i.e., we put $\mu=1$ in Eq. (17) and neglect back scattering in the boundary condition. Then we solve the equation

$$\frac{\partial f}{\partial\tau}=\frac{1}{\lambda}\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial f}{\partial\mu}$$

with the boundary condition (12). The solution,

$$f(\tau,\mu)=\frac{F}{4}\sum_{r=0}^{\infty}(2r+1)P_r(\mu)\exp\left[-\frac{r(r+1)}{\lambda}\tau\right]$$

is, of course, an approximate form of the Goudsmit-Saunderson solution (13). This type of small-angle approximation leads exactly to Williams' theory of multiple scattering.

III. ISOTROPIC SCATTERING

Here we treat the case of a semi-infinite, plane-parallel, isotropically scattering material without incident current. The appropriate integral equation for this problem is, from Eq. (6),

$$\mu(\partial f/\partial\tau)+f=\tfrac{1}{2}\int_{-1}^{1}f(\tau,\mu)d\mu \qquad (21)$$

and the boundary conditions are

$$f(0,\mu)=0,\quad \mu>0, \qquad (22a)$$

$$f(\tau,\mu)e^{-\tau}\rightarrow 0,\quad \tau\rightarrow\infty. \qquad (22b)$$

To solve this boundary value problem, we first develop the unknown function $f(\tau, \mu)$ in a series of Legendre polynomials in μ with unknown functions of τ as coefficients, i.e., let

$$f(\tau,\mu)=\tfrac{1}{2}\sum_{r=0}^{\infty}(2r+1)f_r(\tau)P_r(\mu). \qquad (23)$$

Substituting (23) into Eq. (21) and using both the recursion formula,

$$(2r+1)\mu P_r(\mu)=(r+1)P_{r+1}(\mu)+rP_{r-1}(\mu),$$

and the orthogonality relation of Legendre polynomials, we get

$$\sum_{r=0}^{\infty}\left[r\frac{df_{r-1}}{d\tau}+(r+1)\frac{df_{r+1}}{d\tau}\right]P_r(\mu)=\sum_{r=0}^{\infty}\left[\delta_{r0}-(2r+1)\right]f_rP_r(\mu).$$

[23] We are indebted to G. E. Uhlenbeck for suggesting this argument.

Since this is an identity in μ, we can equate coefficients of the $P_r(\mu)$'s and thus obtain the infinite system

$$r\frac{df_{r-1}}{d\tau}+(2r+1-\delta_{r0})f_r+(r+1)\frac{df_{r+1}}{d\tau}=0, \qquad r=0,1,2\cdots, \quad (24)$$

or, in matrix notation,

$$\begin{bmatrix} 0 & D & 0 & 0 & \cdots \\ D & 3 & 2D & 0 & \cdots \\ 0 & 2D & 5 & 3D & \cdots \\ 0 & 0 & 3D & 7 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix}\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}; \text{ or } \mathbf{\Delta f}=0, \quad (25)$$

where $D=d/d\tau$, $\mathbf{\Delta}=\mathbf{\Delta}(D)$, and $\mathbf{f}$ is a column-matrix.

The infinite system of interconnected Eqs. (25) is hardly manageable. We can approximate, however, this infinite system by a finite system which is much easier to handle. Clearly, there is no unique way of defining the successive approximations. The following choice recommends itself by the ease with which the nth approximation can be obtained explicitly. We define the nth approximation as the solution obtained by retaining the left upper corner submatrix of $2n$ rows and $2n$ columns, and putting all the other elements equal to zero. It is then obvious that the nth approximation approaches the exact solution as $n\rightarrow\infty$. This finite system of $2n$ linear differential equations with constant coefficients can be easily solved. All the f_r's are solutions of the differential equation

$$\Delta_n(D)f_r=0 \quad r=0,1,2\cdots(2n-1), \quad (26)$$

where

$$\Delta_n(D)=\begin{vmatrix} 0 & D & 0 & 0 & \cdots & \cdots \\ D & 3 & 2D & 0 & \cdots & \cdots \\ 0 & 2D & 5 & 3D & \cdots & \cdots \\ 0 & 0 & 3D & 7 & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots & (4n-1) \end{vmatrix}. \quad (27)$$

Since the determinant $\Delta_n(D)$ has an even number of rows, $\Delta_n(-D)$ is equivalent to $\Delta_n(D)$ with all the diagonal elements negative instead of positive. But it can be easily proved that $\Delta_n(D)$ does not change its value if all the diagonal elements change their signs, because all terms have even number of diagonal elements as factors. Therefore, $\Delta_n(D)=\Delta_n(-D)$, and its roots are in pairs with same absolute value but opposite in sign. One pair of the roots is obviously $D=0$, and let us denote the other $n-1$ pairs of nonvanishing roots by k_α with $k_{-\alpha}=-k_\alpha$ and $\alpha=\pm1, \pm2, \cdots\pm(n-1)$.

Then the general solution of Eq. (26) is

$$f_r=\sum_{\alpha=-(n-1)}^{n-1} A_{r\alpha}\exp(-k_\alpha\tau)+B_r\tau+C_r, \qquad r=0,1,2\cdots(2n-1), \quad (28)$$

where the $A_{r\alpha}$'s B_r's, and C_r's are arbitrary constants, but they are not all independent.

To find the relations among the constants, we substitute Eq. (28) back into Eq. (24), and equate the coefficients of $\exp(-k_\alpha\tau)$, etc., since the resulting equation is an identity in τ. We get from the coefficient of $\exp(-k_\alpha\tau)$

$$k_\alpha rA_{r-1,\alpha}-(2r+1-\delta_{r0})A_{r\alpha}+k_\alpha(r+1)A_{r+1,\alpha}=0, \quad (29a)$$

from the coefficients of τ

$$(2r+1-\delta_{r0})B_r=0, \quad (29b)$$

from the constant term

$$rB_{r-1}+(2r+1-\delta_{r0})C_r+(r+1)B_{r+1}=0. \quad (29c)$$

Equation (29a) is a recursion formula for the $A_{r\alpha}$'s. The $2n$ linear homogeneous equations ($r=0, 1, 2\cdots(2n-1)$) of $2n+1$ unknowns $A_{0\alpha}, A_{1\alpha}\cdots A_{2n,\alpha}$ cannot determine all the unknowns uniquely. But we know that $A_{2n,\alpha}=0$, since for this nth approximation we have put $f_{2n}=0$. There remains one arbitrary unknown, which we shall take to be $A_{0\alpha}$. All the other $A_{r\alpha}$'s will then be proportional to $A_{0\alpha}$, so let us define $\rho_r(k_\alpha)$ by

$$A_{r\alpha}=\rho_r(k_\alpha)A_{0\alpha}. \quad (30)$$

Then Eq. (29a) becomes a recursion formula for ρ_r, that is

$$\rho_{r+1}(k_\alpha)-\frac{2r+1-\delta_{r0}}{k_\alpha(r+1)}\rho_r(k_\alpha)+\frac{r}{r+1}\rho_{r-1}(k_\alpha)=0, \qquad \rho_0=1. \quad (31a)$$

Equation (29b) shows that

$$B_r=0 \text{ for all } r, \text{ except } B_0 \text{ is arbitrary.} \quad (31b)$$

Equation (29c) becomes, using (31b),

$$rB_{r-1}+(2r+1-\delta_{r0})C_r=0,$$

which gives

$$\left.\begin{array}{ll} r=0, & C_0 \text{ is arbitrary} \\ r=1, & C_1=-B_0/3 \\ r\geqq2, & C_r=0 \end{array}\right\}. \quad (31c)$$

Therefore, we have altogether $2n$ independent arbitrary constants, that is

$$A_{0\alpha}, B_0, C_0, \quad \alpha=\pm1, \pm2\cdots\pm(n-1),$$

among which one is the proportionality factor determined by the assigned constant net flux in the material. The remaining $2n-1$ constants will be determined by the boundary conditions (22). The boundary condition

TABLE I. Values of $(2/B_{\theta})f(0, -\mu)$ from Eq. (36b).

		Gaussian quadrature method			Spherical Harmonic method[a] $r=0, 1, 2, \cdots n-1$			Spherical Harmonic method[a] $r=n, n+1, \cdots 2n-1$			Spherical Harmonic method[a] r even			Spherical Harmonic method[a] r odd			Chandrasekhar's SH method[b]		
μ	Exact	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$
0	0.5774	0.5774	0.5774	0.5774	0.5	0.5620	0.5656	0.6667	0.8452	5.080	0.5	0.5532	0.5632	0.6667	0.6026	0.5918	0.6667	0.5862	0.5920
0.1	0.7202		0.6965	0.7048		0.6896	0.7000		0.9689	5.393		0.6769	0.6949		0.7195	0.7166		0.7091	0.7047
0.2	0.8373		0.8103	0.8219		0.8094	0.8210		1.0858	5.627		0.7938	0.8146		0.8316	0.8322		0.8224	0.8147
0.3	0.9483		0.9207	0.9338		0.9243	0.9351		1.1986	5.820		0.9066	0.9281		0.9407	0.9432		0.9351	0.9231
0.4	1.0561		1.0288	1.0422		1.0359	1.0453		1.3086	5.989		1.0166	1.0380		1.0478	1.0511		1.0456	1.0301
0.5	1.1621		1.1352	1.1493		1.1452	1.1530		1.4165	6.143		1.1245	1.1455		1.1535	1.1573		1.1543	1.1359
0.6	1.2668		1.2405	1.2547		1.2528	1.2590		1.5231	6.285		1.2311	1.2516		1.2582	1.2623		1.2617	1.2410
0.7	1.3706		1.3450	1.3591		1.3591	1.3639		1.6285	6.420		1.3365	1.3565		1.3620	1.3664		1.3681	1.3454
0.8	1.4738		1.4487	1.4627		1.4645	1.4679		1.7332	6.549		1.4412	1.4604		1.4653	1.4697		1.4736	1.4493
0.9	1.5765		1.5519	1.5658		1.5691	1.5712		1.8371	6.673		1.5451	1.5638		1.5682	1.5726		1.5784	1.5527
1.0	1.6788	1.5774	1.6547	1.6684	1.5	1.6731	1.6740	1.6667	1.9405	6.794	1.5	1.6485	1.6667	1.6667	1.6706	1.6751	1.6667	1.6827	1.6557

[a] The different r means the different choice of the n equations among the system (33) of $2n$ equations to determine the arbitrary constants.
[b] The last three columns are from reference 3.

(22b) forces us to discard all terms with $\exp(+k_{\alpha}\tau)$, so that

$$A_{0\alpha}=0 \quad \text{for} \quad \alpha=-1, -2\cdots-(n-1). \tag{32}$$

Thus, finally, we have n constants left to be determined by the other condition (22a), which cannot be strictly satisfied in this approximation, of course. There are an infinite number of ways to construct the corresponding approximate boundary condition, just as the ways of defining the nth approximation of the solution were not unique. Again, the logical choice should be based upon simplicity and the convergence of the result. The simple approximation of boundary condition (22a) appropriate for this method is

$$\int_0^1 P_r(\mu)f(0, \mu)d\mu=0, \quad r\leq 2n-1. \tag{33}$$

The system (33) has $2n$ equations, which is double the number of unknowns to be determined. Here we choose to take those equations with odd r, because this choice gives the most convergent approximation, as shown in Table I, among other reasonable choices. Thus the n equations are

$$\sum_{m=0}^{2n-1} \frac{2m+1}{2} f_m(0) \int_0^1 P_r(\mu)P_m(\mu)d\mu=0, \qquad r=1, 3\cdots(2n-1) \tag{34}$$

in which we have made the substitution

$$f(0, \mu)=\tfrac{1}{2}\sum_{m=0}^{2n-1}(2m+1)P_m(0)P_m(\mu).$$

The function J in the formal solution (10) for this case is

$$J(\tau)=\tfrac{1}{2}\int_{-1}^{1} f(\tau, \mu)d\mu$$

which is just $\frac{1}{2}f_0(\tau)$. Therefore, using (28) and (32), we have

$$J(\tau)=\tfrac{1}{2}\left[\sum_{\alpha=1}^{n-1} A_{0\alpha}\exp(-k_{\alpha}\tau)+B_0\tau+C_0\right]. \tag{35}$$

Substituting this expression into (10) with $F=0$ and $t\rightarrow\infty$, we get the final solution for the nth approximation

$$f(\tau, +\mu)=\tfrac{1}{2}\left[\sum_{\alpha=1}^{n-1} A_{0\alpha}\frac{\exp(-k_{\alpha}\tau)-e^{-\tau/\mu}}{1-k_{\alpha}\mu}+B_0\tau + (C_0-B_0\mu)(1-e^{-\tau/\mu})\right] \tag{36a}$$

$$f(\tau, -\mu)=\tfrac{1}{2}\left[\sum_{\alpha=1}^{n-1} A_{0\alpha}\frac{\exp(-k_{\alpha}\tau)}{1+k_{\alpha}\mu} + B_0\tau+(C_0+B_0\mu)\right]. \tag{36b}$$

Such a simple form for the nth approximation is not easily obtained from another, otherwise quite reasonable, way of defining successive approximations due to Chandrasekhar. He defines the nth approximation as the solution obtained by retaining $2n+1$ rows and columns of the infinite matrix Δ defined by Eq. (25).

The function $f(0, -\mu)$ (known as the law of darkening in astronomy) has its exact closed form worked out by many authors.[24] They present different ways of deriving the same final solution in the form of a complex integral,[25] which can be evaluated by numerical integration for particular cases. Halpern, Lueneberg, and Clark[26] have obtained an exact solution for the function $f(0, -\mu)$ for the case with incident current in the form of a complex integral, and Chandrasekhar[27] got the same solution later by a different method.

IV. ANISOTROPIC SCATTERING

We generalize the simple problem treated in the previous section in two respects. First, we admit a general anisotropic single scattering function as defined

[24] E. Hopf, *Mathematical Problem of Radiative Equilibrium* (Cambridge University Press, London, 1934); E. A. Schuchard and E. A. Uehling, Phys. Rev. **58**, 611 (1940); M. M. Crum, Quart. J. Math. **18**, 244 (1947); and reference 6, Chap. III and V.
[25] G. Placzek and W. Seidel have given a simplified derivation in Phys. Rev. **72**, 550 (1947).
[26] Halpern, Lueneberg, and Clark, Phys. Rev. **53**, 173 (1938).
[27] See reference 6, Chap. III and V.

by Eq. (1); second, we consider the case with an incident current. It is convenient to reformulate the boundary value problem defined by Eqs. (6) and (7). Let

$$f(\tau,\mu)=I(\tau,\mu)+\pi F\delta(\mu-\mu_0)e^{-\tau/\mu_0}. \quad (37)$$

Putting (37) into Eq. (6), we get the integral equation for $I(\tau,\mu)$,

$$\mu\partial I/\partial\tau+I=\tfrac{1}{2}\int_{-1}^{1}I(\tau,\mu')p(\mu,\mu')d\mu' +\tfrac{1}{4}Fp(\mu,\mu_0)e^{-\tau/\mu_0}. \quad (38)$$

The boundary conditions, Eq. (7), become

$$I(0,\mu)=0, \quad \mu>0, \quad (39a)$$

$$I(t,\mu)=0, \quad \mu<0. \quad (39b)$$

Here we define $J(\tau,\mu)$ as

$$J(\tau,\mu)=\tfrac{1}{2}\int_{-1}^{1}I(\tau,\mu')p(\mu,\mu')d\mu'+\tfrac{1}{4}Fp(\mu,\mu_0)e^{-\tau/\mu_0}. \quad (40)$$

Then the formal solution, similar to Eqs. (10), is

$$I(\tau,+\mu)=e^{-\tau/\mu}\int_0^\tau J(\tau,\mu)e^{\tau/\mu}d\tau/\mu, \quad (41a)$$

$$I(\tau,-\mu)=e^{\tau/\mu}\int_\tau^t J(\tau,-\mu)e^{-\tau/\mu}d\tau/\mu. \quad (41b)$$

(A) Approximate Boltzmann Equation

We have made a small-angle approximation for Eqs. (6) and (7) to get Eqs. (11) and (12). Now if we make the same approximation for Eqs. (38) and (39), we get

$$(\partial I/\partial\tau)+I=\tfrac{1}{2}\int_{-1}^{1}I(\tau,\mu')p(\mu,\mu')d\mu'+\tfrac{1}{4}Fp(\mu,1)e^{-\tau} \quad (42)$$

and

$$I(0,\mu)=0, \quad -1\leq\mu\leq 1. \quad (43)$$

Although we know the exact solution of this approximate boundary value problem, yet we shall work out the approximations of this problem by both the SH and the GQ methods, and compare them with the corresponding approximations of the exact problem later.

In the SH method we develop both functions I and p in Eq. (42) in series of Legendre polynomials. The development for I is

$$I(\tau,\mu)=\tfrac{1}{2}\sum_{r=0}^{\infty}(2r+1)I_r(\tau)P_r(\mu), \quad (44)$$

and that for p is given by Eq. (5). Substituting both developments into (42), and equating coefficients of $P_r(\mu)$, we get the infinite system, similar to (24),

$$dI_r/d\tau=-k_rI_r+\tfrac{1}{2}F(1-k_r)e^{-\tau}, \quad r=0,1,2\cdots, \quad (45)$$

where $k_r=1-\omega_r/(2r+1)$. The system (45) can be immediately integrated, and, for the nth approximation, we have

$$I_r(\tau)=\tfrac{1}{2}F(C_re^{-k_r\tau}-e^{-\tau}), \quad r=0,1,2\cdots(2n-1), \quad (46)$$

where the C_r's are integration constants to be determined by the boundary condition (43). From (44) we have

$$I_r(0)=\int_{-1}^{1}I(0,\mu)P_r(\mu)d\mu,$$

which, in view of (43), gives $I_r(0)=0$. Therefore, $C_r=1$. Putting (5) and (44) with finite upper limit into (40) and using (46), one gets

$$J(\tau,\mu)=\frac{F}{4}\sum_{r=0}^{2n-1}\omega_rP_r(\mu)e^{-k_r\tau}. \quad (47)$$

The same small-angle approximation by the GQ method will be treated in a later section.

The formal solution, corresponding to Eq. (41), in this case is

$$I(\tau,+\mu)=e^{-\tau}\int_0^\tau J(\tau,\mu)e^{\tau}d\tau, \quad (48a)$$

$$I(\tau,-\mu)=e^{-\tau}\int_0^\tau J(\tau,-\mu)e^{\tau}d\tau. \quad (48b)$$

Putting the expression (47) for $J(\tau,\mu)$ into Eq. (48), and then from Eq. (37) (with $\mu_0=1$) we obtain the final solution,

$$f(\tau,+\mu)=\frac{F}{4}\sum_{r=0}^{2n-1}(2r+1)P_r(\mu)(e^{-k_r\tau}-e^{-\tau}) +\pi F\delta(\mu-1)e^{-\tau}, \quad (49a)$$

$$f(\tau,-\mu)=\frac{F}{4}\sum_{r=0}^{2n-1}(-1)^r(2r+1)P_r(\mu)(e^{-k_r\tau}-e^{-\tau}). \quad (49b)$$

When $n\rightarrow\infty$, the two expressions of (49) reduce to the identical form

$$f(\tau,\mu)=\frac{F}{4}\sum_{r=0}^{\infty}(2r+1)P_r(\mu)e^{-k_r\tau}, \quad -1\leq\mu\leq 1, \quad (50)$$

which is the exact solution (13).

(B) Exact Boltzmann Equation

In this section we will get the solution of the exact integral equation (38) with boundary condition (39) by the SH approximation. The procedure is the same as in the isotropic case, except here we have an inhomogeneous term in the integral equation, and the scattering function $p(\mu,\mu')$ is a series given by Eq. (5) instead of just the first term, unity. So now if we develop the function $I(\tau,\mu)$ and equate coefficients, we get the infinite system, corresponding to (24),

$$r(dI_{r-1}/d\tau)+S_rI_r+(r+1)dI_{r+1}/d\tau=\tfrac{1}{2}Fe^{-\tau/\mu_0}\omega_rP_r(\mu_0),\quad r=0,1,2\cdots,\qquad(51)$$

where

$$S_r=2r+1-\omega_r,\quad S_0=0.\qquad(52)$$

Since Eq. (24) is identical with the homogeneous part of Eq. (51) if δ_{r0} is replaced by ω_r, the general solution of the associated homogeneous system of Eq. (51) in the nth approximation must be of the form (28), i.e.,

$$I_r=\sum_{\alpha=-(n-1)}^{n-1}A_{r\alpha}\exp(-k_\alpha\tau)+B_r\tau+C_r,\quad r=0,1,2\cdots(2n-1),\qquad(53)$$

where the constants are related by

$$A_{r\alpha}=\rho_r(k_\alpha)A_{0\alpha},\quad A_{0\alpha}\text{ arbitrary},\qquad(54a)$$

$$\rho_{r+1}(k_\alpha)-\frac{2r+1-\omega_r}{k_\alpha(r+1)}\rho_r(k_\alpha)+\frac{r}{r+1}\rho_{r-1}(k_\alpha)=0,\quad \rho_0=1,\qquad(54b)$$

$$B_0\text{ arbitrary},\quad B_1=B_2=\cdots=B_{2n-1}=0,\qquad(54c)$$

$$C_0\text{ arbitrary},\quad C_1=-B_0/(3-\omega_1),\quad C_2=C_3=\cdots C_{2n-1}=0,\qquad(54d)$$

equations similar to Eqs. (30) to (31c). The k_α's are the nonvanishing roots of the determinant

$$\Delta_n(D)=\begin{vmatrix}0 & D & 0 & 0 & \cdots & \cdots\\ D & S_1 & 2D & 0 & \cdots & \cdots\\ 0 & 2D & S_2 & 3D & \cdots & \cdots\\ 0 & 0 & 3D & S_3 & \cdots & \cdots\\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdots\\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdots\\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdots\\ \cdot & \cdot & \cdot & \cdot & \cdots & S_{2n-1}\end{vmatrix}.\qquad(55)$$

From the form of the nonhomogeneous part of the system (51), it is obvious that the particular integral must be of the form

$$I_r=g_re^{-\tau/\mu_0},\quad r=0,1,2\cdots(2n-1),\qquad(56)$$

where the g_r's are constants, but not arbitrary. To determine the g_r's, we substitute (56) into Eq. (51) and get the recursion formula for g_r

$$(r/\mu_0)g_{r-1}-S_rg_r+[(r+1)/\mu_0]g_{r+1}=-\tfrac{1}{2}F\omega_rP_r(\mu_0).\qquad(57)$$

We first split the inhomogeneous term as follows

$$\begin{aligned}\tfrac{1}{2}F\omega_rP_r(\mu_0)&=\tfrac{1}{2}(F/\mu_0)[(2r+1)\mu_0P_r(\mu_0)-S_r\mu_0P_r(\mu_0)]\\&=\tfrac{1}{2}F[(r/\mu_0)P_{r-1}(\mu_0)-S_rP_r(\mu_0)\\&\qquad+\{(r+1)/\mu_0\}P_{r+1}(\mu_0)].\end{aligned}$$

Then Eq. (57) can be put into the form

$$(r/\mu_0)h_{r-1}-S_rh_r+\{(r+1)/\mu_0\}h_{r+1}=0,\qquad(58)$$

where

$$h_r=g_r+\tfrac{1}{2}FP_r(\mu_0).\qquad(59)$$

Since the recursion formula (29a) (with δ_{r0} replaced by ω_r) reduces to (58) with $k_\alpha=1/\mu_0$, h_r must be connected with h_0 by an equation similar to (30), that is,

$$h_r=\rho_r(1/\mu_0)h_0,\qquad(60)$$

where the ρ_r's obey the recursion formula (54b). Now $g_{2n}=0$ in this approximation, so that, by Eq. (59),

$$h_{2n}=\tfrac{1}{2}FP_{2n}(\mu_0).\qquad(61)$$

With Eq. (61) the system of Eqs. (58) determines uniquely all the remaining h_r's, but it is necessary only to determine one of them, say h_0, from which all the other h_r's can be obtained by Eq. (60). It is easier to visualize the result which we are going to obtain, if the system of Eq. (58) is written in the matrix form

$$\begin{pmatrix}0 & 1/\mu_0 & 0 & 0 & \cdots & \cdot\\ 1/\mu_0 & -S_1 & 2/\mu_0 & 0 & \cdots & \cdot\\ 0 & 2/\mu_0 & -S_2 & 3/\mu_0 & \cdots & \cdot\\ 0 & 0 & 3/\mu_0 & -S_3 & \cdots & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdots & -S_{2n-1}\end{pmatrix}\begin{pmatrix}h_0\\h_1\\h_2\\h_3\\ \cdot\\ \cdot\\ \cdot\\ h_{2n-1}\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\ \cdot\\ \cdot\\ \cdot\\ -{}_nFP_2n(\mu_0)/u_0\end{pmatrix},\qquad(62)$$

where we have made use of (61). Solving this inhomogeneous system, it is easy to see that

$$h_0=nFP_{2n}(\mu_0)(2n-1)!/[\mu_0^{2n}\Delta_n(1/\mu_0)],\qquad(63)$$

where Δ_n is defined by (55). Therefore, combining (60) and (63),

$$h_r=\frac{FP_{2n}(\mu_0)(2n)!}{2\mu_0^{2n}\Delta_n(1/\mu_0)}\rho_r(1/\mu_0).\qquad(64)$$

The complete solution is the sum of (53) and (56) with g_r given by (59) and (64). Thus we have

$$I_r=\sum_{\alpha=-(n-1)}^{n-1}A_{r\alpha}\exp(-k_\alpha\tau)+B_r\tau+C_r+\frac{F}{2}\left[\frac{P_{2n}(\mu_0)(2n)!}{\mu_0^{2n}\Delta_n(1/\mu_0)}\rho_r\left(\frac{1}{\mu_0}\right)-P_r(\mu_0)\right]e^{-\tau/\mu_0}.\qquad(65)$$

TABLE II.

Values of β_{rk}							Values of ϵ_{rk}						
$r \backslash k$	0	1	2	3	4	5	$r \backslash k$	0	1	2	3	4	5
1	$\frac{-1}{4}$	$\frac{1}{2}$	$\frac{-5}{16}$	0	$\frac{3}{32}$	0	1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{5}{16}$	0	$\frac{-3}{32}$	0
3	$\frac{1}{16}$	0	$\frac{-5}{16}$	$\frac{1}{2}$	$\frac{-81}{256}$	0	3	$\frac{-1}{16}$	0	$\frac{5}{16}$	$\frac{1}{2}$	$\frac{81}{256}$	0
5	$\frac{-1}{32}$	0	$\frac{25}{256}$	0	$\frac{-81}{256}$	$\frac{1}{2}$	5	$\frac{1}{32}$	0	$\frac{-25}{256}$	0	$\frac{81}{256}$	$\frac{1}{2}$

The constants in the solution obey the conditions (54a) to (54d), which leaves only $2n$ independent arbitrary constants to be determined by the boundary conditions. We shall again approximate the boundary conditions (39) by the $2n$ equations

$$\left.\begin{aligned} &\int_{-1}^{0} P_r(\mu) I(t,\mu) d\mu = 0, \\ &\int_{0}^{1} P_r(\mu) I(0,\mu) d\mu = 0, \end{aligned}\right\} \quad r = 1, 3, \cdots (2n-1),$$

or

$$\left.\begin{aligned} &\sum_{k=0}^{2n-1} \beta_{rk} I_k(t) = 0, \\ &\sum_{k=0}^{2n-1} \epsilon_{rk} I_k(0) = 0, \end{aligned}\right\} \quad r = 1, 3 \cdots (2n-1), \tag{66}$$

where

$$\beta_{rk} = \tfrac{1}{2}(2k+1) \int_{-1}^{0} P_k(\mu) P_r(\mu) d\mu,$$

$$\epsilon_{rk} = \tfrac{1}{2}(2k+1) \int_{0}^{1} P_k(\mu) P_r(\mu) d\mu.$$

The constants β_{rk}'s and ϵ_{rk}'s can be easily calculated. Table II is sufficient for the first three approximations. The system (66) provides $2n$ equations to determine the $2n$ independent arbitrary constants, $A_{0\alpha}$ ($\alpha = \pm 1, \pm 2 \cdots \pm (n-1)$), B_0 and C_0. The function $J(\tau, \mu)$ defined by (40) is, in this approximation,

$$J(\tau,\mu) = \tfrac{1}{2} \sum_{r=0}^{2n-1} \omega_r P_r(\mu) I_r(\tau) + \frac{F}{4} e^{-\tau/\mu_0} \sum_{r=0}^{2n-1} \omega_r P_r(\mu) P_r(\mu_0). \tag{67}$$

Putting the expression (65) for I_r into Eq. (67) and making use of Eqs. (54a), (54b), and (54c), we have

$$J(\tau,\mu) = \tfrac{1}{2}\Bigg[\sum_{\alpha=-(n-1)}^{n-1} A_{0\alpha} \exp(-k_\alpha \tau) \sum_{r=0}^{2n-1} \omega_r \rho_r(k\alpha) P_r(\mu) + B_0 \tau + C_0 - B_0 \frac{\omega_1}{3-\omega_1}\mu + \frac{F P_{2n}(\mu_0)(2n)!}{2\mu_0^{2n}\Delta_n(1/\mu_0)} e^{-\tau/\mu_0} \times \sum_{r=0}^{2n-1} \omega_r \rho_r(1/\mu_0) P_r(\mu) \Bigg]. \tag{68}$$

Then we get the final solution by substituting the expression (68) into Eq. (41) and in turn into (37)

$$f(\tau, +\mu) = \tfrac{1}{2}\Bigg[\sum_{\alpha=-(n-1)}^{n-1} \frac{A_{0\alpha}(\exp(-k_\alpha\tau) - e^{-\tau/\mu})}{1-\mu k_\alpha} \sum_{r=0}^{2n-1} \omega_r \rho_r(k_\alpha) P_r(\mu) + B_0\tau + \left(C_0 - \frac{3B_0}{3-\omega_1}\mu\right)(1-e^{-\tau/\mu}) + \frac{F P_{2n}(\mu_0)(2n)!}{2\mu_0^{2n}\Delta_n(1/\mu_0)} \frac{e^{-\tau/\mu_0} - e^{-\tau/\mu}}{1-\mu/\mu_0} \sum_{r=0}^{2n-1} \omega_r \rho_r(1/\mu_0) P_r(\mu) \Bigg] + \pi F \delta(\mu-\mu_0) e^{-\tau/\mu_0} \tag{69a}$$

$$f(\tau, -\mu) = \tfrac{1}{2}\Bigg[\sum_{\alpha=-(n-1)}^{n-1} \frac{A_{0\alpha}\left[\exp(-k_\alpha\tau) - \exp\left(-k_\alpha t - \frac{t-\tau}{\mu}\right)\right]}{1+\mu k_\alpha} \sum_{r=0}^{2n-1} (-1)^r \omega_r \rho_r(k_\alpha) P_r(\mu) + B_0(\tau - t e^{-(t-\tau)/\mu}) + \left(C_0 + \frac{3B_0}{3-\omega_1}\mu\right)(1-e^{-(t-\tau)/\mu}) + \frac{F P_{2n}(\mu_0)(2n)!}{2\mu_0^{2n}\Delta_n(1/\mu_0)} \times \frac{\exp\left(-\frac{\tau}{\mu_0}\right) - \exp\left(-\frac{t}{\mu_0} - \frac{t-\tau}{\mu}\right)}{1+\mu/\mu_0} \sum_{r=0}^{2n-1} (-1)^r \omega_r \rho_r(1/\mu_0) P_r(\mu) \Bigg]. \tag{69b}$$

V. COMPARISON WITH GAUSSIAN QUADRATURE METHOD

(A) Isotropic Scattering

Wick and Chandrasekhar have worked out the problem defined by Eqs. (21) and (22) with the GQ method. Chandrasekhar's expression for the function $J(\tau)$[28] is of the same form as (35), and we shall prove that the k_α's are really identical.

First we shall get a relation between our f_r and

[28] Reference 6, Chapter III, Eq. (44).

Chandrasekhar's I_i, which in our notation is $f(\tau, \mu_i)$. In the nth approximation, the development (23) becomes

$$f(\tau, \mu)=\tfrac{1}{2}\sum_{r=0}^{2n-1}(2r+1)f_rP_r(\mu) \tag{70}$$

where $P_r(\mu)$ can certainly be expressed by the Lagrange interpolation formula

$$P_r(\mu)=\sum_{i=-n}^{n}P_r(\mu_i)\frac{P_{2n}(\mu)}{(\mu-\mu_i)P_{2n}'(\mu_i)}. \tag{71}$$

Here the μ_i's are the roots of $P_{2n}(\mu)$. Putting the expression (71) in Eq. (70) and interchanging the sums, we get

$$f(\tau, \mu)=\sum_{i=-n}^{n}\sum_{r=0}^{2n-1}\frac{2r+1}{2}f_rP_r(\mu_i)\frac{P_{2n}(\mu)}{(\mu-\mu_i)P_{2n}'(\mu_i)}. \tag{72}$$

Now in the GQ method, one simply approximates the function $f(\tau, \mu)$ by the interpolation formula

$$f(\tau, \mu)=\sum_{i=-n}^{n}f(\tau, \mu_i)\frac{P_{2n}(\mu)}{(\mu-\mu_i)P_{2n}'(\mu_i)}$$

$$=\sum_{i=-n}^{n}I_i\frac{P_{2n}(\mu)}{(\mu-\mu_i)P_{2n}'(\mu_i)}. \tag{73}$$

Comparing Eqs. (72) and (73) we have the required relation,

$$I_i=\sum_{r=0}^{2n-1}\tfrac{1}{2}(2r+1)f_rP_r(\mu_i),\quad i=\pm 1, \pm 2\cdots\pm n, \tag{74}$$

or, in matrix notation,

$$\mathbf{I}=\mathbf{S}\mathbf{f},$$

where

$$S_{ir}=\tfrac{1}{2}(2r+1)P_r(\mu_i). \tag{75}$$

The inverse transformation

$$\mathbf{f}=\mathbf{S}^{-1}\mathbf{I} \tag{76}$$

has its matrix elements

$$S_{rk}^{-1}=a_kP_r(\mu_k), \tag{77}$$

where the a_k's are the gaussian weights. Using Eqs. (75) and (77), it can be shown that

$$(S^{-1}S)_{ij}=\sum_{k=-n}^{n}\tfrac{1}{2}(2j+1)a_kP_i(\mu_k)P_j(\mu_k)=\delta_{ij}, \tag{78a}$$

$$(SS^{-1})_{ij}=\sum_{r=0}^{2n-1}\tfrac{1}{2}(2r+1)a_jP_r(\mu_i)P_r(\mu_j)=\delta_{ij}. \tag{78b}$$

These are just the orthogonality and the closure relations of Legendre polynomials in finite dimensions.

Now the two sets of linear homogeneous constant coefficient differential equations of this problem for the two methods are related by the same linear transformation. In the GQ method the set of $2n$ equations[29] is

$$\mu_i(\partial I_i/\partial\tau)+I_i=\tfrac{1}{2}\sum_{j=-n}^{n}a_jI_j,\quad i=\pm 1, \pm 2\cdots\pm n,$$

or, in matrix notation,

$$\boldsymbol{\alpha}\mathbf{I}=0, \tag{79}$$

where

$$\alpha_{ij}=(\mu_jD+1)\delta_{ij}-a_j/2,\quad D\equiv\partial/\partial\tau. \tag{80}$$

In the SH method, the set is (24) or

$$\frac{r}{2r+1}\frac{df_{r-1}}{d\tau}+(1-\delta_{r0})f_r+\frac{r+1}{2r+1}\frac{df_{r+1}}{d\tau}=0,$$

$$r=0, 1, 2\cdots(2n-1),$$

which, in matrix notation, can be written as

$$\boldsymbol{\beta}\mathbf{f}=0, \tag{81}$$

with

$$\beta_{rm}=(1-\delta_{r0})\delta_{rm}+\frac{D}{2r+1}[(m+1)\delta_{r,m+1}+m\delta_{r,m-1}]. \tag{82}$$

Using the definitions (75) and (77) of the matrices $\mathbf{S}$ and $\mathbf{S}^{-1}$, and the orthogonality relation (78a), one can easily verify that

$$\mathbf{S}^{-1}\boldsymbol{\alpha}\mathbf{S}=\boldsymbol{\beta}.$$

Therefore, in view of (76), we have

$$\mathbf{S}^{-1}\boldsymbol{\alpha}\mathbf{S}\mathbf{S}^{-1}\mathbf{I}=\boldsymbol{\beta}\mathbf{f}.$$

Thus, the set of equations (79) are transformed to the set (81).

In order to prove that the k_α's given by the two methods are identical, one notices that the vanishing of the determinant $|\alpha|$ leads immediately to the characteristic equation in the GQ method, and the vanishing of the determinant $|\beta|$, which is essentially our Δ_n defined by (27), gives the characteristic equation in the SH method. Now since $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are connected by the transformation $\mathbf{S}$, it is well known that the characteristic roots must be identical. Chandrasekhar[30] has also derived a characteristic equation which does not explicitly involve the a_i's and μ_i's. This is essentially our equation $|\beta|=0$ with only a different constant factor.

The approximate boundary conditions in the two methods are not identical, so the arbitrary constants and the final solutions too, or course, are slightly

[29] The difference in sign between this system of equations and Chandrasekhar's system (reference 6, Chapter III, Eq. (2)) is because he measures angle from the negative instead of the positive x axis. Consequently all his μ's should be replaced by $-\mu$'s before comparison can be made with our results.

[30] Reference 6, Chapter III, Eq. (33).

different. We shall prove now that the difference vanishes when $n\to\infty$ as it should be since both approach the exact solution.

The boundary condition in the GQ method is when $\tau=0$, $I_i=0$ for $i=1, 2\cdots n$. Using Eq. (74), that means

$$\sum_{r=0}^{2n-1} S_{ir}f_r(0)=0, \quad i=1, 2\cdots n, \tag{83}$$

where S_{ir} is defined by (75). The boundary condition in the SH method is given by Eq. (34), i.e.,

$$\sum_{r=0}^{2n-1} R_{kr}f_r(0)=0, \quad k=1, 3, 5\cdots(2n-1), \tag{84}$$

where

$$R_{kr}=\tfrac{1}{2}(2r+1)\int_0^1 P_k(\mu)P_r(\mu)d\mu. \tag{85}$$

Now if we premultiply th ϵ coefficient matrix of the system (83) by any nonsingular constant matrix, say **T**, of n rows and n columns to form a new coefficient matrix for the system, nothing is changed essentially. Let the elements of the T-matrix be defined as

$$T_{ki}=b_iP_k(\mu_i), \quad \begin{matrix} i=1, 2\cdots n, \\ k=1, 3, 5\cdots(2n-1), \end{matrix}$$

where the μ_i's are the positive roots of $P_{2n}(\mu)$, and

$$b_i=\frac{1}{F_n'(\mu_i)}\int_0^1 \frac{F_n(\mu)}{\mu-\mu_i}d\mu, \tag{86}$$

with

$$F_n(\mu)=\prod_{j=1}^{n}(\mu-\mu_j).$$

Then the new coefficient matrix for (83) will have elements

$$Q_{kr}=\sum_{i=1}^{n} T_{ki}S_{ir}=\tfrac{1}{2}(2r+1)\sum_{i=1}^{n} b_iP_k(\mu_i)P_r(\mu_i). \tag{87}$$

We wish to show that $Q_{kr}=R_{kr}$ as $n\to\infty$. With b_i defined by (86) we can construct the quadrature formula

$$\int_0^1 g(\mu)d\mu=\sum_{i=1}^{n} b_ig(\mu_i),$$

which is exact for any function $g(\mu)$ if $n\to\infty$. Thus it follows that the two expressions (85) and (87) are the same as $n\to\infty$. Therefore, Eqs. (83) and (84) are identical in the limit, and so are the final solutions.

Table I gives the exact value and the approximations of the function $f(0, -\mu)$ for this case. Wick has already plotted the first four columns in his paper mentioned above. Among the different choices of the approximate boundary condition for our SH approximation, the odd-r choice gives the best result, and it converges faster to the exact solution than the GQ approximation also, except at the point $\mu=0$, where the GQ method gives the exact result in all approximations. The last three columns are Chandrasekhar's SH approximation, which is different from ours both in the way of defining the nth approximation and in the choice of the equations of the boundary condition. His successive approximations converge slower than the GQ approximations, and, a fortiori, slower than the SH approximations with the odd-r boundary conditions.

An advantage of the GQ boundary conditions is that they permit one to express the nth approximation in a closed form. Thus, heuristically, the transition to an exact solution by means of an H-function is facilitated, though, as we have seen, the odd-r condition leads to faster convergence. For the anisotropic case, the SH procedure is analytically simpler than the GQ procedure.

(B) Anisotropic Scattering

To solve the approximate problem defined by Eqs. (42) and (43) by means of the GQ method, we, following Chandrasekhar, first approximate the integral in Eq. (42) by a sum. For the nth approximation, put

$$\tfrac{1}{2}\int_{-1}^{1} I(\tau, \mu')p(\mu, \mu')d\mu'=\tfrac{1}{2}\sum_{j=-n}^{n} a_jI_jp(\mu, \mu_j), \tag{88}$$

where the symbols have the usual meaning. Substituting (88) into Eq. (42) and using (5), we get the following system of equations:

$$\frac{\partial I_i}{\partial\tau}+I_i=\tfrac{1}{2}\sum_{r=0}^{2n-1}\omega_rP_r(\mu_i)\sum_{j=-n}^{n} a_jI_jP_r(\mu_j)$$
$$+\frac{F}{4}e^{-\tau}\sum_{r=0}^{2n-1}\omega_rP_r(\mu_i), \quad i=\pm1, \pm2\cdots\pm n. \tag{89}$$

To get the solution of the associated homogeneous system of (89),

$$\frac{\partial I_i}{\partial\tau}+I_i=\tfrac{1}{2}\sum_{r=0}^{2n-1}\omega_rP_r(\mu_i)\sum_{j=-n}^{n} a_jI_jP_r(\mu_j), \tag{90}$$

let

$$I_i=g_ie^{-k\tau}$$

where g_i and k are constants. From Eq. (90) we have

$$g_i=\frac{1}{1-k}\sum_{r=0}^{2n-1}\omega_rP_r(\mu_i)\rho_r, \tag{91}$$

where

$$\rho_r=\tfrac{1}{2}\sum_{j=-n}^{n} a_jg_jP_r(\mu_j). \tag{92}$$

Eliminating the g_i's from Eqs. (91) and (92), we get

$$\rho_r=\frac{1}{2}\sum_{j=-n}^{n} a_jP_r(\mu_j)\frac{1}{1-k}\sum_{\lambda=0}^{2n-1}\omega_\lambda P_\lambda(\mu_j)\rho_\lambda$$

$$=\frac{1}{1-k}\sum_{\lambda=0}^{2n-1}\omega_\lambda\rho_\lambda\frac{\delta_{\lambda r}}{2r+1}=\frac{1}{1-k}\frac{\omega_r\rho_r}{2r+1},$$

or

$$\left(\frac{1}{1-k}\frac{\omega_r}{2r+1}-1\right)\rho_r=0,\quad r=0,1,2\cdots(2n-1).$$

This is the characteristic equation for k, so the roots are

$$k_r=1-\omega_r/(2r+1),\quad r=0,1,2\cdots(2n-1).\tag{93}$$

For a certain root of k, say k_α, the only nonvanishing ρ_r is ρ_α which is arbitrary. Therefore, the corresponding g_i given by (91) reduces to only one term, that is,

$$g_i=(1-k_\alpha)^{-1}\omega_\alpha P_\alpha(\mu_i)\rho_\alpha;$$

and it follows that

$$I_i=(1-k_\alpha)^{-1}\omega_\alpha P_\alpha(\mu_i)\rho_\alpha\exp(-k_\alpha\tau),$$
$$\alpha=0,1,2\cdots(2n-1).\tag{94}$$

The general solution is a linear combination of (94), i.e.,

$$I_i=\sum_{\alpha=0}^{2n-1}\frac{\rho_\alpha}{1-k_\alpha}\omega_\alpha P_\alpha(\mu_i)\exp(-k_\alpha\tau)$$
$$=\sum_{r=0}^{2n-1}\rho_r(2r+1)P_r(\mu_i)e^{-k_r\tau},\tag{95}$$

where we have made use of Eq. (93).

To get the particular integral of the nonhomogeneous system (89), let

$$I_i=\tfrac{1}{4}Fh_ie^{-\tau}.\tag{96}$$

Then we get for h_i the following system of linear equations:

$$\sum_{i=-n}^{n}h_i\frac{a_i}{2}\sum_{r=0}^{2n-1}\omega_rP_r(\mu_i)P_r(\mu_j)=-\sum_{r=0}^{2n-1}\omega_rP_r(\mu_j),$$
$$j=\pm1,\pm2\cdots\pm n.\tag{97}$$

It can be easily verified that

$$h_i=-\sum_{r=0}^{2n-1}(2r+1)P_r(\mu_i),\quad i=\pm1,\pm2\cdots\pm n,\tag{98}$$

is the solution of the system (97).

Combining (95) and (96), we get the complete solution for Eq. (89):

$$I_i=\sum_{r=0}^{2n-1}\rho_r(2r+1)P_r(\mu_i)e^{-k_r\tau}-\frac{F}{4}e^{-\tau}\sum_{r=0}^{2n-1}(2r+1)P_r(\mu_i),$$
$$i=\pm1,\pm2\cdots\pm n.\tag{99}$$

The $2n$ constants ρ_0, $\rho_1\cdots\rho_{2n-1}$ are determined by the boundary condition (43), which in this approximation becomes when

$$\tau=0,\quad I_i=0\quad\text{for}\quad i=\pm1,\pm2\cdots\pm n.\tag{100}$$

With condition (100), Eq. (99) gives

$$\rho_r=\tfrac{1}{4}F,\quad r=0,1,2\cdots(2n-1).$$

Thus finally we have

$$I_i=\frac{F}{4}\sum_{r=0}^{2n-1}(2r+1)P_r(\mu_i)(e^{-k_r\tau}-e^{-\tau}).\tag{101}$$

One then substitutes the expression (101) into (88) and in turn into Eq. (40) with $\mu_0=1$ to get the function $J(\tau,\mu)$ as follows:

$$J(\tau,\mu)=\frac{F}{4}\sum_{r=0}^{2n-1}\omega_rP_r(\mu)e^{-k_r\tau}.\tag{102}$$

This is identical with (47), since the difference in boundary condition in the SH and GQ method does not show up if we neglect back scattering. However, the analytical procedure is, obviously, simpler in the SH than in the GQ method. This is only natural, since also in the GQ method one develops the scattering function into a Legendre series and then it is clearly advantageous to develop $f(\tau,\mu)$ also into a Legendre series.

The GQ approximation of the exact integral equation (38) has been worked out by Chandrasekhar.[31] Using his expression of I_j in (88), one gets from (40) an expression for $J(\tau,\mu)$ which can be shown to be identical with (68). To prove the identity of these two expressions, we have to redefine the arbitrary constants in the following way

$$A_{0\alpha}=\tfrac{1}{2}FL_\alpha,\quad B_0=\tfrac{1}{2}F(1-\tfrac{1}{3}\omega_1)L_0,\quad C_0=\tfrac{1}{2}FL_n$$

and to show that the k_α's are the same and

$$\frac{P_{2n}(\mu_0)(2n)!}{\mu_0^{2n}\Delta_n(1/\mu_0)}=H(\mu_0)H(-\mu_0).\tag{103}$$

The proof for the identity of the k_α's in this case is essentially the same as that given for the case of isotropic scattering, except that here

$$\alpha_{ij}=(\mu_jD+1)\delta_{ij}-\frac{a_j}{2}\sum_{r=0}^{2n-1}\omega_rP_r(\mu_i)P_r(\mu_j)$$

instead of (80), and in the expression (82) for β_{rm} one should replace δ_{r0} by $\omega_r/(2r+1)$, so that

$$\beta_{rm}=\left(1-\frac{\omega_r}{2r+1}\right)\delta_{rm}+\frac{D}{2r+1}[(m+1)\delta_{r,m+1}+m\delta_{r,m-1}].$$

[31] See reference 6, Sec. 48.

Using the same transformation matrix **S**, the rest of the proof goes through exactly the same. Thus we have shown in this general case how one gets a characteristic equation which does not explicitly involve the a_i's and μ_i's. However, it does not mean that the characteristic roots are independent of the particular quadrature used. In fact, the values of the roots are different if one uses different quadrature formula. The identity of the roots for the two methods in the present case in obviously due to the fact that both are developments of the same polynomial, namely the Legendre polynomial.

To prove the identity (103), we need the explicit expression for $H(\mu_0)H(-\mu_0)$ as given by Chandrasekhar, i.e.,

$$H(\mu_0)H(-\mu_0)=(-1)^n\prod_{i=1}^{n-1}\frac{1}{1-k_i^2\mu_0^2}\prod_{i'=1}^{n}\frac{\mu_0^2-\mu_{i'}^2}{\mu_{i'}^2}.$$

Now since the μ_i's are the $2n$ roots of $P_{2n}(\mu)$, it is clear that

$$\prod_{i=1}^{n}\frac{\mu_0^2-\mu_i^2}{\mu_i^2}=\frac{P_{2n}(\mu_0)}{(-1)^nP_{2n}(0)}=\frac{2\cdot4\cdot6\cdots2n}{1\cdot3\cdot5\cdots(2n-1)}P_{2n}(\mu_0).$$

Since the k_i's are the $2n-2$ roots of $\Delta_n(D)/(-D^2)$,

$$\prod_{i=1}^{n-1}(1-k_i^2\mu_0^2)=(\mu_0^2)^{n-1}\prod_{i=1}^{n-1}\left(\frac{1}{\mu_0^2}-k_i^2\right)$$

$$=\frac{\mu_0^{2n-2}\Delta_n(1/\mu_0)/(-1/\mu_0^2)}{(-1)^{n-1}[3\cdot5\cdot7\cdots(2n-1)]^2},$$

where the denominator is the coefficient of the highest power of D in $-\Delta_n(D)/D^2$. Combining these results we get (103) immediately.

Since we have proved that the GQ approximation of Chandrasekhar gives an expression for $J(\tau,\mu)$ identical in form with our Eq. (68), it follows that the final solution $f(\tau,+\mu)$ and $f(\tau,-\mu)$ also must be identical in form with our Eqs. (69a) and (69b). However, the numerical values of the arbitrary constants are slightly different because of the different ways of approximating the boundary condition in the two methods. Consequently, for any finite approximation, the two methods will give slightly different numerical values for the final solution too. Table III gives the forward scattering in the first three approximations for both methods, and the corresponding approximations for the approximate integral equation.

The parameters used in constructing Table III coincide with the first case of Goudsmit-Saunderson's Table I[32] (i.e., $\log\xi=4$, $\mu=0.0025$).

[32] S. Goudsmit and J. L. Saunderson, Phys. Rev. **58**, 36 (1940).

TABLE III. Values of $(4/F)[f(t,+\mu)-\pi F\delta(\mu-1)e^{-t}]$ from Eq. (69a).[a]

	$n=1$			$n=2$			$n=3$		
μ	GQ	SH	Ap[b]	GQ	SH	Ap[b]	GQ	SH	Ap[b]
0	2.292	2.430	1.0	−3.60	−4.01	−1.26	3.35	3.89	1.29
0.1				−4.39	−4.65	−1.76	4.13	4.47	1.86
0.2				−4.85	−4.95	−2.04	4.05	4.20	1.82
0.3				−4.86	−4.83	−2.01	2.99	3.01	1.05
0.4				−4.31	−4.17	−1.59	1.08	1.07	−0.34
0.5				−3.07	−2.85	−0.67	−1.24	−1.17	−1.97
0.6				−1.03	−0.79	0.81	−3.09	−2.84	−3.09
0.7				1.93	2.13	2.96	−3.11	−2.62	−2.55
0.8				5.94	6.02	5.85	0.67	1.33	1.30
0.9				11.10	10.98	9.58	10.87	11.49	10.69
1.0	6.256	6.075	3.9	17.54	17.11	14.23	30.90	31.04	28.47

[a] Parameters: $\omega_1=2.9941$, $\omega_2=4.9741$, $\omega_3=6.9343$, $\omega_4=8.8702$, $\omega_5=10.778$, $t=19.46$.
[b] The values in these three columns are from the solution of the approximate integral equation (Eq. (49a) instead of Eq. (69a)).

VI. REDUCTION OF THE ANISOTROPIC PROBLEM TO A QUASI-ISOTROPIC ONE

(A) Forward Scattering

(1) Theoretically the expression (69a) gives the forward scattering to any degree of accuracy one wants. But if the single scattering function is extremely forward, one has to carry the approximation to a large n in order to get some sensible result. This means a tremendous amount of numerical work, so it is not too desirable in practice.

Now, since the Goudsmit-Saunderson solution (50) gives a pretty good approximation for forward scattering, we can consider this as the first approximation of a perturbation treatment. First we rewrite the integral equation (6) as follows:

$$(\partial f/\partial\tau)+f=\tfrac{1}{2}\int_{-1}^{1}f(\tau,\mu')p(\mu,\mu')d\mu'+\epsilon(1-\mu)\partial f/\partial\tau. \quad (104)$$

The parameter ϵ, which is inserted here to indicate that the term is small, will be set equal to unity eventually. Let

$$f=f_1+\epsilon f_2.$$

Putting this into (104), we get one equation with terms free from ϵ

$$(\partial f_1/\partial\tau)+f_1=\tfrac{1}{2}\int_{-1}^{1}f_1(\tau,\mu')p(\mu,\mu')d\mu', \quad (105)$$

and another one involving ϵ (with ϵ put equal to unity)

$$\mu(\partial f_2/\partial\tau)+f_2=\tfrac{1}{2}\int_{-1}^{1}f_2(\tau,\mu')p(\mu,\mu')d\mu'+(1-\mu)\partial f_1/\partial\tau. \quad (106)$$

If f_1 and f_2 satisfy the boundary conditions,

$$f_1(0,\mu)=\pi F\delta(\mu-1),\quad -1\leq\mu\leq1, \quad (107a)$$

$$f_2(0,\mu)=0,\quad \mu>0, \quad (107b)$$

$$f_2(t,\mu)=-f_1(t,\mu),\quad \mu<0, \quad (107c)$$

TABLE IV. Forward scattering.[a]

θ	Goudsmit-Saunderson solution[b]	Correction term[c]
0°	120.9	2.57
6°	72.6	2.45
12°	26.3	2.08
18°	8.3	1.55
24°	2.7	0.97
30°	1.1	0.44
45°	0.1	−0.08

[a] Parameters have same values as in Table III.
[b] Values from reference 32.
[c] Difference of last two columns in Table III.

f will satisfy the original boundary condition (7) with $\mu_0=1$. Up to this point everything is still exact. Equations (105), (106), and (107) are just another way of defining the original problem (6) and (7).

The solution of Eq. (105) with boundary condition (107a) is just the Goudsmit-Saunderson solution (50). With f_1 given, one then wishes to find the correction term f_2 satisfying (106), (107b), and (107c). Here we will make the approximation by taking the finite series solution (49) for f_1 instead of (50) in Eq. (106), and then solve the problem to the same SH approximation with the corresponding approximation in boundary condition. This approximate solution for f_2 is simply the difference of the two series solutions, (69a)−(49a) with $\mu_0=1$ and $\tau=t$. Thus we have for forward scattering

$$f(t,\mu)=\text{Goudsmit-Saunderson solution (50)} +[(69a)-(49a)]. \quad (108)$$

Table IV gives both the Goudsmit-Saunderson value (or f_1) and the correction (or f_2) for the special case which is treated in Table III. In calculating f_2, one does not need to carry the approximation to a large n as in the case of calculating the original f, because the extremely anisotropic part has been taken care of by the first approximation f_1. Here the correction term is calculated with $n=3$, which is, of course, just the difference of the last two columns of Table III. One notices from Table III that in the region with positive values (the negative values in the table are meaningless), the percentage difference between the solutions for the exact and the approximate integral equations decreases steadily with the increase of approximation, and for the third approximation it is only about 8 percent for $\mu=1$. Therefore, the Goudsmit-Saunderson value there probably would not be more than 8 percent off from the exact value. Our correction in Table IV is about 2 percent at $\mu=1$ or $\theta=0°$.

(2) Another simple approximation for f_2 is to make use of the fact that the single scattering function is extremely forward, which means the function $p(\mu,\mu')$ has a sharp peak at $\mu'=\mu$ approximately.[33] So we develop the function f_2 under the integral of Eq. (106) in a Taylor series around $\mu'=\mu$, that is

$$f_2(\tau,\mu')=f_2(\tau,\mu)+[\partial f_2(\tau,\mu)/\partial\mu](\mu'-\mu)\cdots. \quad (109)$$

Then we get

$$\frac{1}{2}\int_{-1}^{1} f_2(\tau,\mu')p(\mu,\mu')d\mu'$$
$$=f_2(\tau,\mu)+\frac{1}{2}\frac{\partial f_2}{\partial\mu}\sum_{r=0}^{\infty}\omega_r P_r(\mu)\int_{-1}^{1}P_r(\mu')(\mu'-\mu)d\mu'\cdots$$
$$=f_2(\tau,\mu)-\mu k_1\frac{\partial f_2}{\partial\mu}\cdots, \quad (110)$$

where $k_1=(1-\frac{1}{3}\omega_1)$. Using (110), Eq. (106) becomes

$$\mu(\partial f_2/\partial\tau)+\mu k_1(\partial f_2/\partial\mu)=(1-\mu)(\partial f_1/\partial\tau). \quad (111)$$

For first approximation, we omit the second term of Eq. (111), since $k_1\ll 1$. Then the solution which satisfies the boundary conditions (107b) and (107c) is

$$f_2(\tau,\mu)=\begin{cases}[(1-\mu)/\mu]f_1(\tau,\mu), & \mu>0;\\ [(1-\mu)/\mu]f_1(\tau,\mu)-(1/\mu)f_1(t,\mu), & \mu<0.\end{cases}$$

For second approximation, we let

$$f_2=g_1+g_2, \quad (112)$$

where $g_1\gg g_2$. Putting Eq. (112) into Eq. (111), but omitting g_2 in the second term, we get the first- and second-order equations

$$\partial g_1/\partial\tau=[(1-\mu)/\mu]\partial f_1/\partial\tau, \quad (113a)$$

and

$$\partial g_2/\partial\tau=-k_1(\partial g_1/\partial\mu), \quad (113b)$$

respectively. The solutions of Eqs. (113a) and (113b) are

$$g_1=[(1-\mu)/\mu]f_1+A(\mu), \quad (114a)$$

$$g_2=-k_1\int_0^{\tau}(\partial g_1/\partial\mu)d\tau+B(\mu)-A(\mu), \quad (114b)$$

where $A(\mu)$ and $B(\mu)$ are arbitrary functions. To satisfy the boundary conditions (107b) and (107c), we find that

$$A(\mu)=0 \text{ and } B(\mu)=\begin{cases}0, & \mu>0;\\ k_1\int_0^{t}(\partial g_1/\partial\mu)d\tau-(1/\mu)f_1(t,\mu), & \mu<0.\end{cases} \quad (115)$$

Combining Eqs. (112), (114), and (115), we have, for

[33] The peak is usually not at $\mu'=\mu$, but a little distance off. For example, in the case we used for constructing Table III, $\mu'=1.000082\mu$ for values of μ^2 not too near unity.

the second approximation,

$$f_2(\tau,\mu)=\begin{cases}\dfrac{1-\mu}{\mu}f_1(\tau,\mu)-k_1\displaystyle\int_0^\tau\left(\frac{1-\mu}{\mu}\frac{\partial f_1}{\partial\mu}-\frac{f_1}{\mu^2}\right)d\tau, & \mu>0;\\[2ex] \dfrac{1-\mu}{\mu}f_1(\tau,\mu)-\dfrac{1}{\mu}f_1(t,\mu)-k_1\left[\displaystyle\int_0^\tau-\int_0^t\right]\left(\frac{1-\mu}{\mu}\frac{\partial f_1}{\partial\mu}-\frac{f_1}{\mu^2}\right)d\tau, & \mu<0.\end{cases}$$

Then we have the final solution

$$f(\tau,\mu)=f_1+f_2=\begin{cases}\dfrac{f_1(\tau,\mu)}{\mu}-k_1\displaystyle\int_0^\tau\left(\frac{1-\mu}{\mu}\frac{\partial f_1}{\partial\mu}-\frac{f_1}{\mu^2}\right)d\tau, & \mu>0; \quad (116a)\\[2ex] \dfrac{f_1(\tau,\mu)-f_1(t,\mu)}{\mu}-k_1\left[\displaystyle\int_0^\tau-\int_0^t\right]\left(\frac{1-\mu}{\mu}\frac{\partial f_1}{\partial\mu}-\frac{f_1}{\mu^2}\right)d\tau, & \mu<0, \quad (116b)\end{cases}$$

where f_1 is the solution (50). The forward scattering is given by (116a) with $\tau=t$, i.e.,

$$f(t,\mu)=\frac{f_1(t,\mu)}{\mu}-k_1\int_0^t\left(\frac{1-\mu}{\mu}\frac{\partial f_1}{\partial\mu}-\frac{f_1}{\mu^2}\right)d\tau,\quad \mu>0. \quad (117)$$

The result given by (116) diverges for $\mu=0$. Since, as $\mu\to 0$, the terms neglected in (111) might be larger than the ones retained, the whole approximation breaks down. But away from $\mu=0$, the solution (116) might give a reasonable approximation. Of course, it is not allowable to make the same kind of development (109) in the original equation (104), since the function $f(\tau,\mu')$ is a highly peaked function too. In fact such a development applied to $f(\tau,\mu')$ would lead just to a Fokker-Planck differential equation [see Eq. (14)–(17)], which is valid only for small angles. On the other hand, $f_2(\tau,\mu')$ in Eq. (106) is a much more smooth function, so we think it is justified to make such a development.

(3) A third way of getting an approximate expression for f_2 is by an iteration method. To illustrate how one gets an exact solution by iteration, we will use the problem defined by Eqs. (11) and (12), whose exact solution is known to be the series (13). The solution of (11) and (12) is equivalent to that of the integral equation

$$f(\tau,\mu)=e^{-\tau}\int_0^\tau J(\tau,\mu)e^\tau d\tau+e^{-\tau}\delta(\mu-1),$$

where

$$J(\tau,\mu)=\tfrac{1}{2}\int_{-1}^1 f(\tau,\mu')p(\mu,\mu')d\mu'.$$

Now suppose we start with Williams' approximate solution (given at the end of Sec. II) of this problem as a trial solution in the last integral equation defining the function $J(\tau,\mu)$, and then substitute in turn into the integral in $f(\tau,\mu)$, we get

$$f^{(1)}(\tau,\mu)=f^{(0)}(\tau,\mu)+\frac{F}{4}\sum_{r=0}^{\infty}(2r+1)P_r(\mu)(g_r-1)\times\left[e^{-r(r+1)\tau/\lambda}-e^{-\tau}\right],$$

where

$$g_r=\frac{1-k_r}{1-r(r+1)/\lambda}$$

and $f^{(0)}(\tau,\mu)$ is the Williams' solution we start with. If we start with $f^{(1)}(\tau,\mu)$ and iterate once more, we get

$$f^{(2)}=f^{(1)}+\frac{F}{4}\sum_{r=0}^{\infty}(2r+1)P_r(\mu)g_r(g_r-1)\times\left\{\exp\left[-\frac{r(r+1)\tau}{\lambda}\right]-e^{-\tau}-\left[1-\frac{r(r+1)}{\lambda}\right]\tau e^{-\tau}\right\}.$$

One can go on and get the series

$$f^{(\infty)}(\tau,\mu)=\frac{F}{4}\sum_{r=0}^{\infty}(2r+1)P_r(\mu)\left\{\exp\left[-\frac{r(r+1)}{\lambda}\tau\right]+(g_r-1)\sum_{n=0}^{\infty}g_r^n\left[\exp\left[-\frac{r(r+1)}{\lambda}\tau\right]-e^{-\tau}\sum_{s=0}^{n}\frac{\left(1-r\dfrac{r+1}{\lambda}\right)^s\tau^s}{s!}\right]\right\},$$

which can be easily shown to be identical with Eq. (13).

For the present case we first calculate the function $J(\tau,\mu)$ from Eq. (8) by using f_1 as the approximation for f under the integral; then we obtain $f(t,+\mu)$ by (10a) with $\mu_0=1$. The result is

$$f(t,+\mu)=f_1(t,+\mu)-\frac{F}{4}\sum_{r=0}^{\infty}(2r+1)k_rP_r(\mu)\times\frac{1-\mu}{1-\mu k_r}\left(e^{-k_rt}-e^{-t/\mu}\right). \quad (118)$$

(B) Back Scattering

(1) The expression (69b) gives the back scattering, but here again it is unpractical for cases with very anisotropic single scattering function. The perturbation treatment that we are going to give for this case is based upon the extremely small cross section of the back scattering.

First we make a Taylor development of the scattering function $p(\cos\alpha)$ defined by Eq. (1) around any backward direction, say $\cos\alpha=\cos\alpha_0$ with $\alpha_0>\pi/2$; i.e.,

$$p(\cos\alpha)=p(\cos\alpha_0)+p'(\cos\alpha_0)(\cos\alpha-\cos\alpha_0)\cdots. \quad (119)$$

Rearrange the terms to make a series of Legendre polynomials,

$$p(\cos\alpha)=\epsilon_n\sum_{r=0}^{n}\omega_r^*P_r(\cos\alpha), \quad (120)$$

where $\omega_0^*=1$. Though the function $p(\cos\alpha)$ is extremely forward, it is much more isotropic in back scattering. Therefore, we need only a few terms of the development (119) or (120); i.e., n will be small. The factor ϵ_n depends upon n and α_0, and it is small for small n. From (120) we get a development for $p(\mu,\mu')$ similar to (5):

$$p(\mu,\mu')=\epsilon_n\sum_{r=0}^{n}\omega_r^*P_r(\mu)P_r(\mu'). \quad (121)$$

Putting Eq. (121) into Eq. (38), we have (setting $\mu_0=1$)

$$\mu\frac{\partial I}{\partial\tau}+I=\frac{F}{4}p(\mu,1)e^{-\tau}+\frac{\epsilon_n}{2}\sum_{r=0}^{n}\omega_r^*P_r(\mu)\int_{-1}^{1}I(\tau,\mu')P_r(\mu')d\mu'. \quad (122)$$

Now multiply (122) by $\exp(-sr)d\tau$ and integrate τ from 0 to t. Also let

$$\varphi(s,\mu)=\int_0^t d\tau e^{-s\tau}I(\tau,\mu).$$

Then Eq. (122) becomes

$$\mu[I(t,\mu)e^{-st}-I(0,\mu)]+s\mu\varphi(s,\mu)+\varphi(s,\mu)=\frac{F}{4}p(\mu,1)\frac{1-e^{-(1+s)t}}{1+s}+\frac{\epsilon_n}{2}\sum_{r=0}^{n}\omega_r^*P_r(\mu)\int_{-1}^{1}\varphi(s,\mu')P_r(\mu')d\mu'. \quad (123)$$

In first approximation we will omit the sum in (123). Then we put $s=-1/\mu$, and introduce the proper boundary condition (39) for $I(\tau,\mu)$. Thus we get

$$I(0,\mu)=0,\quad I^{(1)}(t,\mu)=-\frac{F}{4}p(\mu,1)\frac{1-e^{(1-\mu)t/\mu}}{1-\mu}e^{-t/\mu},\quad \mu>0, \quad (124a)$$

$$I(t,\mu)=0,\quad I^{(1)}(0,\mu)=\frac{F}{4}p(\mu,1)\frac{1-e^{(1-\mu)t/\mu}}{1-\mu},\quad \mu<0. \quad (124b)$$

$I^{(1)}(0,\mu)$ is, of course, the first approximation for back scattering.

In order to get the second approximation for $I(\tau,\mu)$, we have to get the first approximation of $\varphi(s,\mu)$. One just substitutes the two expressions (124a) and (124b) into (123), still omitting the sum, and gets

$$\varphi^{(1)}(s,\mu)=\frac{F}{4}\frac{p(\mu,1)}{1+s\mu}\left\{\frac{1-e^{-(1+s)t}}{1+s}+\begin{cases}\dfrac{1-e^{t(1-\mu)/\mu}}{(1-\mu)/\mu}e^{-t/\mu-st} & \mu>0\\[2ex] \dfrac{1-e^{t(1-\mu)/\mu}}{(1-\mu)/\mu} & \mu<0.\end{cases}\right\} \quad (125)$$

Putting (125) into the integral in Eq. (123), and carrying out the integration in μ', we get the second approximation by again setting $s=-1/\mu$ and using boundary condition for $I(\tau,\mu)$:

$$I^{(2)}(0,\mu)=I^{(1)}(0,\mu)-\frac{\epsilon_n}{2\mu}\sum_{r=0}^{n}\omega_r^*P_r(\mu)\times\left[\int_{-1}^{1}\varphi^{(1)}(s,\mu')P_r(\mu')d\mu'\right]_{s=-1/\mu},\quad \mu<0. \quad (126)$$

One can go to higher approximations in a similar manner.[34]

For a semi-infinite medium, $t\to\infty$, the first approximation of back scattering becomes obviously

$$I^{(1)}(0,\mu)=\tfrac{1}{4}Fp(\mu,1)/(1-\mu),\quad \mu<0. \quad (127)$$

Now if we make the development (119) around the direction where we wish to calculate the back scattering,

[34] A similar technique was used by L. V. Spencer, "Penetration and diffusion of x-rays," Natl. Bur. Standards (U. S.) Memo Rept. We are indebted to U. Fano for a copy of this report.

that is, to put $\cos\alpha_0=\mu$, then, in view of Eqs. (119), (120), and (121), we have

$$p(\mu, 1)=p(\mu) \qquad (128)$$

exactly. Thus we get finally

$$I^{(1)}(0, \mu)=\tfrac{1}{4}Fp(\mu)/(1-\mu), \quad \mu<0. \qquad (129)$$

If the development is made around a direction different from the one where we calculate the back scattering, then $p(\mu, 1)$ is simply the finite series

$$p(\mu, 1)=\sum_{r=0}^{n}\frac{1}{r!}p^{(r)}(\cos\alpha_0)(\mu-\cos\alpha_0)^r, \qquad (130)$$

where the r in parenthesis means the rth derivative with respect to the argument. In (128) the value of $p(\mu, 1)$, for a fixed μ, does not change with the number of terms taken in the development (119), but in (130) it does. Since (130) is a series in $(\mu-\cos\alpha_0)$, it requires a larger n to approach (128) for μ further away from $\cos\alpha_0$. With $t\rightarrow\infty$, the expression (125) reduces to

$$\varphi^{(1)}(s, \mu)=\frac{F}{4}\frac{p(\mu, 1)}{1+s\mu}\left[\frac{1}{1+s}+\begin{cases}0 & \mu>0,\\ \mu/(1-\mu) & \mu<0,\end{cases}\right]$$

from which one can calculate $I^{(2)}(0, \mu)$ by (126).

To give some idea about the order of magnitude of the first approximation and the convergence of the method, we have calculated a few values for the semi-infinite case with the Rutherford scattering function

$$p(\cos\alpha)=4\beta(1+\beta)/(1+2\beta-\cos\alpha)^2, \qquad (131)$$

which is normalized according to the definition (1). β is the screening constant, usually much smaller than unity. Using (131), we get the first approximation immediately from (129):

$$I^{(1)}(0, \mu)=F\beta(1+\beta)/(1+2\beta-\mu)^2(1-\mu) \simeq F\beta/(1-\mu)^3, \quad \mu<0. \qquad (132)$$

If the second approximation is written in the form

$$I^{(2)}(0, \mu)=I^{(1)}(0, \mu)(1+k_n\beta), \qquad (133)$$

the constant k_n, which determines the convergence, changes with n, the number of terms taken in the development (119) besides the first constant term. Table V gives the values of k_n for $n\leq 4$ at $\mu=-1$ and $\cos\alpha_0=-1$ too. The last two columns, giving the difference between and the ratio of consecutive k_n's, show that the k_n probably would not diverge as n increases.

The calculation for finite t is a little longer, but there is no essential difficulty, but some of the integrals involved must be understood in the sense of cauchy principal value. We have calculated the first and second approximation only for $n=0$ and $\mu=-1$. We get

TABLE V. k_n of Eq. (126).

n	k_n	$k_{n+1}-k_n$	k_{n+1}/k_n
0	0.69	1.89	3.72
1	2.58	2.31	1.89
2	4.89	2.26	1.46
3	7.15	2.06	1.29
4	9.21		

$$I^{(1)}(0, -1)=\tfrac{1}{8}F\beta(1-e^{-2t}),$$

$$I^{(2)}(0, -1)=I^{(1)}(0, -1)\left\{1+\frac{\beta}{1-e^{-2t}}[\ln 2+E_1(2t) -2E_1(t)e^{-t}-(\gamma+\ln t)e^{-2t}]\right\}, \qquad (134)$$

where γ is the euler constant and

$$E_1(x)=\int_x^\infty e^{-u}du/u.$$

For the case we treated in constructing Tables III and IV, t is of the order 20. Then Eq. (134) gives practically the same result as the case of infinite t.

From a physical point of view one would expect that such a development is good only for thin foils. However, the results of Table V seem to show that this development may be useful for thicker foils too. Further numerical work would be necessary, however, to make this conclusion safe. Physically, this conclusion would mean that in back scattering a large number of small angle deflections is less probable than a small number of large deflections.

(2) Equation (116b) gives another approximation for backscattering, i.e.,

$$f(0, \mu)=-\frac{f_1(t, \mu)}{\mu}+k_1\int_0^t\left(\frac{1-\mu}{\mu}\frac{\partial f_1}{\partial\mu}-\frac{f_1}{\mu^2}\right)d\tau, \quad \mu<0. \qquad (135)$$

The function $f_1(t, \mu)$ in the form of the infinite series (50) is not too practical for numerical calculation, since for an extremely forward single scattering function one has to take a tremendous number of terms. For the case of small t, several authors[14, 16, and 17] have made developments of this function which are more suitable for numerical calculation than the series. However, they all made the small-angle approximation, so their developments are not suitable for $\mu<0$ as required by (135). We shall give another development here which approximates better for smaller μ.

The function $f_1(\tau, \mu)$ is a solution of (105) and (107a).

We first develop $f_1(\tau,\mu)$ in a Taylor series of τ, i.e.,

$$f_1(\tau,\mu)=\pi F\delta(\mu-1)+\tau(\partial f_1/\partial\tau)_{\tau=0}+\tfrac{1}{2}\tau^2(\partial^2 f_1/\partial\tau^2)_{\tau=0}\cdots,\quad(136)$$

in which we have made use of the boundary condition (107a). Putting (136) into (105) and equating coefficients of different powers of τ, we get

$$\begin{aligned}(\partial f_1/\partial\tau)_{\tau=0}&=\tfrac{1}{4}Fp(\mu)-\pi F\delta(\mu-1),\\ (\partial^2 f_1/\partial\tau^2)_{\tau=0}&=\tfrac{1}{4}F\left\{\tfrac{1}{2}\int_{-1}^{1}p(\mu,\mu')p(\mu')d\mu'-2p(\mu)\right\}+\pi F\delta(\mu-1),\end{aligned}\quad(137)$$

where $p(\mu)$, which is written for $p(\mu,1)$, is just the single scattering function defined by (1). Combining (136) and (137), we get

$$f_1(t,\mu)=(1-t+\tfrac{1}{2}t^2)\pi F\delta(\mu-1)+\tfrac{1}{4}F\left\{tp(\mu)+\tfrac{1}{2}t^2\left[\tfrac{1}{2}\int_{-1}^{1}p(\mu,\mu')p(\mu')d\mu'-2p(\mu)\right]\right\}.\quad(138)$$

One gets, of course, exactly the same series if one develops the solution $f_1(t,\mu)$, as given by (50), into power series of t, only not immediately in this form. In the form of (138), one notices that, besides the singular function at $\mu=1$, the coefficient of t contains only a single-collision term, and that of t^2 contains both single and double-collision terms. The t^3 term will have single-, and double-, and triple-collision terms, and so on.

The first term inside the curly brackets of (138) is the well-known single scattering tail for a thin foil. To evaluate the integral, we notice that the integrand is the product of two extremely peaked functions with one peak at $\mu'=\mu$ and the other at $\mu'=1$. So we break the integral at $\mu'=(1+\mu)/2$, and develop the slowly varying function in both intervals into Taylor series around the peak of the other function. Using the single scattering function (131), and keeping terms of first power of β, we get, combining with other terms of (138),

$$f_1(t,\mu)=(1-t+\tfrac{1}{2}t^2)\pi F\delta(\mu-1)+\tfrac{1}{4}Ftp(\mu)\times\left\{1+\frac{\beta t}{1-\mu}\left[2(3+\mu)\ln\frac{1}{2\beta}-(3+\mu)\ln\frac{2}{1-\mu}-\frac{27\mu^3-15\mu^2-27\mu+47}{4(1-\mu)^2}\right]\right\}.\quad(139)$$

In calculating (139) we have taken the Taylor series up to terms with third derivative. The above result is, of course, incorrect if $\beta/(1-\mu)$ is not small. The deviation from single scattering, given by the term with square brackets, reduces to

$$2\beta t(\ln 1/\beta-\ln 2-1/2)\quad(140)$$

for $\mu=-1$ in our approximate expression (139). For this special value of μ, the integral in (138) can be evaluated exactly. We found the exact value for the deviation up to first power of β to be

$$2\beta t(\ln 1/\beta-1)$$

instead of (140).

If we apply the development (136) to the exact Boltzmann equation (6), we get, instead of (138),

$$f(t,\mu)=\left(1-t+\frac{t^2}{2}\right)\pi F\delta(\mu-1)+\frac{F}{4}\left\{\frac{tp(\mu)}{\mu}+\frac{t^2}{2\mu}\left[\frac{1}{2}\int_{-1}^{1}\frac{p(\mu,\mu')p(\mu')d\mu'}{\mu'}-\frac{1+\mu}{\mu}p(\mu)\right]\right\}.$$

(3) We can get an expression for back scattering similar to (118) for forward scattering by iteration. The only difference in calculation is that here we use (10b) instead of (10a). Thus, we get

$$(0,-\mu)=\frac{F}{4}\sum_{r=0}^{\infty}(-1)^r\omega_rP_r(\mu)\frac{1-e^{-(1+\mu k_r)t/\mu}}{1+\mu k_r},\quad \mu>0.\quad(141)$$

(C) Final Remark

Making use of the development (138), one can show, in the case of forward scattering, that for small t

$$f(t,\mu)\simeq[1-t(1-k_1)]\pi F\delta(\mu-1)+Ftp(\mu)/4\mu\cdots\quad(142)$$

in the solution (117), and

$$f(t,\mu)\simeq(1-t)\pi F\delta(\mu-1)+Ftp(\mu)/4\mu\cdots\quad(143)$$

in the solution (118). The two expressions (142) and (143) are approximately the same, since $k_1\ll 1$. In the case of back scattering, the three kinds of approximation lead to the solutions (124b), (135), and (141). They represent the same quantity, the angular distribution of back scattering, since $I(0,\mu)$ and $f(0,\mu)$ are identical for $\mu<0$ by definition (37). Now one can also prove that for small t the three solutions approach the same expression

$$f(0,\mu)=-Ftp(\mu)/4\mu,\quad \mu<0.$$

All our approximations described in this Sec. VI, except the one in subsection (B1), are based upon the Goudsmit-Saunderson solution (50) as a first approximation. Therefore, if we want to have some numerical results, it is necessary to tabulate the solution (50) first. For small t, there are some developments in the literature[28] suitable for numerical calculation in the range θ not too large, and we have given the development (139) for large θ. For arbitrary t and θ very small, one can replace the sum (50) by an integral, as done by Moliere and Snyder and Scott[14,16] and get some approximate values. But when θ is not very small, in fact very large as in the case of backscattering, we are not able to convert the sum to any form more suitable for numerical calculation, except for small t.

In calculating the series (50), one major job is to compute the k_r's, which involves some elementary integrations. For the scattering function (131), we have calculated the first seven k_r's and found, by inspection, the general rule

$$(1-k_r)=\frac{1}{2^r A_0}\sum_{i=0}^{m}(-1)^i\frac{(2r-2i)!}{i!(r-i)!(r-2i)!}A_{r-2i}, \quad (144)$$

where $m=\frac{1}{2}r$ or $\frac{1}{2}(r-1)$ for r even or odd, and

$$A_n=\frac{nx^{n-1}}{2}\ln\frac{x-1}{x+1}+\frac{x^n}{x^2-1}+\frac{n-1}{1}x^{n-2}+\frac{n-3}{3}x^{n-4}+\frac{n-5}{5}x^{n-6}\cdots \quad (145)$$

with $x=1+2\beta$. In using the series (145), all terms, after the second, with negative powers of x have to be discarded. Snyder and Scott, and Lewis[16,19] have given some approximate expressions for k_r.

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The Primeval Lead Isotopic Abundances and the Age of the Earth's Crust*

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Nier's determinations of lead isotopic abundances in common lead ores have been the subject of considerable study in connection with attempts to calculate the age of the earth. The importance of the age of the earth in fixing the age of the elements has led us to remark on the very high precision frequently attributed to the former age determinations. A calculation is presented which yields a rough maximum age of the earth, namely, $t(\max)=5.3$ billion years. The primeval lead isotopic abundances have been estimated and briefly discussed in the light of nuclear systematics.

INTRODUCTION

In several recent papers dealing with the subject of the age of the elements, use has been made of the important value for the age of the earth's crust due to Holmes,[1] namely, 3.350×10^9 yr. In view of the rather unreasonably high accuracy assigned to this value by many authors[2] and the great interest in such ages for cosmological and cosmogonical problems, it seems pertinent to discuss briefly the analyses leading to such ages, and to point out that, at the present time, the limit of speculations on this subject due to the nature of the data may be a rough estimate of the maximum age of the earth's crust, and of the primeval lead isotopic abundances.

Katcoff, Schaeffer, and Hastings[3] have recently calculated the time between element formation and formation of the earth's atmosphere to be $\Delta t=0.27\times10^9$ yr. Their calculation was based on the addition of Xe^{129} to the atmosphere by the decay of I^{129}, as suggested earlier by Suess.[4] More recently Suess and Brown[5] have pointed out that this calculation actually leads to an approximate lower limit for the value of the time interval Δt. They find a value of $\Delta t\cong0.4\times10^9$ yr. The age of the elements is then given by the sum of this time and the age of the earth's crust, assuming that the time between formation of the crust and of the atmosphere, and between formation of the earth as an entity and its crust, may be neglected. These suggested values of Δt are sufficiently small that, if they are correct even only as to order of magnitude, then the important calculation, in so far as the age of the elements is concerned, is that yielding the age of the earth's crust. However, the determination of Δt is otherwise of great interest in connection with the early history of the universe.

FORMULATION

The determinations of the age of the earth's crust by various investigators are all based on a set of data due to Nier.[6] These data are accurate mass spectro-

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[1] A. Holmes, Nature **157**, 680 (1946); **159**, 127 (1947); **163**, 453 (1949); Endeavour **6**, 99 (1947). See also H. Jeffreys, Nature **162**, 822 (1948); **164**, 1046 (1949), as well as E. C. Bullard and J. P. Stanley, Suomen Geodeetisen Laitoksen Julkaisuja, Veröffentlichungen des Finnischen Geodatischen Institutes, No. 36, 33 (1949). The latter authors obtains 3.29×10^9 yr using the same data as Holmes but employing a least squares analysis.

[2] For example, Fleming, Ghiorso, and Cunningham, Phys. Rev **82**, 967 (1951), suggest that their new value of the decay constant for U^{235} will alter Holmes' value for the age of the crust, *viz.*, 3.350×10^9 yr, by three percent. It is very difficult to understand how this small change in the crustal age was estimated, in light of the involved nature of Holmes' analysis leading to this age, and its concomitant approximate nature.

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