An Operator Calculus Having Applications in Quantum Electrodynamics

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An alteration in the notation used to indicate the order of operation of noncommuting quantities is suggested. Instead of the order being defined by the position on the paper, an ordering subscript is introduced so that $A_sB_{s'}$ means AB or BA depending on whether s exceeds s' or *vice versa*. Then A_s can be handled as though it were an ordinary numerical function of s. An increase in ease of manipulating some operator expressions results. Connection to the theory of functionals is discussed in an appendix. Illustrative applications to quantum mechanics are made. In quantum electrodynamics it permits a simple formal understanding of the interrelation of the various present day theoretical formulations.

The operator expression of the Dirac equation is related to the author's previous description of positrons. An attempt is made to interpret the operator ordering parameter in this case as a fifth coordinate variable in an extended Dirac equation. Fock's parametrization, discussed in an appendix, seems to be easier to interpret.

In the last section a summary of the numerical constants appearing in formulas for transition probabilities is given.

IN this paper we suggest an alteration in the mathematical notation for handling operators. This new notation permits a considerable increase in the ease of manipulation of complicated expressions involving operators. No results which are new are obtained in this way, but it does permit one to relate various formulas of operator algebra in quantum mechanics in a simpler manner than is often available. In particular, it is applied to quantum electrodynamics to permit an easier way of seeing the relationships among the conventional formulations, that of Schwinger and Tomanaga,[1] and that of the author.[2] These relationships have already been discussed by many people, particularly Dyson.[3] The connection was shown by means of a re-ordering of operators in each term of a perturbation power series. Here, the same end is achieved in much the same way without having to resort to such an expansion.

It is felt, in the face of daily experimental surprises for meson theory, that it might be worth while to spend one's time expressing electrodynamics in every physical and mathematical way possible. There may be some hope that a thorough understanding of electrodynamics might give a clue as to the possible structure of the more complete theory to which it is an approximation. This is one reason that this paper is published, even though it is little more than a mathematical re-expression of old material. A second reason is the desire to describe a mathematical method which may be useful in other fields.

The mathematics is not completely satisfactory. No attempt has been made to maintain mathematical rigor. The excuse is not that it is expected that rigorous demonstrations can be easily supplied. Quite the contrary, it is believed that to put the present methods on a rigorous basis may be quite a difficult task, beyond the abilities of the author.

The mathematical ideas are described and are illustrated with simple applications to quantum mechanics, in the first four sections. Some possible mathematical relations between the operator calculus described here and the theory of functionals is described in Appendix A, with further specific mathematical applications in Appendixes B and C. Section 5, and more particularly Secs. 6 to 9, apply specifically to quantum electrodynamics and may be omitted without loss by those whose interest is limited to mathematical questions. The use of a fifth variable to parametrize the Dirac equation is discussed in Secs. 8 and 9. An alternative procedure due to V. Fock[4] appears in Appendix D. Section 10 gives a summary of the rules for computing matrix elements.

1. DESCRIPTION OF THE NOTATION

The order of operation of operators is conventionally represented by the position in which the operators are written on the paper. Thus, the product AB of two operators A and B is to be distinguished from the product in reverse order BA. The algebra of operators is noncommutative, so that all of the ordinary algebra, calculus, and analysis with ordinary numbers becomes of small utility for operators. Thus, for a single operator, α, ordinary functions of this operator, such as $A=\exp\alpha$, can be defined, for example, by power series. These functions obey the rules of ordinary analysis even though α is an operator. But if another operator β is introduced with which α does not commute, the question of functions of the two variables α, β is beset with commutation difficulties and the simplest theorems of analysis are lost. For example, if $B=\exp\beta$, it is not true

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[1] See J. Schwinger, Phys. Rev. **76**, 790 (1949), and S. Tomonaga, Phys. Rev. **74**, 224 (1948), where additional references to previous work may be found.

[2] The author's previous papers will hereafter be designated as follows: R. P. Feynman, Revs. Modern Phys. **20**, 367 (1948)—C; Phys. Rev. **76**, 749 (1949)—I; Phys. Rev. **76**, 769 (1949)—II; and Phys. Rev. **80**, 440 (1950)—III.

[3] F. Dyson, Phys. Rev. **75**, 486, 1736 (1949).

[4] V. Fock, Physik. Z. Sowjetunion **12**, 404 (1937).

that BA, that is, $\exp\beta \exp\alpha$, is equal to $\exp(\beta+\alpha)$. Thus, the law of addition of exponents fails. Consequently, the principles of elementary calculus are no longer operative in a simple way. For example, expand $\exp(\alpha+\beta)$ to first order in β, assuming β small. The zero-order term is $\exp\alpha$, but the first-order term is neither $\beta \exp\alpha$ nor $(\exp\alpha)\beta$ nor the average of the two. From the theory of time-dependent perturbations in quantum mechanics we learn that it is

$$\exp(\alpha+\beta) = \exp\alpha + \int_0^1 \exp[(1-s)\alpha]\beta \exp(s\alpha)ds + \cdots. \quad (1)$$

The appearance of the integral in this analytic result appears surprising and its derivation does not indicate clearly how to differentiate or expand other functions of $\alpha+\beta$. Further, the simple integral on s is not easy to perform, although the results can be given in several ways as power series. That the integral cannot be done in a general fashion is clearly due to a weakness of notation, for in a representation in which α is diagonal with eigenvalues α_n we can of course verify directly the usual result,

$$\langle \exp(\alpha+\beta)\rangle_{mn} = (\exp\alpha_n)\delta_{mn} + (\exp\alpha_m - \exp\alpha_n)\beta_{mn}(\alpha_m-\alpha_n)^{-1} + \cdots \quad (2)$$

of the perturbation theory of stationary states.

We shall change the usual notation of the theory of operators and indicate the order in which operators are to operate by a different device. We attach an index to the operator with the rule that the operator with higher index operates later. Thus, BA may be written B_1A_0 or A_0B_1. The order no longer depends on the position on the paper, so that all of the ordinary processes of analysis may be applied as though A_0 and B_1 were commuting numbers. It is only at the end of a calculation, when the quantities are to be interpreted as operators, that the indices 0 and 1 are of importance if one wishes to reconvert an expression to the usual notation. Thus, if $A=\exp\alpha$ and $B=\exp\beta$, we can now safely write $BA=\exp(\alpha_0+\beta_1)$, as there is only one way to interpret the latter expression. Other analytic processes then become valid. For example,

$$\exp(\alpha_0+\beta_1) = 1+(\alpha_0+\beta_1)+\frac{1}{2!}(\alpha_0+\beta_1)^2+\cdots = 1+\alpha+\beta+\tfrac{1}{2}(\alpha^2+2\beta\alpha+\beta^2)+\cdots$$

in the conventional notation. For on squaring

$$(\alpha_0+\beta_1)^2 = \alpha_0^2+2\alpha_0\beta_1+\beta_1^2,$$

we must interpret the quantity $\alpha_0\beta_1$ as $\beta\alpha$ in accordance with our convention. The quantity β_1^2 alone (that is, not multiplied by any other expression with an index, such as α_0) is simply β^2, since the index is no longer necessary to define the order of operations, there being only one operator in the term.

The notation is to be extended so that the index need not be integral, for example, $A_{-\frac{1}{2}}B_{3.2}=BA$, since $3.1>-\frac{1}{2}$, and in general $A_sB_{s'}=BA$ if $s'>s$ and AB if $s>s'$ and is undefined if $s=s'$.

How can we work with an expression such as $\exp(\alpha+\beta)$ so as to free the α and β of their noncommutative aspects and thus utilize the theory of functions for rearranging the expressions? Take a quantity N very large and write

$$\exp(\alpha+\beta) \approx \left[\exp\frac{1}{N}(\alpha+\beta)\right]^N \approx \left[1+\frac{1}{N}(\alpha+\beta)\right]^N = \left[1+\frac{1}{N}(\alpha+\beta)\right]\left[1+\frac{1}{N}(\alpha+\beta)\right]\cdots \times\left[1+\frac{1}{N}(\alpha+\beta)\right] \text{ for } N \text{ factors.}$$

In each factor we replace $\alpha+\beta$ by $\alpha_i+\beta_i$, where i is an index running to N, and write

$$\exp(\alpha+\beta) = \lim_{N\to\infty}\prod_{i=1}^{N}\left[1+\frac{1}{N}(\alpha_i+\beta_i)\right] = \lim_{N\to\infty}\exp\left[\frac{1}{N}\sum_{i=1}^{N}(\alpha_i+\beta_i)\right],$$

where the last expression is written in accordance with the new convention that the index i controls the order of operation. (The ambiguity arising from α_i and β_i with the same index can only cause trouble in a product $\alpha_i\beta_i$, and such products are of vanishing importance as $N\to\infty$.) More simply, calling $s=i/N$, we can take the limit and write

$$\exp(\alpha+\beta) = \exp\left[\int_0^1 (\alpha_s+\beta_s)ds\right]. \quad (3)$$

That this is valid is, of course, evident, since we could call $\alpha_s+\beta_s=\gamma_s$ with γ a definite operator operating at order s, so that

$$\exp\left(\int_0^1 \gamma_s ds\right) = \exp\left(\int_0^1 \gamma ds\right),$$

for in this expression the order index is unnecessary, only one operator γ being involved. The integral is just γ,

$$\int_0^1 \gamma ds = \gamma\int_0^1 ds = \gamma,$$

since γ does not now depend on s. Therefore, Eq. (3) is trivial as it stands; but what is not trivial is the fact

that the right-hand side of Eq. (3) may be manipulated just as though α_s and β_s were numerical functions of s, with the assurance that now the order of operations will always be automatically specified by the index. For example, from Eq. (3) we have the legitimate relation

$$\exp(\alpha+\beta)=\exp\left(\int_0^1 \alpha_s ds\right)\exp\left(\int_0^1 \beta_s ds\right). \quad (4)$$

As an example, showing that such manipulations do not destroy the validity of equations, consider the term of first order in both α and β on both sides of Eq. (4). Expanding the left side as $1+(\alpha+\beta)+\frac{1}{2}(\alpha+\beta)^2+\cdots$, we see that the term in question is $\frac{1}{2}(\alpha\beta+\beta\alpha)$, while expansion of the right side gives for the corresponding term $(\int_0^1 \alpha_s ds)(\int_0^1 \beta_s ds)$. This can be simplified by being written as

$$\int_0^1\int_0^1 \alpha_s ds\beta_{s'}ds'=\int_0^1\int_0^s \alpha_s\beta_{s'}dsds'+\int_0^1\int_s^1 \alpha_s\beta_{s'}dsds'.$$

In the first integral we have $s>s'$, so that $\alpha_s\beta_{s'}$ is equal to $\alpha\beta$, while in the second $s<s'$, so it is $\beta\alpha$. Hence, there results $\alpha\beta\int_0^1\int_0^s dsds'+\beta\alpha\int_0^1\int_s^1 dsds'$; thus on performing the integrations we find finally

$$\left(\int_0^1 \alpha_s ds\right)\left(\int_0^1 \beta_s ds\right)=\tfrac{1}{2}(\alpha\beta+\beta\alpha).$$

This process of rearranging the form of expressions involving operators ordered by indices so that they may be written in conventional form we shall call disentangling the operators. The process is not always easy to perform and, in fact, is the central problem of this operator calculus. As a second example of disentangling, consider the problem of expanding Eq. (4) to the first order in β. It is evidently

$$\exp(\alpha+\beta)=\exp\left(\int_0^1 \alpha_s ds\right)+\exp\left(\int_0^1 \alpha_{s'}ds'\right)\int_0^1 \beta_s ds+\cdots. \quad (5)$$

The first term is simply $\exp\alpha$, for α_s is independent of s, as there is no other operator with which α_s does not commute in this term. The next is the integral over s of $\exp(\int_0^1 \alpha_{s'}ds')\beta_s$. In the integral on s' we can split the range, according to whether $s'<s$ or $s'>s$: $\exp(\int_s^1 \alpha_{s'}ds')\exp(\int_0^s \alpha_{s'}ds')\beta_s$. The $\alpha_{s'}$ in the first factor acts after the β_s and is otherwise independent of s', while the $\alpha_{s'}$ in the second factor is to act before the β_s. Hence, if we write these factors respectively after and before the β_s and imply the usual convention, the $\alpha_{s'}$ will be independent of s' in the range 0 to s and we may perform the integral. Hence, the result is the integral on s of $\exp[(1-s)\alpha]\beta\exp(s\alpha)$ in agreement with Eq. (1).

Incidentally, by applying new subscripts in another way the term may be also written as $\int_0^1 \exp[(1-s)\alpha_2]\beta_1 \times\exp(s\alpha_0)ds$, in which case the integral may be immediately performed to give

$$\begin{aligned}[(d/d\epsilon)\exp(\alpha+\epsilon\beta)]_{\epsilon=0} &= \exp\left(\int_0^1 \alpha_{s'}ds'\right)\int_0^1 \beta_s ds \\ &= \int_0^1 \exp[(1-s)\alpha]\beta\exp(s\alpha)ds \\ &= (\exp\alpha_2-\exp\alpha_0)(\alpha_2-\alpha_0)^{-1}\beta_1. \quad (6)\end{aligned}$$

All the four expressions are equivalent as has been shown, but only the first and third are in a form in which the operators are disentangled so that the conventional expressions may be used. In the representation in which α is diagonal, it should be evident that the matrix element of the last expression is that given in Eq. (2)

Any operator function of $\alpha+\beta$ can, by replacing $\alpha+\beta$ by $\int_0^1 \alpha_s ds+\int_0^1 \beta_s ds$, be manipulated in a manifold of ways, many of which lead to useful formulas. In a like manner, more complicated operator expressions can be rewritten using ordering indices. They may then be manipulated using all of the results of ordinary analysis.

A word about notation: Inasmuch as in mathematics and physics there are already many uses of the subscript notation, very often we shall write $\alpha(s)$ for α_s. In a sense, $\alpha(s)$ is a function of s, namely, in the sense that although the operator α may be definite, its order of operation is not—so that the operator plus a prescription of where it is to operate, $\alpha(s)$, is a function of s. Furthermore, there will be many cases in which the operator actually depends explicitly on the parameter of order. In this case we should have strictly to write $\alpha_s(s)$ but will omit the subscript when no ambiguity will result from the change.

We may remark in a general sense about the mathematical character of our expressions. Given an expression such as $\int_0^1 \beta(s)ds$, we are not concerned with evaluating the integral, for the quantity when separated from other factors with which it might be multiplied is incompletely defined. Thus, although $\int_0^1 \beta_s ds$ standing alone is equivalent simply to β, this is far from true when $\int_0^1 \beta_s ds$ is multiplied by other expressions such as $\exp\int_0^1 \alpha_s ds$. Thus, we must consider the complete expression as a complete functional of the argument functions $\alpha(s)$, $\beta(s)$, etc. With each such functional we are endeavoring to associate an operator. The operator depends on the functional in a complex way (the operator is a functional of a functional) so that, for example, the operator corresponding to the product of two functionals is not (in general) the simple product of the operators corresponding to the separate factors. (The corresponding statement equating the sum of two

functionals and the sum of the corresponding operators is true, however.) Hence, we can consider the most complex expressions involving a number of operators M, N, as described by functionals $F[M(s), N(s)\cdots]$ of the argument functions $M(s)$, $N(s)\cdots(\equiv M_s, N_s\cdots)$. For each functional we are to find the corresponding operator in some simple form;[5] that is, we wish to disentangle the functional. One fact we know is that any analytic rearrangement may be performed which leaves the value of the functional unchanged for arbitrary $M(s)$, $N(s)\cdots$ considered as ordinary numerical functions. Besides, there are a few special operations which we may perform on $F[M(s), N(s)\cdots]$, to disentangle the expressions, which are valid only because the functional does represent an operator according to our rules. These special operations (such as extracting an exponential factor discussed in Sec. 3) are, of course, proper to the new calculus; and our powers of analysis in this field will increase as we develop more of them.

2. APPLICATIONS IN QUANTUM MECHANICS

The wave equation $i\partial\psi/\partial t=H\psi$ determines the wave function $\psi(t_2)$ at time t_2 in terms of that at time t_1, $\psi(t_1)$. In fact, they are related by a unitary transformation $\psi(t_2)=\Omega(t_2, t_1)\psi(t_1)$. The unitary operator $\Omega(t_2, t_1)$ can be expressed as $\Omega(t_2, t_1)=\exp(-i(t_2-t_1)H)$ in the case that H is independent of the time. In spite of the simple appearance of the analytic form of Ω in terms of H, little has been done except formally with this expression for the reasons outlined in the previous section. We may readily re-express it as

$$\Omega(t_2, t_1)=\exp\left(-i\int_{t_1}^{t_2} H_t dt\right)$$

and may then find the expression easy to utilize. Further, if H is an explicit function of the time $H(t)$, we can consider the Ω to be developed as a large number of small unitary transformations in succession, so that we have directly

$$\Omega(t_2, t_1)=\exp\left[-i\int_{t_1}^{t_2} H_t(t)dt\right]. \tag{7}$$

Hereafter in this section we shall make the convention that time is the ordering parameter and simply write $H(t)$ for $H_t(t)$.

We can use this expression to derive many results in quantum theory. Thus, if $H(t)$ can be written as the sum of two parts $H^{(0)}(t)$ and $U(t)$, we have

$$\Omega^{(U)}(t_2, t_1)=\exp\left(-i\int_{t_1}^{t_2} H^{(0)}(t)dt\right)\times\exp\left(-i\int_{t_1}^{t_2} U(t)dt\right). \tag{8}$$

[5] This point of view is discussed in further detail in Appendix A.

If $H^{(0)}$ is simple and U is small, an expansion in powers of U is simple. We call $\Omega^{(U)}$ the operator corresponding to the hamiltonian $H^{(0)}+U$ and $\Omega^{(0)}$ that corresponding to $H^{(0)}$. The first-order difference of $\Omega^{(U)}$and $\Omega^{(0)}$ is

$$-i\int_{t_1}^{t_2} U(t)dt\exp\left[-i\int_{t_1}^{t_2} H(t')dt'\right],$$

which may be disentangled as

$$-i\int_{t_1}^{t_2}\exp\left[-i\int_{t}^{t_2} H(t')dt'\right]U(t)\exp\left[-i\int_{t_1}^{t} H(t')dt'\right]dt = -i\int_{t_1}^{t_2}\Omega^{(0)}(t_2, t)U(t)\Omega^{(0)}(t, t_1)dt, \tag{9}$$

as explained in connection with Eq. (6). This is a standard result of time-dependent perturbation theory.

As a second example consider the perturbation term of first order in U and in V arising from the hamiltonian $H^{(0)}(t)+U(t)+V(t)$. It is

$$-\int_{t_1}^{t_2} U(t')dt'\int_{t_1}^{t_2} V(t'')dt''\exp\left[-i\int_{t_1}^{t_2} H(t)dt\right].$$

In order to disentangle this, we can break the t'' integral into two regions, $t''<t'$ and $t''>t'$. The term arising from the first region has V operating before U, while the reverse is true for the other region. (The integral on t for each region is then divided into three parts determined by the relation of t to t', t''.) Thus, the term becomes, when disentangled, the sum of two terms:

$$\begin{aligned}&-\int_{t_1}^{t_2}\int_{t_1}^{t'}\Omega^{(0)}(t_2, t')U(t')\Omega^{(0)}(t', t'')V(t'')\\&\qquad\times\Omega^{(0)}(t'', t_1)dt''dt',\\&-\int_{t_1}^{t_2}\int_{t'}^{t_2}\Omega^{(0)}(t_2, t'')V(t'')\Omega^{(0)}(t'', t')U(t')\\&\qquad\times\Omega^{(0)}(t', t_1)dt''dt'.\end{aligned} \tag{10}$$

This is the way that the various terms corresponding to the different diagrams arise in quantum electrodynamics when an attempt is made to calculate explicitly a single operator expression arising in perturbation theory.

The results here are very similar to those derived from the lagrangian form of quantum mechanics as in III. Here we have the advantage of being able to use the more familiar operator concepts and to work in greater generality from the start. For it is not necessary that H be restricted to coordinate and momentum operators only. Equations (7) and (8) are correct for any H; for example, one containing creation and annihilation operators of second quantization, or Dirac matrices, etc.

The connection of these formulas to those given in I is simple. $K(x_2, t_2; x_1, t_1)$ is just a coordinate integral

kernel representation of the operator $\Omega(t_2, t_1)$ so that, for example, Eq. (9) gives directly

$$-i\int_{t_1}^{t_2} K^{(0)}(x_2, t_2; x, t)U(x, t)K^{(0)}(x, t; x_1, t_1)dxdt,$$

the expression (9) of I, while Eq. (10) translates immediately into the expression (30) of III.

As another type of application, consider two interacting systems whose hamiltonian is $H^{(a)}+H^{(b)}+U[x^{(a)}, x^{(b)}]$, where $H^{(a)}$ involves operators of system (a) only, $H^{(b)}$ involves only those of system (b), and U involves both. Then we may ask for the amplitude, if at t_1 system (a) is in state ψ_1 and system (b) in ϕ_1, that at t_2 they are in ψ_2, ϕ_2. This is the matrix element

$$m=\left\langle \psi_2\phi_2 \middle| \exp\left(-i\int_{t_1}^{t_2} H^{(a)}(t)dt - i\int_{t_1}^{t_2} H^{(b)}(t)dt - i\int_{t_1}^{t_2} U[x^{(a)}(t), x^{(b)}(t)]dt \right) \middle| \psi_1\phi_1 \right\rangle. \quad (11)$$

But this may be split into two problems. We may first find the matrix element

$$T^{(a)}[x^{(b)}(t)]=\left\langle \psi_2 \middle| \exp\left(-i\int_{t_1}^{t_2} H^{(a)}(t)dt - i\int_{t_1}^{t_2} U[x^{(a)}(t), x^{(b)}(t)]dt \right) \middle| \psi_1 \right\rangle \quad (12)$$

for the system (a) alone, considering that in the interaction potential $U[x^{(a)}, x^{(b)}]$, all operators referring to (b) are arbitrary numerical functions of t. (We have been writing as though U depends on (b) only through the coordinate, $x^{(b)}$; but the same method applies if it is also a function of momentum, or spin, or other operators on system (b).) Then the matrix element $T^{(a)}$ depends on the function $x^{(b)}(t)$. As we indicate, it is a functional of $x^{(b)}(t)$. The final answer, m, is then a matrix element $\langle\phi_2|M|\phi_1\rangle$ between the states ϕ_1 and ϕ_2;

$$m=\left\langle \phi_2 \middle| \exp\left(-i\int_{t_1}^{t_2} H^{(b)}(t)dt \right) T^{(a)}[x^{(b)}(t)] \middle| \phi_1 \right\rangle, \quad (13)$$

wherein now the quantities $x^{(b)}(t)$ are considered as ordered operators operating relative to each other and to $H^{(b)}(t)$ in accordance with the time parametrization.

In this way we can analyze one part of a pair of interacting systems without having yet analyzed the other. The influence of a on b is completely contained in the operator functional $T^{(a)}[x^{(b)}(t)]$. This separation may be useful in analysis of the theory of measurement and of quantum statistical mechanics. It is the possibility of such a separation which exists also in the case[2] of the lagrangian form of quantum mechanics, **C**, which makes that form useful in analyzing the quantum properties of the electromagnetic field. We may therefore expect that with the present operator notation it should be equally easy to make this analysis. That this is indeed true we show by example further on. Since this, the main advantage of the lagrangian form, can be so easily managed with the new notation for operators, this may well take the place of the lagrangian form in many applications. It is in some ways a more powerful and general form than the lagrangian. It is not restricted to the nonrelativistic mechanics in any way. A possible advantage of the other form at present might be a slight increase in *anschaulichkeit* offered for the interpretation of the nonrelativistic quantum mechanics.

3. DISENTANGLING AN EXPERIMENTAL FACTOR

There is one theorem which is very useful in disentangling operator expressions. We shall give it in this section. Suppose we have several operators M, N, etc. (which may also be functions of time, or more generally, the ordering parameter s), which are ordered in some way.

Let us say the functional $F[M(s), N(s)\cdots]$ defines the ordered operator. Now suppose we replace $M(s)$ by $M'(s)=U^{-1}M(s)U$, $N(s)$ by $N'=U^{-1}NU$, etc., where U is some constant operator. Then, as is well known, in $F[M(s), N(s)\cdots]$ in any product of successive operators, such as $M(s+ds)N(s)$, the UU^{-1} cancel out in between (that is, $MN=U^{-1}MUU^{-1}NU=U^{-1}MNU$, etc.), so that there results

$$F[M'(s), N'(s)\cdots]=U^{-1}F[M(s), N(s)\cdots]U, \quad (14)$$

where the U's are written to operate in the correct order. (If we wish to be more specific, we can imagine the range of the ordering parameter to be $s=0$ to 1 and write the right-hand side as $U_1^{-1}FU_0$.)

This is a simple rewriting of a well-known theorem of equivalence transformations. However, a much more interesting case is that in which $U(s)$ is actually a function of the ordering parameter. That is, we contemplate performing different transformations on the operators $M(s)$ depending on the value of s at which they are to operate. Then in a product of successive operators such as $M'(s+ds)N'(s)$, where

$$M'(s+ds)=U^{-1}(s+ds)M(s+ds)U(s+ds)$$

(operating in the order indicated by the position of U^{-1}, M and U) and $N'(s)=U^{-1}(s)N(s)U(s)$, the factors $U(s+ds)$ and $U^{-1}(s)$ will not cancel out, but we will find the operator $U(s+ds)U^{-1}(s)$ operating between times s and $s+ds$, say at $s+\frac{1}{2}ds$. If we assume U continuous, we can imagine $U(s)$ differs from $U(s+ds)$ to the first order in ds, and hence that $U(s+ds)U^{-1}(s)$ equals to first order in ds:

$$U(s+ds)U^{-1}(s)=1+P(s)ds,$$

where $P(s)$ is an operator defined by this relation in the

limit $ds\to 0$. We may write this relation

$$dU(s)/ds = P(s)U(s) \tag{15}$$

with positional ordering. Hence, between s and $s+ds$ there should operate an additional factor $1+P(s)ds$, which for convenience we may write, valid to first order, as $\exp[P(s+\frac{1}{2}ds)ds]$. The $s+\frac{1}{2}ds$ in $P(s+\frac{1}{2}ds)$ will automatically locate the factor in the correct order. But there is a factor of this kind appearing between the operators for each value of s, or multiplying the factors all together, we obtain the net factor $\exp\int_0^1 P(s)ds$, the product becoming a sum, or integral in the exponent. Hence, we have the general theorem:

$$F[M'(s), N'(s)\cdots] = U^{-1}(1)F[M(s), N(s)\cdots] \times \exp\left(\int_0^1 P(s)ds\right)U(0),$$

where

$$M'(s) = U^{-1}(s)M(s)U(s) \tag{16}$$

and

$$U(s) = \exp\left[\int_0^s P(s')ds'\right]U(0), \tag{17}$$

this last coming from integrating Eq. (15). We shall use the theorem by writing it in the form

$$\exp\int_0^1 P(s)ds F[M(s), N(s)\cdots] = U(1)F[M'(s), N'(s)\cdots]U^{-1}(0), \tag{18}$$

in which form it serves as a rule for disentangling an exponential factor from another expression. A word of caution is necessary in reading Eqs. (18), (16), and (17), for three different notations are used in the expressions. In Eq. (18) the new ordered notation is used in its complete form; for example, the s in $\exp\int_0^1 P(s)ds$ gives the order in which the P is to operate relative to the M, N of the functional F which it multiplies. In Eq. (16), however, all the operators are to operate at s, but the relative order in M' of U, M, and U^{-1} is as given by the usual position convention. Finally, Eq. (17) would be less ambiguous if it were replaced by the differential equation (15). For in the solution (17), the s' are to bear no relation to the s in Eq. (16) or Eq. (18). The operator $U(s)$ is to be computed from P by Eq. (17) first, then the whole operator $U(s)$ is to operate in Eq. (16), and then in Eq. (18) at the position s.

We shall use this theorem in several applications related to quantum electrodynamics. Most particularly, we shall find a certain special case useful enough to warrant special mention. It is the case that $P(s)$ is of the form $\alpha(s)P_s$, where $\alpha(s)$ is a simple numerical function, and P_s is an operator whose form does not depend on s but whose order of operation does. Then if we call $a(s)=\int_0^s \alpha(s')ds'$, so that $a(s)$ is also a numerical function, Eq. (17) gives $U(s)=\exp[a(s)P_s]U(0)$. We shall further choose to specialize $U(0)=1$. (The more general case corresponds to a final simple constant equivalence transformation (14) with $U(0)$.) Then our theorem may be written

$$\exp\left[\int_0^1 \alpha(s)P_s ds\right]F[M(s), N(s)\cdots] = \exp\left[P_1\int_0^1 \alpha(s)ds\right]F[M'(s), N'(s)\cdots], \tag{19}$$

where

$$M'(s) = \exp\left[-P_s\int_0^s \alpha(s')ds'\right]M(s) \times \exp\left[+P_s\int_0^s \alpha(s')ds'\right]. \tag{20}$$

Further, since this theorem with Eq. (20) substituted into Eq. (19) is valid when $\alpha(s)$ is an *arbitrary* numerical function, it is also true if $\alpha(s)$ is any ordered operator $\alpha(s)$ commuting with P for all s, provided that in *all* expressions involving α, the parameter s or s' is consistently interpreted as giving the order in which the α operates.[6]

The mathematical proof of the theorems (18) and (19) offered here is admittedly very sketchy; but since the theorems are true, it should not be hard to supply them with more satisfactory demonstrations (see Appendix A for an alternative demonstration).

There are a number of other interesting relations which we may derive from Eq. (19), but which we shall not need in this paper. One is included here because it has been found useful in certain other applications. If $\alpha(s)$ is considered infinitesimal in Eqs. (19) and (20), expansion in first order in α gives the following result (or differentiate each side with respect to $\alpha(t)$ and set $\alpha(s)=0$),

$$P_t F[M(s)] = P_1 F[M(s)] - \int_t^1 (PM-MP)_s \delta F/\delta M(s)ds$$

(if we assume F can be represented by a functional having a derivative $\delta F/\delta M(s)$). We have taken F to depend only on one operator $M(s)$, but the generalization is clear. Here, $(PM-MP)_s$ is conventional ordering is $PM(s)-M(s)P$ and is considered to act as an entity at s. The differential form

$$(dP_t/dt)F[M(s)] = (PM-MP)_t \delta F/\delta M(t) \tag{21}$$

is also useful.

[6] To simplify such descriptions, in a situation involving two sets of operators, any one of the first set commuting with any one of the second, it is often convenient to generalize to the use of two different ordering parameters—one for first set, and one for the second.

4. THE INTERACTION REPRESENTATION

As a first simple direct application of our theorem consider again the perturbation problem (8) of computing the operator

$$\Omega^{(U)}(t_2, t_1)$$

$$=\exp\left[-i\int_{t_1}^{t_2} H^{(0)}(t)dt\right]\exp\left[-i\int_{t_1}^{t_2} U(t)dt\right]. \quad (8)$$

If we suppose the properties of $H^{(0)}$ to be known and simple, the right side of Eq. (8) may be disentangled by means of our theorem (18). We consider $-iH^{(0)}(t)$ as an operator $P(s)$ and

$$\exp\left[-i\int_{t_1}^{t_2} U(t)dt\right]$$

as the functional F from which the

$$\exp\left[-i\int H^{(0)}(t)dt\right]$$

is to be disentangled. Hence, a direct application of Eq. (18) gives

$$\exp\left[-i\int_{t_1}^{t_2} H^{(0)}(t)dt\right]\exp\left[-i\int_{t_1}^{t_2} U(t)dt\right]$$

$$=S(t_2)\exp\left[-i\int_{t_1}^{t_2} U'(t)dt\right]S^{-1}(t_1), \quad (22)$$

where

$$S(t)=\exp\left[-i\int_{a}^{t} H^{(0)}(t')dt'\right]$$

(the lower limit a used in defining S is arbitrary; it may be taken as t_1 so $S(t_1)=1$ if that is convenient), and

$$U'(t)=S^{-1}(t)U(t)S(t) \quad (23)$$

(operating in positional order). If we take matrix elements not between states ψ_1 and ψ_2 but between $\psi_1'=S^{-1}(t_1)\psi_1$ and $\psi_2'=S^{-1}(t_2)\psi_2$, we may call the Ω-matrix Ω' and omit the $S(t_2)$ and $S^{-1}(t_1)$ factors in Eq. (22). These new time-dependent states ψ' are evidently states that would give rise to ψ_1 at t_1 and ψ_2 at t_2 (from some fixed reference time a) if the perturbation were not acting. Then the time-dependent perturbation theory simply comes to evaluating

$$\Omega'(t_2, t_1)=\exp\left(-i\int_{t_1}^{t_2} U'(t)dt\right). \quad (24)$$

Expansion in power series, substitution of U' from Eq. (23), and use of the relation $\Omega^{(0)}(t', t'')=S(t')S^{-1}(t'')$ leads immediately to the formulas (9) and (10), so that Eqs. (23) and (24) give the simplest form to the time-dependent perturbation theory. Of course, the same results may be obtained by a unitary transformation in the conventional way. Ordinarily, result (24) is not written in this way, for it involves the time convention on the ordering of the operators. (It is usually expressed as a differential equation for Ω'.) If the perturbation U represents an interaction between some systems described by $H^{(0)}$, the reduction of Eq. (8) to Eq. (24) is called passing to the interaction representation.

5. SYSTEM COUPLED TO AN HARMONIC OSCILLATOR

As a further example of the use of the notation we solve completely the problem of a particle or system of particles coupled linearly to an harmonic oscillator. This problem in greater generality is the main problem of quantum electrodynamics. It has been thoroughly studied in III, but we solve it again as an illustration of the new notation. Let the hamiltonian of the combined system be

$$H=H_p(t)+H_{\text{osc}}-\Gamma(t)q,$$

where H_{osc} is the hamiltonian of the oscillator alone,

$$H_{\text{osc}}=(1/2m)(p^2+\omega^2q^2),$$

where p is the momentum conjugate to q, the coordinate of the oscillator. Further, H_p, which may depend explicitly on time, is the hamiltonian of the particles, and Γ may contain any operators pertaining to the particles as well as possibly being an explicit function of the time. We ask for the matrix element for finding the particles in state $\chi_{t''}$ and the oscillator in some eigenstate m at time t'', if at a previous time t' the particles are in state $\chi_{t'}$, and the oscillator in its nth eigenstate. It is the matrix element

$$\left\langle \chi_{t''}\phi_m\left|\exp-i\int_{t'}^{t''}[H_p(t)+H_{\text{osc}}(t)-\Gamma(t)q(t)]dt\right|\phi_n\chi_{t'}\right\rangle$$

using the time ordering convention. As already discussed in Sec. 2, this can be considered as the matrix element between states $\chi_{t'}$ and $\chi_{t''}$ of the matrix

$$M=\exp\left(-i\int_{t'}^{t''} H_p(t)dt\right)G_{mn}, \quad (25)$$

where G_{mn} (the analog of T of Sec. 2), a functional of $\Gamma(t)$, serves to define the net effect on the particles of their interaction with the oscillator. Calculation of G_{mn} means evaluating

$$G_{mn}=\left\langle\phi_m\left|\exp-i\int_{t'}^{t''} H_{\text{osc}}(t)dt\right.\right.$$

$$\left.\left.\exp i\int_{t'}^{t''}\Gamma(t)q(t)dt\right|\phi_n\right\rangle \quad (26)$$

in a general way as a functional of $\Gamma(t)$. We are to consider $\Gamma(t)$ here as a simple numerical function, and

only later utilize the fact that it is an operator involving the particles when we go to evaluate the matrix M, Eq. (25), between the particle states $\chi_{t'}$ and $\chi_{t''}$. The evaluation of G_{mn} for an arbitrary numerical function $\Gamma(t)$ may be performed in a variety of ways. One is by the lagrangian form of quantum mechanics given explicitly in III, Sec. 3, with the sole difference (which is unessential for this part of the problem) that there $\Gamma(t)$ was called $\gamma(t)$ and was a functional of the coordinates $x(t)$ of the particles, while here we see we are in a more general position as $\Gamma(t)$ may be a functional of any ordered operators referring to the particles. We find, for example [III, Eq. (14)],

$$G_{00}=\exp\Big[-(1/4\omega)\int_{t'}^{t''}\int_{t'}^{t''}\exp(-i\omega|t-s|)\times\Gamma(t)\Gamma(s)dtds\Big]. \quad (27)$$

The same result may also be obtained by a direct solution of the Schrödinger equation for the forced oscillator. The great advantage of the operator notation is to allow this formal solution for an oscillator forced by an *arbitrary potential function* $\Gamma(t)$ to be equally useful when the oscillator is actually in interaction with a quantum-mechanical system!

Thus, we have the answer for G_{mn} in III, Eq. (57), using Γ for γ. It is, however, interesting to see how this expression for G_{mn} could be worked out directly using the methods of the ordered operator calculus. We want to disentangle the operator

$$G=\exp\Big(-i\int_{t'}^{t''}H_{\mathrm{osc},t}dt\Big)\exp\Big(i\int_{t'}^{t''}\Gamma(t)q_t dt\Big). \quad (26')$$

Let us call, in the usual way, $Q^*=(\frac{1}{2}\omega)^{\frac{1}{2}}(q-i\omega^{-1}p)$ and $Q=(\frac{1}{2}\omega)^{\frac{1}{2}}(q+i\omega^{-1}p)$ the creation and annihilation operators. They satisfy the commutation relation

$$QQ^*-Q^*Q=1. \quad (28)$$

In terms of them $H_{\mathrm{osc}}=\frac{1}{2}\omega(Q^*Q+QQ^*)$ and

$$q=(2\omega)^{-\frac{1}{2}}(Q+Q^*). \quad (29)$$

Now as a first step we pass to the interaction representation (Sec. 4). We use the theorem (20) with $P_s=-iH_{\mathrm{osc}}$, $\alpha(s)=1$, to disentangle the $\exp[-i\int H_{\mathrm{osc}}(t)dt]$ factor, obtaining

$$G=S(t'')\exp\Big\{i(2\omega)^{-\frac{1}{2}}\int_{t'}^{t''}\Gamma(t)[Q'(t)+Q'^*(t)]dt\Big\}S^{-1}(t'),$$

where $Q'(t)=S^{-1}(t)QS(t)$, $Q'^*(t)=S^{-1}(t)Q^*S(t)$, and $S(t)=\exp(-itH_{\mathrm{osc}})$. By redefining the wave functions so they contain $S(t)$, or, for eigenstates, the energy factors $\exp(-iE_nt)$ for the free oscillator, we can eliminate the $S(t'')$ and $S^{-1}(t')$ and need merely calculate the matrix element of

$$G'=\exp i(2\omega)^{-\frac{1}{2}}\int_{t'}^{t''}\Gamma(t)[Q'(t)+Q'^*(t)]dt.$$

From Eq. (28) we readily calculate that[7]

$$Q'(t)=Qe^{-i\omega t}\quad\text{and}\quad Q'^*(t)=Q^*e^{+i\omega t}, \quad (30)$$

so that the problem becomes the disentanglement of

$$G'=\exp\Big[i(2\omega)^{-\frac{1}{2}}\int_{t'}^{t''}\Gamma(t)e^{+i\omega t}Q_t^*dt\Big]\times\exp\Big[i(2\omega)^{-\frac{1}{2}}\int_{t'}^{t''}\Gamma(s)e^{-i\omega s}Q_s ds\Big].$$

We shall find it most convenient to disentangle this into a form in which all the annihilation operators operate first, and then come the creation operators (since the nth state cannot suffer more than n annihilation operators, $Q^{n+1}\phi_n$ vanishing, the expression will be easy to evaluate in this form). To this end let us use theorem (19) again, this time with $P=Q^*$,

$$\alpha(s)=i(2\omega)^{-\frac{1}{2}}\Gamma(t)e^{+i\omega t}.$$

Calling, temporarily,

$$A(t)=i(2\omega)^{-\frac{1}{2}}\int_{t'}^{t}\Gamma(t)e^{+i\omega t}dt,$$

we find

$$G'=\exp[A(t'')Q_{t''}^*]\exp\Big[i(2\omega)^{-\frac{1}{2}}\int_{t'}^{t''}\Gamma(t)e^{-i\omega t}Q''(t)dt\Big],$$

where

$$Q''(t)=\exp[-A(t)Q^*]Q\exp[+A(t)Q^*]. \quad (31)$$

The commutation relation (28) here gives[8]

$$Q''(t)=Q_t+A(t), \quad (32)$$

so that

$$G'=\exp[A(t'')Q_{t''}^*]\times\exp\Big[i(2\omega)^{-\frac{1}{2}}\int_{t'}^{t''}\Gamma(t)e^{-i\omega t}\{Q_t+A(t)\}dt\Big].$$

In the last factor Q_t can be replaced by $Q_{t'}$ since t is in any case less than t'', so that all the Q_t's come before the $Q_{t''}^*$, and Q_t need not be ordered relative to itself as it is a constant operator. Hence, we may, with a slight rewriting (for example, $i\beta^*$ for $A(t'')$), write

$$G'=\exp(i\beta^*Q_{t''}^*)\exp(i\beta Q_{t'})G_{00} \quad (33)$$

[7] For, $Q'(t)=\exp(itH_{\mathrm{osc}})Q\exp(-itH_{\mathrm{osc}})$ implies $dQ'/dt=iS^{-1}(t)\times(H_{\mathrm{osc}}Q-QH_{\mathrm{osc}})S(t)=-i\omega Q'(t)$, since $H_{\mathrm{osc}}Q-QH_{\mathrm{osc}}=-\omega Q$ by Eq. (28). Thus, since $Q'(0)=Q$, one obtains $Q'(t)=e^{-i\omega t}Q$.

[8] For, differentiation of the expression for $Q''(t)$ gives $dQ''(t)/dt=-A'(t)\exp(-A(t)Q^*)(Q^*Q-QQ^*)\exp(+A(t)Q^*)=A'(t)$ by Eq. (28), so integration gives $Q''(t)=A(t)+Q$, since $Q''(t')=Q$, inasmuch as $A(t')=0$.

with G_{00} equal to $\exp i(2\omega)^{-\frac{1}{2}}\int_{t'}^{t''}\Gamma(t)e^{-i\omega t}A(t)dt$ and therefore identical to Eq. (27), and with

$$\beta=(2\omega)^{-\frac{1}{2}}\int_{t'}^{t''}\Gamma(t)e^{-i\omega t}dt,$$
$$\beta^*=(2\omega)^{-\frac{1}{2}}\int_{t'}^{t''}\Gamma(t)e^{+i\omega t}dt, \tag{34}$$

just as in III, Eq. (58). The operator G' is now completely disentangled. Its matrix element between n and m we call G_{mn}. The matrix element may be evaluated by ordinary methods, since the t' and t'' in Q and Q^*, respectively, in Eq. (33) are unnecessary if the positional notation is used. That the element for $n=0$, $m=0$ is just what we call G_{00} is evident, for if $\exp(i\beta Q)$ be expanded as $1+\beta Q+\beta^2Q^2\cdots$ and the result applied to ϕ_0, all the terms beyond the first give zero for $Q\phi_0=0$. Thus, this exponential may effectively be replaced by unity. Likewise, the second can be replaced by unity for $\phi_0{}^*Q^*=0$.

The case of more general values of m, n, may be worked out by writing

$$\phi_n=(n!)^{-\frac{1}{2}}Q^{*n}\phi_0, \tag{35}$$

so that

$$G_{mn}=\langle\phi_0|\,(m!)^{-\frac{1}{2}}(n!)^{-\frac{1}{2}}Q^m e^{i\beta^*Q^*}e^{i\beta Q}Q^{*n}\,|\,\phi_0\rangle G_{00}. \tag{36}$$

Then, since $e^{i\beta Q}Q^*=(Q^*+i\beta)e^{i\beta Q}$ (as in Eqs. (31), (32)), repetition n times gives $e^{i\beta Q}Q^{*n}=(Q^*+i\beta)^n e^{i\beta Q}$, and likewise $Q^m e^{i\beta^*Q^*}=e^{i\beta^*Q^*}(Q+i\beta^*)^m$. We find

$$G_{mn}=\langle\phi_0|\,(m!)^{-\frac{1}{2}}(n!)^{-\frac{1}{2}}e^{i\beta^*Q^*}(Q+i\beta^*)^m \times(Q^*+i\beta)^n e^{i\beta Q}\,|\,\phi_0\rangle G_{00}. \tag{37}$$

The exponentials may now be replaced by unity as previously discussed. The other factors expanded by the binomial theorem give

$$G_{mn}=(m!)^{-\frac{1}{2}}(n!)^{-\frac{1}{2}}\sum_r\binom{m}{r}\sum_s\binom{n}{s}(i\beta^*)^{m-r} \times(i\beta)^{n-s}\langle\phi_0|\,Q^rQ^{*s}\,|\,\phi_0\rangle G_{00}.$$

The next to last factor by Eq. (35) is

$$(s!)^{\frac{1}{2}}(r!)^{\frac{1}{2}}\langle\phi_r|\,\phi_s\rangle=(s!)^{\frac{1}{2}}(r!)^{\frac{1}{2}}\delta_{rs},$$

so that finally

$$G_{mn}=(m!n!)^{-\frac{1}{2}}\sum_r\binom{m}{r}\binom{n}{r}r!(i\beta^*)^{m-r}(i\beta)^{n-r}G_{00} \tag{38}$$

as in III, Eq. (57).

Having this form for the behavior of a system of particles interacting with a single oscillator, we could go on and discuss the quantum electromagnetic field as a set of such oscillators. It is evident that to do so would be simply to repeat the steps described in III, Sec. 4, using Γ for γ and reinterpreting the symbols as ordered operators rather than as amplitudes associated with a path. The result in general is Eq. (43) below [in agreement with III, Eq. (48)], and there is no need to go into the details again of summing the effects of all the oscillators to obtain this result. We will pass directly to a discussion of the complete electromagnetic field.

6. QUANTUM ELECTRODYNAMICS

There are available several equivalent formulations of quantum electrodynamics.[1,2,9] We shall give a very brief outline of their interrelationships using the ordered operator notation. We can start with the usual formalism of Heisenberg, Pauli, and Dirac.[9] The wave function of the system, consisting of the electron-positron field and of the electromagnetic field in interaction, satisfies a wave equation $i\partial\psi/\partial t=H\psi$, where the hamiltonian for the system may be written $H=H_m+H_f+H_i$, where H_m is that of the electron-positron field free of potentials, H_f is that of the electromagnetic field in empty space, and H_i represents the interaction of the two fields. The problem is to obtain the wave function at time t_2 in terms of its value at a previous time t_1. It is therefore a study of the operator

$$\exp\left\{-i\int_{t_1}^{t_2}[H_m(t)+H_f(t)+H_i(t)]dt\right\}.$$

We can simplify this by first disentangling the exponential factor

$$\exp\left\{-i\int_{t_1}^{t_2}[H_m(t)+H_f(t)]dt\right\}.$$

That is, we go directly to the interaction representation, and find that we must analyze

$$\exp\left\{-i\int_{t_1}^{t_2}H_i'(t)dt\right\}.$$

We shall always use the interaction representation and shall omit the prime here for simplicity of notation. Furthermore, it will be sufficient for our purpose to consider only the case $t_1\to-\infty$ and $t_2\to+\infty$, so that quantum electrodynamics is a study of the operator[10]

$$S=\exp\left[-i\int_{-\infty}^{\infty}j_\mu(\mathbf{x},t)A_\mu(\mathbf{x},t)d^3\mathbf{x}dt\right], \tag{39}$$

where the $A_\mu(1)$ is the operator potential of the electromagnetic field and $j_\mu(1)$ is the operator current of the

[9] See, for example, P. A. M. Dirac, *The Principles of Quantum Mechanics*, (The Clarendon Press, Oxford, 1947), third edition, Chapter 12.

[10] For systems with particles of spin zero or one, S may be written in this same form by use of the Kemmer-Duffin matrices β_μ, as is shown by C. N. Yang and D. Feldman, Phys. Rev. **79**, 972 (1950), for example, Eq. (33). Thus, all of these results given here for the Dirac field are equally correct for spin zero or one if γ_μ is replaced by β_μ. See also M. Neumann and W. H. Furry, Phys. Rev. **76**, 1677 (1949), and R. Moorhouse, Phys. Rev. **76**, 1691 (1949).

Dirac electron-positron field. They, of course, commute with each other, since they refer to different systems. Further,[11] $A_\mu(1)$, $A_\mu(2)$ commute if 1 and 2 are separated by a spacelike interval, as do $j_\mu(1)$, $j_\mu(2)$. In the expression for S the operators are ordered in accordance with the time t.

We may thus define the problem of quantum electrodynamics as a study of the operator S. Let us imagine for purposes of discussion that we disregard the derivation of S given in the preceding paragraph. We imagine the problem is given directly as the analysis of the operator S defined in Eq. (39) (assuming the commutation rules, reference 11). Let us see how the various formalisms are simply different ways of expressing or analyzing S.

First, we might try to define S in some way which would not require the use of the ordering notation. Suppose we split the range of integration of t into two regions $-\infty$ to τ and τ to ∞. Then the integral may be split into two parts. We can write the factors, as

$$S=\exp\left[-i\int_\tau^\infty j_\mu(\mathbf{x},t)A_\mu(\mathbf{x},t)d^3\mathbf{x}dt\right]\times\exp\left[-i\int_{-\infty}^\tau j_\mu(\mathbf{x}',t')A_\mu(\mathbf{x}',t')d^3\mathbf{x}'dt'\right].$$

Now, since $t'<t$, all the operators on the last factor act before those of the first factor, so they are disentangled relative to the first factor. Hence, we are led to define an operator function of τ,

$$\Omega(\tau)=\exp\left[-i\int_{-\infty}^\tau j_\mu(\mathbf{x},t)A_\mu(\mathbf{x},t)d^3\mathbf{x}dt\right].$$

If τ is changed to $\tau+d\tau$, an additional factor appears operating in front of all other $t\lessgtr\tau$, namely, $\exp[-i\Delta\tau\int j_\mu(\mathbf{x},\tau)A_\mu(\mathbf{x},\tau)d^3\mathbf{x}]$. Hence, $\Omega(\tau)$ satisfies the differential equation

$$id\Omega/d\tau=\left[\int j_\mu(\mathbf{x},\tau)A_\mu(\mathbf{x},\tau)d^3\mathbf{x}\right]\Omega(\tau),\qquad(40)$$

the operators operating in positional order.

Thus, we are lead to a differential equation, the solution of which can be used to define S (for S is $\Omega(\tau)$ as $\tau\to+\infty$ when $\Omega(\tau)$ is that solution of Eq. (40) which $\to I$ as $\tau\to-\infty$). If we define $\psi(-\infty)$ as an initial state wave function, clearly, $\psi(\tau)=\Omega(\tau)\psi(-\infty)$ satisfies the same equation as Ω. This is the Schrödinger equation in the usual formulation if written in interaction representation. (We probably would not be led to go back to the ordinary representation as this is an unnecessary increase in complexity.)

The apparent lack of covariance implied by using time to define the differential equation can be remedied by analyzing S in a slightly different manner, suggested by Tomanaga and by Schwinger.[1]

The variables $\mathbf{x}$, t over which one integrates in Eq. (39) may be divided into two groups in another way; those previous to and those following an arbitrary spacelike surface σ:

$$S=\exp\left[-i\int_b j_\mu(\mathbf{x},t)A_\mu(\mathbf{x},t)d^3\mathbf{x}dt\right]\times\exp\left[-i\int_a j_\nu(\mathbf{x}',t')A_\nu(\mathbf{x}',t')d^3\mathbf{x}'dt'\right],$$

where the region a of integration of the second factor are those points of space-time previous to σ, while b are those following σ. Now again the factors are disentangled. It might at first be argued that since there are some values of t' greater than t, the corresponding operators in $\int_a$ should follow, not precede, those in $\int_b$. But for those t' which exceed t, the points $\mathbf{x}$, $\mathbf{x}'$ are separated by a spacelike interval (as σ is a spacelike surface); hence the order of the $A_\mu(\mathbf{x},t)$ and $A_\nu(\mathbf{x}',t')$ as well as of $j_\mu(\mathbf{x},t)$ and $j_\nu(\mathbf{x}',t')$ is irrelevant, as these commute. Hence, the operators are, in fact, disentangled; and we can define

$$\Omega(\sigma)=\exp\left[-i\int_a j_\mu(\mathbf{x},t)A_\mu(\mathbf{x},t)d^3\mathbf{x}dt\right]$$

as an operator defined as a functional of the surface σ. A small change in surface at $\mathbf{x}$, t changes the operator by

$$\delta\Omega(\sigma)/\delta\sigma(\mathbf{x},t)=-ij_\mu(\mathbf{x},t)A_\mu(\mathbf{x},t)\Omega(\sigma),\qquad(41)$$

the equation of Schwinger[12] for $\Omega(\sigma)$ (and also for $\psi(\sigma)$ defined by $\Omega(\sigma)\psi(-\infty)$). Again, S is $\Omega(\sigma)$ as the surface σ is removed to $+\infty$.

These differential equations (40) or (41) are therefore needed to define the operator S if one is limited to conventional notation. The form (41) has the advantage of putting the relativistic invariance more into evidence. However, the solution (39) is common to both and is more easily used. It is likewise evidently invariant if we write it

$$S=\exp\left[-i\int j_\mu(1)A_\mu(1)d\tau_1\right]\qquad(42)$$

(with the point 1 representing $\mathbf{x}_1$, t_1 and $d\tau_1=d^3\mathbf{x}_1dt_1$) and assume the convention here that *if two operators in Eq. (42) correspond to points separated by either a timelike or a zero interval, that operates first which corresponds*

[11] See, for example, J. Schwinger, Phys. Rev. **74**, 1439 (1948). In his notation (except that we put a factor e in A_μ rather than j_μ), the commutation relations are [his Eqs. (2.28)and (2.29)]

$$[A_\mu(x),A_\nu(x')]=4\pi e^2i\delta_{\mu\nu}D(x-x')$$

and $j_\mu(x)=\bar\psi(x)\gamma_\mu\psi(x)$ with $\{\psi_\alpha(x),\bar\psi_\beta(x')\}=-iS_{\alpha\beta}(x-x')$ if no external potential is acting. Other combinations commute.

[12] I. Schwinger. Phys. Rev. **74**, 1439 (1948).

to the earlier time. If they are separated by a spacelike interval, no definition is necessary, for they commute.

The other developments consist in methods of actually evaluating Eq. (42), given the commutation relations[11] of the $A_\mu(1)$. The method explained by Dyson[3] consists of making a power series expansion of S and disentangling it term by term. For example, the second-order term is

$$-\tfrac{1}{2}\int j_\mu(1)A_\mu(1)d\tau_1\int j_\nu(2)A_\nu(2)d\tau_2.$$

This term may then be analyzed into the conventional notation by reordering the operators. In this example it is necessary merely to break the region of integration in t_2 up into two, $t_2<t_1$ and $t_2>t_1$. Actually, because of the symmetry they give equal contributions, so that the result is

$$-\int\int j_\mu(1)A_\mu(1)d^3\mathbf{x}_1dt_1\int^{t_1}\int j_\nu(2)A_\nu(2)d^3\mathbf{x}_2dt_2,$$

the ordering now being conventional. From here the matrix elements are computed between given states by use of the commutation relations (46) below. For further details we refer to Dyson's papers.[3] The result is that given by the rules of II.

Another method is to notice that the entire dependence of S on A_μ can be directly evaluated. As far as the states of the field are concerned, the evaluation of matrix elements of S is exactly the same as though $j_\mu(1)$ were a numerical function (since it commutes with all $A_\mu(1)$). Hence, these may be worked out by first obtaining the result for a field interacting with a given unquantized current distribution $j_\mu(1)$. This can be done, for example, by using the lagrangian methods described in III. For example, the matrix taken between states in which the field is empty of photons initially and finally is

$$S_{00}=\exp\left[-\tfrac{1}{2}ie^2\int\int j_\mu(1)j_\mu(2)\delta_+(s_{12}{}^2)d\tau_1d\tau_2\right] \quad (43)$$

as is shown in III (for j_μ a numerical function). This may now be interpreted as follows: The matrix element of S for a transition in which at $t=-\infty$ there are no real photons and the matter is in state χ_-, to the state at $+\infty$ also empty of photons with the matter in state χ_+, is the matrix element of S_{00} between χ_- and χ_+, where S_{00}, given in Eq. (43), operates now only on matter variables, the order of operators $j_\mu(1)$, $j_\mu(2)$ being determined just as in Eq. (42). This expression forms the basis for the author's treatment of virtual photon processes (II).

If an additional unquantized potential $B_\mu(1)$ is present, the expression (42) for S is altered just by the replacement of $A_\mu(1)$ by $A_\mu(1)+B_\mu(1)$.

The matrix corresponding to Eq. (43) would be a functional of $B_\mu(1)$ and a function of e^2:

$$S_{e^2}[B]=\exp\left[-\tfrac{1}{2}ie^2\int\int j_\mu(1)j_\mu(2)\delta_+(s_{12}{}^2)d\tau_1d\tau_2\right]$$
$$\times\exp\left[-i\int j_\nu(1)B_\nu(1)d\tau_1\right]. \quad (44)$$

It is evident by direct substitution, that $S_{e^2}[B]$ satisfies

$$dS_{e^2}/d(e^2)=\tfrac{1}{2}i\int\int\delta^2S_{e^2}/\delta B_\mu(1)\delta B_\mu(2)\delta_+(s_{12}{}^2)d\tau_1d\tau_2. \quad (45)$$

Since the equation is linear, any matrix element of S_{e^2}, say $T_{e^2}[B]$, between two states of the matter satisfies same equation. This is Eq. (45) of III, which is shown in III to be a general statement of the rules given in II for solving electrodynamic problems. Evidently, the case of real photons in initial or final state can be carried through in parallel to the discussion in III, with $j_\mu(1)$ now as an operator.

This completes our discussion in a general way of the relations between the various representations of electrodynamics. However, we wish to add a word concerning the derivation of Eq. (43). We have indicated how this may be done using the lagrangian method. However, we have seen from our example with the single forced oscillator that the same results may be obtained directly with the operator method, in just as simple a manner. Of course, by considering the field as a set of such oscillators we will arrive at Eq. (43), thus completely avoiding the lagrangian formulation. However, since the relation between Eqs. (42) and (43) is so fundamental, we should like to show how the operator method permits a simple direct passage from Eq. (42) to Eq. (43).[13] (We are simply following the steps leading from Eq. (26′) to Eq. (33) for the single harmonic oscillator, but are using $A_\mu(1)$ to replace q.)

The field operator $A_\mu(1)$ can be split into two parts $A_\mu(1)=A_\mu{}^+(1)+A_\mu{}^-(1)$, where the first $A_\mu{}^+(1)$ annihilates photons, and the second $A_\mu{}^-(1)$ creates them.[14] They satisfy the commutation relations (positional

[13] We omit the usual extra complications in all such demonstrations concerned with showing that disregard of the supplementary conditions on $\partial A_\mu/\partial x_\mu$ is legitimate.

[14] Ordinarily, the field operator $A_\mu(1)$ is expanded into modes $A_\mu(1)=\Sigma_iA_{\mu i}(x)(Q_i{}^*e^{i\omega_it}+Q_ie^{-i\omega_it})$, where $A_{\mu i}$ is the numerical function [for example, cosines or sines, III Eq. (1)] describing the classical mode i of frequency ω_i and $Q_i{}^*$, Q_i are the creation and annihilation operators into which q_i, the coordinate of the oscillator of this mode, has been split (29). The factors $e^{\pm i\omega t}$ result from use of interaction representation (30). Then, we have $A_\mu{}^-(1)=\Sigma_iA_{\mu i}(x)Q_i{}^*e^{i\omega_it}$ and $A_\mu{}^+(1)=\Sigma_iA_{\mu i}(x)Q_ie^{-i\omega_it}$. The commutation rule (46) then results from that of the Q and Q^* (28). Using the representation of III Eq. (1), the right-hand side of Eq. (46) comes from Eq. (28) directly in the form

$$(2\pi)^{-3}e^2\delta_{\mu\nu}\int\exp[-ik(t_2-t_1)]\cos(\mathbf{K}\cdot\mathbf{x}_1-\mathbf{K}\cdot\mathbf{x}_2)d^3\mathbf{K}/k,$$

which on integration for $t_2>t_1$ [see III Eq. (22)] is Eq. (46). The separation has been accomplished directly in coordinate space by J. Schwinger, Phys. Rev. **75**, 651 (1949).

ordering)

$$A_\mu^-(1)A_\nu^+(2)-A_\nu^+(2)A_\mu^-(1)=-ie^2\delta_{\mu\nu}\delta_+(s_{12}^2) \quad (46)$$

for[15] $t_1<t_2$. We have set $s_{12}^2=(x_{1\mu}-x_{2\mu})(x_{1\mu}-x_{2\mu})$.

On the basis of this commutation rule, we are to disentangle the operator

$$S=\exp\left[-i\int j_\mu(1)A_{\mu t_1}^-(1)d\tau_1\right]\times\exp\left[-i\int j_\nu(2)A_{\nu t_2}^+(2)d\tau_2\right], \quad (42')$$

where for definiteness we indicate the time of operation by the subscript. This is already of the exponential form of theorem (18), using $-i\int j_\mu(1)A_{\mu t_2}^-(1)d^3\mathbf{x}_1$ as $P(s)$, $s=t_2$. Hence, the result is

$$S=\exp\left[-i\int j_\mu(1)A_{\mu\infty}^-(1)d\tau_1\right]\times\exp\left[-i\int j_\nu(2)A_{\nu t_2}^{\prime\prime+}(2)d\tau_2\right],$$

where

$$A_{\nu t_2}^{\prime\prime+}(2)=\exp\left[+i\int_{-\infty}^{t_2} j_\mu(1)A_\mu^-(1)d\tau_1\right]A_{\nu t_2}^+(2)\times\exp\left[-i\int_{-\infty}^{t_2} j_\mu(1')A_\mu^-(1')d\tau_{1'}\right], \quad (47)$$

where in Eq. (47) we suppress the ordering rules for A^+, A^- and use instead positional notation (but maintain the rules for j_μ).

The commutation rule (46) permits Eq. (47) to be written[16]

$$A_{\nu t_2}^{\prime\prime+}(2)=A_{\nu t_2}^+(2)+\int_{-\infty}^{t_2} e^2\delta_+(s_{12}^2)j_\mu(1)d\tau_1. \quad (48)$$

Hence, we have

$$S=\exp\left[-i\int j_\mu(1)A_{\mu\infty}^-(1)d\tau_1\right]\times\exp\left[-\tfrac{1}{2}ie^2\int\int j_\mu(1)j_\mu(2)\delta_+(s_{12}^2)d\tau_1d\tau_2\right]\times\exp\left[-i\int j_\mu(2)A_{\mu-\infty}^+(2)d\tau_2\right], \quad (49)$$

the A operators being entirely disentangled (the j's are still entangled). The ordering index t_2 on $A_{\mu t_2}^+$ has been changed to $-\infty$ in Eq. (49), since all the A^+ commute and act before $A_{\mu\infty}^-$, so that no ordering is necessary.

Taken between states empty of photons the result is just S_{00} of Eq. (43), for the annihilation operation A^+ on the state of zero photons is zero, and creation operation of A^- has zero amplitude of leaving a state without photons. If there is one photon present initially and we ask that no photons remain, we shall have to annihilate it and create none, so that if the A^- and A^+ exponentials are expanded in power series, we must take only the term linear in A^+ and independent of A^-. This is equivalent to a first-order action of the potential B_μ in Eq. (44) in perturbation. The corresponding rules for higher numbers of real photons are readily derived from Eq. (49). In this way we have completed an independent deduction of all the main formal results in quantum electrodynamics, by use of the operator notation.

7. THE DIRAC EQUATION

Up to now we have discussed the matter system using the description of second quantization. It was pointed out in I in the case of the electron-positron field where a small number of charges is involved, another simple interpretation is available. In this section we should like to discuss this from an operator point of view and to give in the following sections the formulas in this picture for electrons interacting through the agency of the electromagnetic field.

We begin by discussing the behavior of a single charge (plus the virtual pairs produced from it) in an unquantized potential $\boldsymbol{B}=\gamma_\mu B_\mu$, omitting the contributions from closed loop diagrams. This section will therefore constitute a brief summary of I using operator notation.

The behavior of a single charge is obtained by solving the Dirac equation

$$(i\nabla-\boldsymbol{B}-m)\psi=0 \quad (50)$$

with suitable boundary conditions and interpreting the solution as described in I. For convenience we shall always solve, instead,

$$(i\nabla-\boldsymbol{B}-m)\psi=iF, \quad (51)$$

where F is a source function, by writing

$$\psi=(i\nabla-\boldsymbol{B}-m)^{-1}iF \quad (52)$$

and interpreting the reciprocal operator in the definite sense implied by the limit of the operator when m has a vanishingly small negative imaginary part. If, for example, we wish the ordinary solution for $t>t_0$ which at $t=t_0$ has the form $f(\mathbf{x})$ representing an electron (i.e., $f(\mathbf{x})$ has only positive energy components), that solution

[15] This restriction at first sight looks unrelativistic. For $t_1>t_2$ we would have the complex conjugate of $-ie^2\delta_{\mu\nu}\delta_+(s_{12}^2)$, but $-i\delta_+(s_{12}^2)$ is real in spacelike regions (as $\delta_+(x)=\delta(x)-i(\pi x)^{-1}$).

[16] For, if $A_{\nu t_2}^{\prime\prime+}(2)$ of (47) is considered as a functional of j_μ its first variation with respect to $j_\mu(3)$ is $(t_3<t_2)$.

$$i\exp\left(+i\int_{-\infty}^{t_2} j_\mu A_\mu^- d\tau\right)[A_\mu^-(3)A_\nu^+(2)-A_\nu^+(2)A_\mu^-(3)]\times\exp\left(-i\int_{-\infty}^{t_2} j_\mu A_\mu^- d\tau\right)=e^2\delta_{\mu\nu}\delta_+(s_{12})$$

by Eq. (46). The first variation of expression (48) gives the same result, so that Eq. (48) is correct for all j_μ, since it obviously is correct for $j_\mu=0$.

is the ψ obtained from Eq. (52) by setting[17] $F(1)=\gamma_\iota\delta(t_1-t_0)f(\mathbf{x}_1)$. If f contains negative energy components, Eq. (52) gives the desired solution for these components for $t<t_0$.

From the definition of $K_+^{(B)}(2,1)$ in I, Eq. (15), we can write

$$K_+^{(B)}(2,1)=(i\nabla-\boldsymbol{B}-m)^{-1}i\delta(2,1), \quad (53)$$

so that $-iK_+^{(B)}(2,1)$ is the space-time representation of the operator $(i\nabla-\boldsymbol{B}-m)^{-1}$ needed for Eq. (52). Thus, the solution (52) is

$$\psi(2)=\int K_+^{(B)}(2,1)F(1)d\tau_1. \quad (54)$$

The perturbation theory, considering $\boldsymbol{B}$ as a perturbation on the free particle, arises from Eq. (53) from a power series expansion in $\boldsymbol{B}$. For any pair of operators A, B we have

$$(A+B)^{-1}=A^{-1}-A^{-1}BA^{-1}+A^{-1}BA^{-1}BA^{-1}\cdots, \quad (55)$$

so that with $A=(i\nabla-m)$, $B=-\boldsymbol{B}$ we have

$$(i\nabla-\boldsymbol{B}-m)^{-1}=(i\nabla-m)^{-1}+(i\nabla-m)^{-1}\boldsymbol{B}(i\nabla-m)^{-1}$$
$$+(i\nabla-m)^{-1}\boldsymbol{B}(i\nabla-m)^{-1}\boldsymbol{B}(i\nabla-m)^{-1}+\cdots \quad (56)$$

or in space representation (putting

$$K_+(2,1)=i(i\nabla-m)^{-1}\delta(2,1))$$

$$K_+^{(B)}(2,1)=K_+(2,1)-i\int K_+(2,3)\boldsymbol{B}(3)K_+(3,1)d\tau_3$$
$$-\int\int K_+(2,4)\boldsymbol{B}(4)K_+(4,3)\boldsymbol{B}(3)K_+(3,1)d\tau_4 d\tau_3, \quad (57)$$

as in I, Eqs. (13) and (14). The corresponding momentum representation is evident directly from Eq. (56), for $(i\nabla-m)^{-1}$ is $(\not{p}-m)^{-1}$.

If F is to represent an initial state, it is also convenient to use the free particle solution $f(1)=(i\nabla-m)^{-1}iF(1)$ to represent the state. We are often interested in the amplitude that the system is in a final state $g(\mathbf{x})$. In this case, we can define a sink function G and a corresponding free particle solution $\bar{g}=i\bar{G}(i\nabla-m)^{-1}$ (where we write the adjoint so that it will correspond to the solution $g(1)=-i(i\nabla-m')^{-1}G(1)$ corresponding to m' having the opposite sign of the imaginary part to m). The matrix element to go from f to g is then the space-time integral of

$$i\bar{G}(i\nabla-\boldsymbol{B}-m)^{-1}F. \quad (58)$$

The expansion (56) gives for this element

$$-i\bar{g}\boldsymbol{B}f-i\bar{g}\boldsymbol{B}(i\nabla-m)^{-1}\boldsymbol{B}f$$
$$-i\bar{g}\boldsymbol{B}(i\nabla-m)^{-1}\boldsymbol{B}(i\nabla-m)^{-1}\boldsymbol{B}f-\cdots \quad (59)$$

(assuming $\bar{g}$, f are orthogonal states, $\int\bar{g}(1)F(1)d\tau_1=0$ so the leading term vanishes). In space representation the first two terms of this are I, Eq. (22), and I, Eq. (23); in momentum representation the second term is I, Eq. (35).

If more than one real charge is present without interaction, there is an operator $(i\nabla-\boldsymbol{B}-m)^{-1}$ for each charge, operating exclusively on the space and spinor coordinates of that charge. Operators corresponding to distinct charges commute. Matrix elements are taken in the antisymmetric way described in I, Sec. 4, for accord with the exclusion principle.

The contribution from closed loops is a factor $C_v=\exp(-L)$, where L is not very easily defined directly in operators. But the first-order change on changing the potential from $\boldsymbol{B}$ to $\boldsymbol{B}+\Delta\boldsymbol{B}$ is

$$\Delta L=\mathrm{trace}[\{(i\nabla-\boldsymbol{B}-m)^{-1}-(i\nabla-m)^{-1}\}\Delta\boldsymbol{B}], \quad (60)$$

where the "trace" means the diagonal integral in coordinates and the "Sp" on the spinor indices, in space-time representation just I, Eq. (29).

This completes our summary in terms of operators of the results given in paper I on the theory of positrons. The main point is that aside from the problems of closed loops, one is merely analyzing by various techniques the consequences of Eq. (52) and, therefore, in general, the properties of the operator

$$(i\nabla-\boldsymbol{B}-m)^{-1}. \quad (61)$$

We may now turn to the quantum electrodynamics of such a particle, or system of particles. For simplicity we may restrict ourselves to the case of all virtual photons. The real photon case can, of course, as always be obtained by considering also the effects of external potentials. For simplicity further assume, at first, zero external potential. Our central problem, then, is the calculation of the matrix element

$$R={}_0\langle(i\nabla-\boldsymbol{A}-m)^{-1}\rangle_0 \quad (62)$$

of $(i\nabla-\boldsymbol{A}-m)^{-1}$ between states of the field empty of photons initially and finally. Here $\boldsymbol{A}=\gamma_\mu A_\mu$ and $A_\mu(1)$ is the operator $A_\mu^+(1)+A_\mu^-(1)$ acting on the field coordinates and satisfying commutation rules (46). This problem is relatively hard to solve directly. We do have the matrix element of any exponential form in A_μ in (43); but with A_μ involved in a reciprocal, it is another matter. We shall represent the reciprocal as a superposition of exponentials in the next section.

From Eq. (62) we can derive in a simple direct manner the perturbation series results of II. For we know[18] that

$${}_0\langle A_\mu(1)\rangle_0=0, \quad {}_0\langle A_\mu(1)A_\nu(2)\rangle_0=ie^2\delta_{\mu\nu}\delta_+(s_{12}^2), \quad \text{etc.} \quad (63)$$

[17] For $t<t_0$, the ψ from Eq. (52) would be zero and would not be the solution desired; it can be obtained only from F with a different definition of the poles of the reciprocal operator. We assume we are only interested in the solutions in regions of space-time later than the time the "initial" electron wave functions are specified and earlier than the "initial" positron function is given. Since we will take matrix elements between two states, this represents no real limitation (see I).

[18] The order of the A_μ operators in Eq. (63) is according to the time convention. If we put $A=A^++A^-$ and use Eq. (46) to rearrange factors, they are evident, since A_μ^+ on the initial (and A_μ^- into the final) photon-free state vanishes. They are identical to the lagrangian relations III, Eq. (52), and form the basis of Dyson's description.

Hence, if we expand the reciprocal in power series as Eq. (56), in coordinate representation (57), with $\boldsymbol{B}$ replaced by $\boldsymbol{A}$ (or by $\boldsymbol{A}+\boldsymbol{B}$ if an unquantized potential $\boldsymbol{B}$ is present along with $\boldsymbol{A}$), we may readily write down the zero-zero matrix element of each term. For example, third term of Eq. (57) gives the self-energy contribution in accord with II, Eq. (6). The other results of II are just further consequences of this series expansion as we have emphasized in III.

8. USE OF A FIFTH PARAMETER IN DIRAC'S EQUATION

In this section we discuss the representation of the reciprocal operator in exponential form. Since $\int_0^\infty \exp(iWx)dW = i/x$ (or rather $\lim_{\epsilon\to 0} i/(x+i\epsilon)$), we we may write

$$i(i\nabla-\boldsymbol{B}-m)^{-1}=\int_0^\infty \exp[i(i\nabla-\boldsymbol{B}-m)W]dW, \quad (64)$$

the definition of the singularity (as the limit with m having an infinitesimal negative imaginary part) being automatically represented. (See, however, the remarks at the end of this section.) We can also write this in the ordered operator form,

$$\int_0^\infty \exp\left[i\int_0^W \{i\nabla(w)-\boldsymbol{B}(w)\}dw\right]\exp(-imW)dW, \quad (65)$$

where we have written $\boldsymbol{B}(w)$ for $\gamma_\mu(w)B_\mu(x_\nu(w))$, where $x_\nu(w)$ are the four ($\nu=1$ to 4) coordinate operators, of which B_μ is a function, ordered by the ordering parameter w and $\gamma_\mu(w)$ are the four Dirac matrices similarly ordered. Likewise, we have $i\nabla(w)=\gamma_\mu(w)i\partial_\mu(w)$, where $i\partial_\mu(w)$ are the four ordered momentum operators conjugate to x_μ.

In this form $\boldsymbol{B}$ can be replaced by $\boldsymbol{A}$ and the expectation value for virtual transitions can be taken. This is done in the next section. We continue here with a more complete discussion of Eq. (65).

The perturbation expansion in $\boldsymbol{B}$ of Eq. (65) should lead, of course, to Eq. (56). For example, the term first order in $\boldsymbol{B}$ in Eq. (65) is evidently

$$-i\int_0^\infty \exp(imW) \times\exp\left[i\int_0^W i\nabla(w')dw'\right]\int_0^W \boldsymbol{B}(w)dwdW. \quad (66)$$

Now the range of the w' may be divided into two regions and the quantities reordered (just as in Eqs. (5) and (6)) to

$$-i\int_0^\infty\int_0^W dWdw\exp[i(W-w)(i\nabla-m)]\boldsymbol{B} \times\exp[iw(i\nabla-m)],$$

so that changing the order of integration, one finds immediately from

$$\int_0^\infty \exp[iv(i\nabla-m)]dv=i(i\nabla-m)^{-1}$$

the result $-i(i\nabla-m)^{-1}\boldsymbol{B}(i\nabla-m)^{-1}$ as required.

The contribution L of a closed loop can be written directly in exponential form. It is easily shown from Eqs. (60) and (65) that

$$L=\int_0^\infty \text{trace}\left[\exp\left\{i\int_0^W (i\nabla(w)-\boldsymbol{B}(w))\right\}dw\right]\frac{dW}{W} \times\exp(-imW). \quad (67)$$

(The second term in Eq. (60) actually has zero trace and was added only to make convergence problems appear less difficult. It has been omitted in writing Eq. (67). Also, the value of Eq. (67) when $\boldsymbol{B}=0$ may be subtracted away, if desired, for a constant addition on L changes only the normalization of all probabilties.)

Incidentally, the method of rendering this expression convergent (see II, Sec. 7) for further calculations, is to call its value for mass m, $L(m^2)$ and then to calculate

$$L^P=\int_0^\infty [L(m^2)-L(m^2+\lambda^2)]G(\lambda)d\lambda,$$

where

$$\int_0^\infty G(\lambda)d\lambda=1 \quad \text{and} \quad \int_0^\infty \lambda^2 G(\lambda)d\lambda=0$$

and to assume that L^P is to be used as the correct value of L in place of $L(m^2)$. This is equivalent to replacing the factor $\exp(-imW)$ in the integrand of Eq. (67) by another function $F(W)$, where

$$F(W)=\int_0^\infty [\exp(-imW)-\exp\{-i(m^2+\lambda^2)^{\frac{1}{2}}W\}] \times G(\lambda)d\lambda.$$

For large W this approaches $\exp(-imW)$, but for small W it falls off, the real part of it, at least, varying as W^4. This renders Eq. (67) convergent. (The imaginary part of $F(W)$ does not seem to lead to momentum space integrals whose convergence would be in question.) This suggests a general method of maintaining convergence; by keeping processes corresponding to small intervals of w from occurring with large amplitude. This is briefly discussed in III in reference 22. What is said there applies qualitatively as well to the Dirac case analyzed here, with u replaced by w.

If there are several charges in the system, we must associate a separate w, for each, say w_n for the nth. Each must have its own set of matrices $\gamma_\mu{}^{(n)}$ and coordinates $x^{(n)}$ (γ's for different charges commute). If we call

$$\boldsymbol{B}^{(n)}=\gamma_\mu{}^{(n)}(w_n)B_\mu(x^{(n)}),$$

the total matrix for all N particles is

$$\prod_{n=1}^{N}\int_0^\infty \exp\left\{i\int_0^{W_n}[i\nabla^{(n)}(w_n)-\boldsymbol{B}^{(n)}(w_n)]dw_n\right\}\times\exp(-imW_n)dW_n. \quad (68)$$

If in addition there are present a number of closed loops, the corresponding number of factors L must be multiplied in.

One might try to give a kind of physical or, rather, mathematical view by which the form of Eq. (65) can be appreciated, in the following manner:

We may deal with the Dirac equation somewhat in analogy to the method used in the discussion of the Klein-Gordon equation in the Appendix A of III. Consider a fifth variable w in addition to the four x, and that we have a wave function $\phi(x, w)$, which is to satisfy

$$-i\partial\phi/\partial w=(i\nabla-\boldsymbol{B})\phi. \quad (69)$$

Then since the potentials $B_\mu(x)$ are independent of w, the equation is separable in w, so that $\phi(x, w)=\exp(imw)\psi(x)$ is a solution of Eq. (69), if $\psi(x)$ is a solution of the Dirac Eq. (50). Also, if we have any special solution of Eq. (69), $\phi(x, w)$, we may obtain a solution of Eq. (50) by finding

$$\psi(x)=\int_{-\infty}^{\infty}\phi(x, w)\exp(-imw)dw. \quad (70)$$

Hence, by studying Eq. (69), we are at the same time studying the Dirac equation.

Given the wave function $\phi(x, 0)$ for $w=0$, the wave function at $w=W$ is given by

$$\phi(x, W)=\theta(W)\phi(x, 0), \quad (71)$$

where the operator $\theta(W)$ is

$$\theta(W)=\exp i\int_0^W[i\nabla(w)-\boldsymbol{B}(w)]dw \quad (72)$$

for $W>0$ and, for convenience,[17] we take $\theta=0$ for $W<0$. The important operator for the Dirac equation, in view of Eqs. (70) and (71), is

$$\int_{-\infty}^{\infty}\theta(W)\exp(-imW)dW,$$

which is just Eq. (65)

This interpretation suffers from a difficulty, however. For a free particle the operator $\theta(W)$ in momentum space is $\theta(W)=\exp(iW\not{p})=\cos(Wp)+i(\not{p}/p)\sin(pW)$, where $p=(\not{p}^2)^{\frac{1}{2}}$. The integral of this times $\exp(-imW)$ is really not always defined, even if m has a small negative imaginary part, for in intermediate states $\not{p}^2$ may be negative and p imaginary, so that θ contains positive exponentials in W and the integrand is oscillating with ever increasing amplitude. We therefore look at Eq. (64) as a *formal* definition of the value of the integral in all cases. Although this is satisfactory in a formal way for operators, it means that our interpretation cannot be taken literally. For example, we cannot obtain an unambiguous integral representation of $\theta(W)$ in coordinate space, for the requisite integral $\int\exp(-i\not{p}W)\exp(-ip\cdot x)d^4p$ is undefined. This is because it is probably not possible to obtain the wave function (71) at any value of W from that at $W=0$ from Eq. (69) without further definitions. At least, the corresponding second-order equation $(\partial^2\phi/\partial t^2)-\nabla^2\phi-\partial^2\phi/\partial W^2=0$ is apparently not of the kind for which this type of Huygens principle applies.

An alternative method of parametrizing the equation which does not seem to suffer from this interpretational difficulty is given in Appendix D. It leads, however, to more complicated (although algebraically equivalent) expressions for matrix elements than does Eq. (64).

9. DIRAC ELECTRONS IN QUANTUM ELECTRODYNAMICS

Returning now to quantum electrodynamics, for a single charge we want the expectation between photon free states of R in Eq. (62). This by Eq. (65) is the integral over all positive W of $\exp(-imW)$ times

$$\left\langle \exp i\int_0^W[i\nabla(w)-\boldsymbol{A}(w)]dw\right\rangle_0.$$

This is just $\exp[i\int_0^W i\nabla(w)dw]$ times (the $\langle\ \rangle_0$ refers to the photon states, that is, affects A_μ only)

$$\left\langle \exp-i\int_0^W\gamma_\mu(w)A_\mu(x(w))dw\right\rangle_0,$$

which is of the form

$$\left\langle \exp-i\int A_\mu(1)j_\mu(1)d\tau_1\right\rangle_0$$

of Eq. (42) with

$$j_\mu(1)=\int_0^W\gamma_\mu(w)\delta^4(x_\nu(w)-x_{\nu 1})dw, \quad (73)$$

where $x_{\nu 1}$ is the field point at which j_μ is calculated and $\delta^4(x_{\nu 2}-x_{\nu 1})$ means $\delta(2, 1)$. Thus, we may find the expectation value with the relation (43). With this value of j_μ substituted on the right side of Eq. (43), the δ^4 functions are immediately integrable, and we find finally

$$R=\int_0^\infty\exp\left[-\int_0^W\nabla(w)dw\right]\times\exp\left[-\tfrac{1}{2}ie^2\int_0^W\int_0^W\gamma_\mu(w')\gamma_\mu(w'')\times\delta_+(s^2_{w'w''})dw'dw''\right]\exp(-imW)dW, \quad (74)$$

in which we have written $s^2{}_{w'w''}$ for $[x_\mu(w')-x_\mu(w'')]\times[x_\mu(w')-x_\mu(w'')]$.

This expression then contains a description of a Dirac electron interacting with itself. If an extra factor $\exp(-i\int_0^W \boldsymbol{B}(w)dw)$ is included, it describes such an electron also in an external potential. The terms may be expanded in powers of $\boldsymbol{B}$ and e^2, and each term may then be simplified in the way we have described many times before, for example, in connection with Eq. (66).

When several charges are present, the result from Eq. (68) is the integral[19]

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty \exp\left\{i\sum_n\int_0^{W_n}[i\nabla^{(n)}(w_n)-m]dw_n\right\}$$
$$\times\exp\left[-\tfrac{1}{2}i\sum_n\sum_m e^2\int_0^{W_n}\int_0^{W_m}\gamma_\mu{}^{(n)}(w_n)\gamma_\mu{}^{(m)}(w_m)\right.$$
$$\left.\times\delta_+(s^2{}_{w_nw_m})dw_ndw_m\right]dW_1dW_2\cdots dW_m. \quad (75)$$

The contributions from closed loops may be obtained from this by choosing some value of n, say, $n=i$, to represent matrices applying to a loop, dividing under the integral sign by W_i, and taking the trace with respect to the variables i.

The various present-day meson theories of nuclear interaction may be set up in quite analogous ways. For example, a nucleon interacting with itself through the agency of neutral pseudoscalar mesons with pseudoscalar coupling is evidently described by Eq. (74); but with m replaced by the proton mass, and the interaction term altered by the replacement of e^2 by g^2, γ_μ by γ_5, and $\delta_+(s^2)$ by $4\pi I_+(s^2)$, the appropriate propagation function for mesons of mass μ (I_+ is defined in I, Eq. (32), but $m=\mu$). Charged mesons may be represented by the use of isotopic spin operators also ordered by w.

10. SUMMARY OF NUMERICAL FACTORS FOR TRANSITION PROBABILITIES

The exact values of the numerical factors appearing in the rules of II for computing transition probabilities are not clearly stated there, so we give a brief summary here.[20]

The probability of transition per second from an initial state of energy E to a final state of the same total energy (assumed to be in a continuum) is given by ($\hbar=c=1$),

$$\text{Prob. trans/sec}=2\pi N^{-1}|\mathfrak{M}|^2\rho(E),$$

where $\rho(E)$ is the density of final states per unit energy range at energy E and $|\mathfrak{M}|^2$ is the square of the matrix element taken between the initial and final state of the transition matrix $\mathfrak{M}$ appropriate to the problem. N is a normalizing constant. For bound states conventionally normalized it is 1. For free particle states it is a product of a factor N_i for each particle in the initial and for each in the final energy state. N_i depends on the normalization of the wave functions of the particles (photons are considered as particles) which is used in computing the matrix element of $\mathfrak{M}$. The simplest rule (which does not destroy the apparent covariance of $\mathfrak{M}$), is[21] $N_i=2\epsilon_i$, where ϵ_i is the energy of the particle. This corresponds to choosing in momentum space, plane waves for photons of unit vector potential, $\boldsymbol{e}^2=-1$. For electrons it corresponds to using $(\bar{u}u)=2m$ (so that, for example, if an electron is deviated from initial $\boldsymbol{p}_1$ to final $\boldsymbol{p}_2$, the sum over all initial and final spin states of $|\mathfrak{M}|^2$ is $\mathrm{Sp}[(\boldsymbol{p}_2+m)\mathfrak{M}(\boldsymbol{p}_1+m)\overline{\mathfrak{M}}]$). Choice of normalization $(\bar{u}\gamma_t u)=1$ results in $N_i=1$ for electrons. The matrix $\mathfrak{M}$ is evaluated by making the diagrams and following the rules of II, but with the following definition of numerical factors. (We give them here for the special case that the initial, final, and intermediate states consist of free particles. The momentum space representation is then most convenient.)

First, write down the matrix directly without numerical factors. Thus, electron propagation factor is $(\boldsymbol{p}-m)^{-1}$, virtual photon factor is $\boldsymbol{k}^{-2}$ with couplings $\gamma_\mu\cdots\gamma_\mu$. A real photon of polarization vector e_μ contributes factor $\boldsymbol{e}$. A potential (times the electron charge, e) $A_\mu(x)$ contributes momentum $\boldsymbol{q}$ with amplitude $\boldsymbol{a}(q)$, where $a_\mu(q)=\int A_\mu(1)\exp(iq\cdot x_1)d^4x_1$. (Note: On this point we deviate from the definition of $\boldsymbol{a}$ in I which is there $(2\pi)^{-2}$ times as large.) A spur is taken on the matrices of a closed loop. Because of the Pauli principle the sign is altered on contributions corresponding to an exchange of electron identity, and for each closed loop. One multiplies by $(2\pi)^{-4}d^4p=(2\pi)^{-4}dp_tdp_xdp_ydp_z$ and integrates over all values of any undetermined momentum variable $\boldsymbol{p}$. (Note: On this point we again differ.[20])

The correct numerical value of $\mathfrak{M}$ is then obtained by multiplication by the following factors. (1) A factor $(4\pi)^{\frac{1}{2}}e$ for each coupling of an electron to a photon. Thus, a virtual photon, having two such couplings, contributes $4\pi e^2$. (In the units here, $e^2=1/137$ approximately and $(4\pi)^{\frac{1}{2}}e$ is just the charge on an electron in heaviside units.) (2) A further factor $-i$ for each virtual photon.

For meson theories the changes discussed in II, Sec. 10 are made in writing $\mathfrak{M}$, then further factors are

[19] This equation with its interpretation was proposed as a formulation of the laws of quantum electrodynamics (for virtual photons) by the author at the Pocono Conference of Theoretical Physics (1948). The notation for ordering operators was explained there. However, at this time, the author had no complete formal derivation of Eq. (75) from the conventional electrodynamics, nor did he know of a satisfactory method of dealing with the closed loop divergences.

[20] In I and II the unfortunate convention was made that d^4k means $dk_4dk_1dk_2dk_3(2\pi)^{-2}$ for momentum space integrals. The confusing factor $(2\pi)^{-2}$ here serves no useful purpose, so the convention will be abandoned. In this section d^4k has its usual meaning, $dk_4dk_1dk_2dk_3$.

[21] In general, N_i is the particle density. It is $N_i=(\bar{u}\gamma_t u)$ for spin one-half fields and $i[(\phi^*\partial\phi/\partial t)-\phi\partial\phi^*/\partial t]$ for scalar fields. The latter is $2\epsilon_i$ if the field amplitude ϕ is taken as unity.

(1) $(4\pi)^{\frac{1}{2}}g$ for each meson-nucleon coupling and (2) a factor $-i$ for each virtual spin one meson, but $+i$ for each virtual spin zero meson.

This suffices for transition probabilities, in which only the absolute square of $\mathfrak{M}$ is required. To get $\mathfrak{M}$ to be the actual phase shift per unit volume and time, additional factors of i for each virtual electron propagation, and $-i$ for each potential or photon interaction, are necessary. Then, for energy perturbation problems the energy shift is the expected value of $i\mathfrak{M}$ for the unperturbed state in question divided by the normalization constant N_i belonging to each particle comprising the unperturbed state.

The author has profited from discussions with M. Peshkin and L. Brown.

APPENDIX

In this Appendix (A, B, C) an attempt will be made to discuss some of the properties of ordered operators and of functionals in a somewhat more general way.

Almost certainly many of the equations will be incorrect in their general form. This is especially true of those involving fourier transforms in function space. However, it is expected that they are correct in the special cases in which the formulas have been applied in the main part of the paper. Therefore, at least at first, when new results using these methods are derived, care should be taken to check the final result in some independent way. It is analogous to using power series expansions, or fourier transforms, in a calculation in a situation in which the conditions for the validity of the power expansions or of the transform have not been checked, or are not known to be satisfied. The physicist is very familiar with such a situation and usually satisfied with it, especially since he is confident that he can tell if the answer is physically reasonable. But mathematicians may be completely repelled by the liberties taken here. The liberties are taken not because the mathematical problems are considered unimportant. On the contrary, this appendix is written to encourage the study of these forms from a mathematical standpoint. In the meantime, just as a poet often has license from the rules of grammar and pronunciation, we should like to ask for "physicists' license" from the rules of mathematics in order to express what we wish to say in as simple a manner as possible. (These remarks do not apply to Appendix D.)

A. Relation to Theory of Functionals

In this section we would like to suggest how a general theory of ordered operators might be built up, and in particular, to point out certain relations to the theory of functionals. For clarity of exposition in this Sec. A, only, we represent all operators by bold-faced letters $\mathbf{M}$ and ordinary functions in regular type M. We have mentioned that with every functional $F[M(s), N(s)\cdots]$ of the argument functions $M(s)$, $N(s)$ we wish to associate an operator (by identifying $M(s)$ with an operator $\mathbf{M}(s)$ interpreting s as an ordering parameter with the operators $\mathbf{M}(s)$, $\mathbf{N}(s)$ supposedly known and with known commutation relations). The general theory of these associations might instead have begun by defining the meaning for the special case of the exponential functional $\exp\int_0^1 M(s)ds$ (we assume throughout this section, for convenience, that the range of s is 0 to 1). The corresponding operator

$$\mathbf{R}=\exp\int_0^1 \mathbf{M}(s)ds \tag{1-a}$$

is defined as the value $\mathbf{G}(1)$ at $s=1$ of that solution of the operator differential equation

$$d\mathbf{G}(s)/ds=\mathbf{M}(s)\mathbf{G}(s), \tag{2-a}$$

which is the identity operator at $s=0$, i.e., $\mathbf{G}(0)=\mathbf{I}$. We have thereby defined the operators corresponding to more complex functionals such as $F=\exp\int_0^1[\mu(s)\mathbf{M}(s)+\nu(s)\mathbf{N}(s)+\cdots]ds$, where $\mu(s)$, $\nu(s)\cdots$ are numerical functions and $\mathbf{M}$, $\mathbf{N}$ arbitrary operators (which need not commute) as the $\mathbf{G}(1)$ from

$$d\mathbf{G}(s)/ds=[\mu(s)\mathbf{M}(s)+\nu(s)\mathbf{N}(s)+\cdots]\mathbf{G}(s) \tag{3-a}$$

with $\mathbf{G}(0)=\mathbf{I}$. For clearly $\mu(s)\mathbf{M}(s)+\nu(s)\mathbf{N}(s)+\cdots$ can be considered as a single operator function of s, the $\mathbf{M}(s)$ in Eqs. (1-a) and (2-a).

Next we make the general definition that the operator to be associated with the sum of two or more functions $F_1[M(s), N(s)\cdots]+F_2[M(s), N(s)\cdots]$ is the sum of the operators $F_1[\mathbf{M}(s), \mathbf{N}(s)\cdots]+F_2[\mathbf{M}(s), \mathbf{N}(s)\cdots]$ corresponding to each separately.

Considering a derivative as the limit of a difference, we can use this idea of superposition to further extend the range of functionals for which operators are defined. As an example, in virtue of the fact that $\int_0^1 M(s)ds\int_0^1 N(s)ds$ is the first derivative with respect to both μ, ν of $\exp\int_0^1[\mu M(s)+\nu N(s)]ds$ evaluated at μ, $\nu=0$ we may define the operator corresponding to $\int_0^1 \mathbf{M}(s)ds\int_0^1 \mathbf{N}(s)ds$ as the corresponding derivative of the operator $\exp\int_0^1[\mu\mathbf{M}(s)+\nu\mathbf{N}(s)]ds$. Then from a study of the properties of the solution of Eq. (3-a) expanded in powers of μ, ν we may readily verify that $\int_0^1 \mathbf{M}(s)ds\int_0^1 \mathbf{N}(s)ds$ could also be evaluated directly by considering s as an ordering index on the operators.

Thus, the superposition rule permits a wide increase in the class of functionals for which we have defined operators. In fact, with some mathematical license, we have defined the operator for any functional. We wish to imagine that any functional can be represented as a superposition of exponential ones in a manner analogous to the representation of an arbitrary function as a superposition of exponential functions. Thus, we expect to be able to write for any functional $F[M(s)]$ (the true mathematical restrictions are completely unknown to me)

$$F[M(s)]=\int \exp\left[i\int_0^1 \mu(s)M(s)ds\right]\mathfrak{F}[\mu(s)]\mathfrak{D}\mu(s), \tag{4-a}$$

where $\mathfrak{F}[\mu(s)]$ is a new (complex) functional, the functional transform of $F[M(s)]$, and $\int\cdots\mathfrak{D}\mu(s)$ represents (some kind of an) integration over the space of functions $\mu(s)$. For simplicity we take the case of just one argument function $M(s)$. If $F[M(s)]$ is given, $\mathfrak{F}$ can be determined perhaps from

$$\mathfrak{F}[\mu(s)]=\int \exp\left[-i\int_0^1 \mu(s)M(s)ds\right]F[M(s)]\mathfrak{D}M(s) \tag{5-a}$$

with suitable normalization. Then, if $\mathfrak{F}$ is known, we define the operator $F[\mathbf{M}(s)]$ as

$$F[\mathbf{M}(s)]=\int \exp\left[i\int_0^1 \mu(s)\mathbf{M}(s)ds\right]\mathfrak{F}[\mu(s)]\mathfrak{D}\mu(s), \tag{6-a}$$

where $\mu(s)$ is a numerical function. Since we have already defined the operator $\exp[i\int_0^1 \mu(s)\mathbf{M}(s)ds]$ (by Eq. (3-a) with μ replaced by $i\mu$), we now simply require superposition of such operators for various $\mu(s)$. The extension to functionals of several variables is evident.

With these definitions of operators in terms of exponential functionals, the various theorems are easily proved. For example, the theorem (18) of Sec. 3 is first readily demonstrated for the special case that F is an exponential (1-a). Thus, to calculate

$$\mathbf{R}=\exp\int_0^1 \mathbf{P}(s)ds\exp\int_0^1 \mathbf{M}(s)ds$$

we must solve $d\mathbf{G}(s)/ds=[\mathbf{P}(s)+\mathbf{M}(s)]\mathbf{G}(s)$. We try a solution $\mathbf{G}(s)=\mathbf{U}(s)\mathbf{X}(s)$, so that $d\mathbf{G}/ds=(d\mathbf{U}/ds)\mathbf{X}+\mathbf{U}d\mathbf{X}/ds=\mathbf{PG}+\mathbf{U}d\mathbf{X}/ds$ in virtue of Eq. (15). Thus, we have a solution if $d\mathbf{X}/ds=\mathbf{U}^{-1}\mathbf{MG}=\mathbf{U}^{-1}\mathbf{MUX}=\mathbf{M}'\mathbf{X}$ with $\mathbf{M}'$ as in Eq. (16). Since $\mathbf{G}(0)=1$, if $\mathbf{U}(0)=1$, we must have $\mathbf{X}(0)=1$ so the solution of the $\mathbf{X}$ equation is $\exp\int_0^1 \mathbf{M}'(s)ds$ in accordance with the definition (1-a), (2-a). (If $\mathbf{U}(0)\neq 1$, replace $\mathbf{X}$ by $\mathbf{X}\mathbf{U}^{-1}(0)$ throughout.) Hence, Eq. (18) is established for exponential functionals. And since the theorem involves F linearly, it is therefore true for any

superposition of exponentials and hence for any functional which can be defined by means of such superposition.

B. Momentum and Coordinate Operators

In nonrelativistic quantum mechanics, without spin, all operators can be made up of coordinate operators and their conjugate momentum operators. We show in this section how, at least in principle, all such operator functions can be disentangled.

We can consider the case of one degree of freedom Q, and its momentum P. (When more variables are present, they present no new problem as variables corresponding to different independent coordinates commute.) Thus, we are to disentangle the general operator $F[P(s), Q(s)]$ subject to the condition

$$PQ-QP=-i. \tag{7-a}$$

This is the problem solved in this section. We can satisfy the commutation relation by putting $P=-id/dQ$ (so that our solution may have applications outside quantum mechanics for the combination of operators X, d/dX are of frequent occurrence). Then the operator F can be defined by giving the function g of Q resulting from

$$g(Q)=F[P(s), Q(s)]f(Q) \tag{8-a}$$

for arbitrary functions $f(Q)$, where $Q(s)$ and $P(s)$ are interpreted as multiplication by Q, and $-i$ times differentiation with respect to Q in the order defined by F.

To obtain the relation of g, f suppose the P-dependence of F can be expanded as a functional transform, ($p(s)$, $q(s)$, $v(s)$ are numerical functions)

$$F[p(s), q(s)]=\int \exp\left[-i\int_0^1 p(s)v(s)ds\right]\mathfrak{F}[q(s), v(s)]\mathfrak{D}v(s), \tag{9-a}$$

where $\mathfrak{F}$ is a functional of $v(s)$ and of $q(s)$. Now to evaluate the operator

$$\exp\left[-i\int_0^1 P(s)v(s)ds\right]\mathfrak{F}[Q(s), v(s)] \tag{10-a}$$

we use our theorem (20) to disentangle the $P(s)$ operator. We use $\alpha(s)=-iv(s)$ in Eq. (20), calling $y(s)=\int_0^s v(s')ds'$, so that Eq. (10-a) is

$$e^{-iy(1)P(1)}\mathfrak{F}[Q'(s), v(s)], \tag{11-a}$$

where $Q'(s)=e^{+iy(s)P}Q_s e^{-iy(s)P}$. As is well known from Taylor's theorem, the operator $e^{hd/dx}$ displaces x by h so that[22]

$$Q'(s)=Q_s+y(s). \tag{12-a}$$

Substitution into Eq. (11-a) finds all the Q_s preceding the $P(1)$ so the operators are disentangled and Q_s may be written simply Q_0, whence we have

$$F[P(s), Q(s)]=\int \exp\left[-iP_1\int_0^1 v(s)ds\right]\times\mathfrak{F}\left[Q_0+\int_0^s v(s')ds', v(s)\right]\mathfrak{D}v(s), \tag{13-a}$$

which in principle, at least, solves the problem.

We can go a bit further and assume Eq. (9-a) can be inverted as

$$\mathfrak{F}[q(s), v(s)]=\int \exp\left[i\int_0^1 p(s)v(s)ds\right]F[p(s), q(s)]\mathfrak{D}p(s), \tag{14-a}$$

where $p(s)$ is a numerical function for transforming F. Also, $y(s)=\int_0^s v(s)ds$ is as good a function as $v(s)$ for purposes of integration,[23] and we may write, substituting Eq. (14-a) into Eq. (13-a),

$$F[P(s), Q(s)]=\int\int \exp\left[-iP_1y(1)+i\int_0^1 p(s)\dot{y}(s)ds\right]\times F[p(s), Q_0+y(s)]\mathfrak{D}p(s)\mathfrak{D}y(s), \tag{15-a}$$

the integral extending over all $p(s)$, and all $y(s)$ subject to $y(0)=0$.

Considering P as $-id/dQ$, the operator $F[P(s), Q(s)]$ may be considered to operate on a function $f(Q)$ to produce another function of Q. In particular, we are often interested in quantum mechanics in the projection of this final function into a given "final state" function g; that is, F is often defined through its matrix element

$$(g^*Ff)=\int g^*(Q)F[P(s), Q(s)]f(Q)dQ.$$

If we substitute into this expression (15-a), the P_1 can be considered to act entirely on $g^*(Q)$ and since $\exp(+iyP)g(Q)=g(Q+y)$, we find

$$(g^*Ff)=\int g^*(Q_0+y(1))\exp\left[i\int p(s)\dot{y}(s)ds\right]F[p(s), Q_0+y(s)]\times\mathfrak{D}p(s)\mathfrak{D}y(s)\cdot f(Q_0)dQ_0.$$

Define $q(s)$ as the numerical function $q(s)=Q_0+y(s)$ and write finally ($q_0=Q_0$)

$$(g^*Ff)=\int\int g^*(q_1)\exp\left[i\int p(s)\dot{q}(s)ds\right]F[p(s), q(s)]\times\mathfrak{D}p(s)\mathfrak{D}q(s)f(q_0)dq_0dq_1, \tag{16-a}$$

where the integral $\mathfrak{D}p(s)$ is over all $p(s)$ and the integral $\mathfrak{D}q(s)$ is over all trajectories $q(s)$ which go between the initial position q_0 and the final one q_1, the final integration on dq_1 being represented explicitly. This represents a complete reduction of an ordered operator $F[P(s), Q(s)]$ involving conjugate operators P, Q (7-a)to a property of the corresponding numerical functional $F[p(s), q(s)]$, for in Eq. (16-a) $p(s)$, $q(s)$ are numerical functions so that all the operators have been eliminated.

This is obviously related to the lagrangian form of quantum mechanics of **C**. In fact, for transitions, we are interested in the operator $S=\exp[-i\int_0^T H(t)dt]$, where, for example, $H=(1/2m)P^2+V(Q, t)$. The matrix elements of this, according to Eq. (16-a) are (use t for s in range 0 to T)

$$(g^*Sf)=\int\int g^*(q_T)\exp\left[i\int_0^T p(t)\dot{q}(t)dt-i\int_0^T (1/2m)p(t)^2dt-i\int_0^T V(q(t), t)dt\right]f(q_0)\mathfrak{D}p(t)\mathfrak{D}q(t)dq_0dq_T.$$

The integral on $p(t)$ is easily done. (See Appendix C for a more general discussion of gaussian integrals.) Substitute $p(t)=m\dot{q}+p'(t)$, so that

$$\int_0^T [(1/2m)p(t)^2-p(t)\dot{q}(t)]dt=(1/2m)\int_0^T p'(t)^2dt-\tfrac{1}{2}m\int_0^T \dot{q}(t)^2dt.$$

Also, $\mathfrak{D}p(t)=\mathfrak{D}p'(t)$, since p and p' differ by a constant at each t (keeping $q(t)$ integration until later). The $\mathfrak{D}p'(t)$ integral then separates out and integrates to some constant. Hence, within such a normalizing constant, the matrix element is

$$(g^*Sf)=\int g^*(q_T)\exp\left\{+i\int_0^T [\tfrac{1}{2}m\dot{q}(t)^2-V(q(t), t)]dt\right\}\times f(q_0)\mathfrak{D}q(t)dq_0dq_T. \tag{17-a}$$

That is, the transition amplitude from point q_0 at $t=0$ to q_T at $t=T$ is the integral over all trajectories connecting these points of $\exp i\int_0^T L[\dot{q}(t), q(t)]dt$, L being the lagrangian for this problem. This is the fundamental theorem on which the interpretation of **C** is based.

The fact that the nonrelativistic quantum mechanical operators (other than spin) can be expressed in terms of an integral over

[22] Or, differentiating Q'(s) with respect to s, find $dQ'(s)/ds=ie^{iy(s)P}(PQ-QP)e^{-iy(s)P}dy/ds$. If we use the commutation relation, this is dy/ds, whence $Q'(s)$ differs from $y(s)$ by the constant operator Q, the evident value of Q' for $y(s)=0$, establishing Eq. (12-a).

[23] For, if $\mathfrak{D}v(s)$ be considered as the limit as $\Delta\to 0$ of an integration over all the variables $v_i=v(s_i)$ with $s_{i+1}-s_i=\Delta$, then the change is from the variables $v_i=(y_{i+1}-y_i)\Delta^{-1}$. Integration over y_i for all $i>0$ is equivalent to integration on all v_i. (Since $dv_i=\Delta^{-1}dy_{i+1}$, the jacobian of the transformation is $\Delta^{-(1/\Delta)}$, which is only a change of normalization, and we are disregarding normalization factors.)

trajectories is based on the fact that the operators involved satisfy Eq. (7-a). If other operators are involved, such as Pauli's spin operators $\boldsymbol{\sigma}$, or Dirac matrices γ_μ, which satisfy different commutation rules, a complete reduction eliminating all the operators is not nearly so easily affected. It is possible to eliminate the p operators in the Dirac or Pauli equation and get forms like Eq. (17-a), but the amplitude for a single trajectory is then a hypercomplex quantity in the algebra of the γ_μ or $\boldsymbol{\sigma}$. We give an example of this.

Without disentangling the γ_μ operators we shall disentangle the momentum operators $p_\mu=i\partial/\partial x_\mu$ from the operator $\theta(W)$ of Eq. (72), which is a key operator in the analyses of the Dirac equation.

If we write

$$\theta(W)=\exp\left[i\int_0^W \gamma_\mu(w)p_\mu(w)dw\right]\exp\left[-i\int_0^W \boldsymbol{B}\{x(w),w\}dw\right],$$

the p_μ operators are already in exponential form and no fourier transforms are necessary. We may disentangle the p_μ in the first integral by a direct use of the theorem (20) with $P(s)=p_\mu(w)$ (p_μ for each value μ is disentangled separately) and $\alpha(s)=i\gamma_\mu(w)$. The resulting $x_\mu'(w)$ operator is $x_\mu(w)+\int_0^w \gamma_\mu(w')dw'$ just as in Eq. (12-a) so that we obtain

$$\theta(W)=\exp\left[ip_\mu(W)\int_0^W \gamma_\mu(w)dw\right]\times\exp\left[-i\int_0^W \boldsymbol{B}\left\{x_\mu(0)+\int_0^W \gamma_\mu(w')dw',w\right\}dw\right]. \quad (18\text{-a})$$

Here the x_μ and p_μ are completely separated, but the γ_μ are thoroughly entangled. The w in $\boldsymbol{B}$ keeps track of the fact that the γ_μ in its definition acts at the order w; thus, it is $\gamma_\mu(w)B_\mu[x_\mu(0)+\int_0^w \gamma_\mu(w')dw']$. A similar separation may be made in the operator for self-action (74) which now is

$$R=\int_0^\infty \exp[ip_{\mu,\infty}C_\mu(W)]\exp\left\{-\tfrac{1}{2}ie^2\int_0^W\int_0^W \dot{C}_\mu(w')\dot{C}_\mu(w'')\times\delta_+[\{C_\mu(w')-C_\mu(w'')\}^2]dw'dw''\right\}\exp(-imW)dW, \quad (19\text{-a})$$

where one must substitute $C_\mu(w)=\int_0^w \gamma_\mu(w')dw'$, $\dot{C}_\mu(w)=\gamma_\mu(w)$ ($p_{\mu,\infty}$ refers to the momentum operator operating on the final state). All reference to space coordinates have disappeared. The problem of self-energy of an electron is reduced to the algebraic one of disentangling a combination of γ_μ in an expression in which, however, they are almost hopelessly tangled up. Not much has been done with this expression. (It is suggestive that perhaps coordinates and the space-time they represent may in some future theory be replaced completely by an analysis of ordered quantities in some hypercomplex algebra).

Since the spin operators are so simple and fundamental to quantum mechanics, they present some interesting unsolved problems. For example, if $F[x(s),y(s),z(s)]$ is a known functional of a three-space trajectory $x(s)$, $y(s)$, $z(s)$, evaluate in terms of this functional, the operator $F[\sigma_x(s),\sigma_y(s),\sigma_z(s)]$, where the σ_x, σ_y, σ_z are the anticommuting Pauli operators of unit square satisfying $\sigma_x\sigma_y\sigma_z=i$. The corresponding problem with Dirac operators is a kind of four-dimensional generalization of this. Alternatively, since the Dirac operators can be represented as the outer product of two commuting sets of Pauli operators, the solution of the problem with Pauli operators could be directly extended to the Dirac case. The Pauli matrices (times i) are the basis for the algebra of quaternions so that the solution of such problems might open up the possibility of a true infinitesimal calculus of quantities in the field of hypercomplex numbers.

C. Gaussian Functionals

In a large number of problems the operators appear in exponentials only up to the second degree. For this reason it is handy to have available a formula for the integration of gaussian functionals. We can define a gaussian functional $G[y(s)]$, of one function $y(s)$, as one of the form, $G[y(s)]=\exp iE[y(s)]$ with $E[y(s)]$ quadratic. Thus, we have

$$E[y(s)]=\tfrac{1}{2}\int_0^1\int_0^1 A(t,s)y(t)y(s)dtds+\int_0^1 B(s)y(s)ds, \quad (20\text{-a})$$

where $A(t,s)$, $B(s)$ are functions independent of y (that is, G is gaussian if the second functional derivative of $\ln G$ is independent of y). Gaussian functionals of several variables are of frequent occurrence. All the quantum field theory hamiltonians and lagrangians are of this form in the field variables. A formula for the integral of $G[y(s)]$ over all paths $y(s)$ has been found useful. It will be developed here. Consider the integral (we suppose $A(t,s)$ real, or at least has a positive definite imaginary part)

$$I[A,B]=\int \exp\{iE[y(s)]\}\mathfrak{D}y(s). \quad (21\text{-a})$$

It is a functional of $A(t,s)$ and $B(s)$. First, the dependence on B may be determined, as follows.

Let $\bar{y}(s)$ be that trajectory which makes the exponent $E[y(s)]$ an extremum. That is, $\bar{y}$ is a solution of (assuming A symmetric)

$$\int_0^1 A(t,s)\bar{y}(s)ds=-B(t). \quad (22\text{-a})$$

Or, if N be the reciprocal kernel to A (which can often most easily be found merely by solving Eq. (22-a)),

$$\bar{y}(s)=-\int^1 N(t,s)B(t)dt. \quad (23\text{-a})$$

Then put $y(s)=\bar{y}(s)+x(s)$. (Note, $\mathfrak{D}y(s)=\mathfrak{D}x(s)$.) In virtue of Eq. (22-a), one finds $E[y]=E[\bar{y}]+\frac{1}{2}\int_0^1\int_0^1 A(t,s)x(t)x(s)dtds$. Here, $E[\bar{y}(s)]$ can also be written explicitly as $-\frac{1}{2}\int_0^1\int_0^1 B(s)\times N(t,s)B(t)dtds$, using Eq. (23-a). Substituting this into Eq. (21-a), we see a factor $\exp\{iE[\bar{y}(s)]\}=G[\bar{y}(s)]$ may be taken outside the integral, as it is independent of $x(s)$. Hence, we have

$$I[A,B]=G[\bar{y}(s)]J[A], \quad (24\text{-a})$$

where

$$J[A]=\exp\left[\tfrac{1}{2}i\int_0^1\int_0^1 A(t,s)x(t)x(s)dtds\right]\mathfrak{D}x(s)=I[A,0] \quad (25\text{-a})$$

does not depend on B, and

$$G[\bar{y}(s)]=\exp\{iE[\bar{y}(s)]\}=\exp\left[-\tfrac{1}{2}i\int_0^1\int_0^1 B(s)N(t,s)B(t)dtds\right]. \quad (26\text{-a})$$

Often this is as far as it is necessary to go, as the dependence of I on B may have been all that is necessary to know, $J[A]$ being a kind of normalizing factor that is not of importance or that can be obtained in some other manner.

Having this form for I, we may obtain other integrals. For example, since

$$\delta I[A,B]/\delta B(t)=i\int G[y(s)]y(t)\mathfrak{D}y(t),$$

this integral can be immediately evaluated by differentiation of the expression (24-a) for I with respect to $B(t)$. Since

$$\delta G[\bar{y}(s)]/\delta B(t)=iG[\bar{y}(s)]\delta E(\bar{y}(s))/\delta B(t)=-i\int_0^1 B(s)N(t,s)ds\cdot G[\bar{y}(s)],$$

we find

$$\int G[y(s)]y(t)\mathfrak{D}y(s)=\bar{y}(t)I[A,B]. \quad (27\text{-a})$$

Differentiating a second time, since $\delta\bar{y}(t)/\delta B(t')=-N(t,t')$, one finds

$$\int G[y(s)]y(t)y(t')\mathfrak{D}y(s)=[\bar{y}(t)\bar{y}(t')+iN(t,t')]I[A,B], \quad (28\text{-a})$$

etc., for higher powers of y.

Incidentally, this permits us to obtain the properties of J. For the left-hand side of (28-a) is also $-2i\delta I[A,B]/\delta A(t,t')$. In the special case $B=0$, we have $\bar{y}=0$ from (23-a), and since $J(A)=I[A,0]$, we find

$$\delta J/\delta A(t,t')=-\tfrac{1}{2}N(t,t')J. \quad (29\text{-a})$$

This property of J determines it to within a numerical factor independent of A.

We have used these theorems, or something like them, on various occasions. One example was the passage from Eq. (16-a) to Eq. (17-a). In more generality, put

$$F=\exp\left[-i\int H(p(s), q(s))ds\right],$$

where H is quadratic in p. The integrations on $p(s)$ in Eq. (16-a) now represent an example of our theorem with $y=p$ and $E[p(s)]=\int[p\dot{q}-H(p, q)]ds$. The extremum requires $\partial H/\partial p=\dot{q}$. If the solution of this is $\bar{p}$, considered as a function of $\dot{q}$, q, then the integral on p produces within unimportant factors an exponential of $\int[\bar{p}\dot{q}-H(\bar{p}, q)]ds$, that is, $i\int Lds$, where L is the lagrangian. This example shows that in our discussion, we have not been sufficiently rigorous mathematically, for the important problem of the order of noncommuting operators p, q, in the original definition of H does not seem to have arisen.

A second example is the integration of $\exp(i\int Ldt)$ when L is quadratic in $\dot{q}$, q. For the forced harmonic oscillator where $L=\frac{1}{2}(\dot{q}^2-\omega^2q^2)+\gamma(t)q(t)$, the integral was carried out in III, footnote 7. The operator $A(t, s)$ is $-[(d^2/dt^2)+\omega^2]\delta(t-s)$, the inverse N of which involves sines and cosines but is not unique. However, in this case boundary conditions exist at the end points $q(0)$, $q(T)$, and these boundary conditions determine N and also restrict the range of y integration. The footnote serves a model of what to do under circumstances and will not be discussed further here.

The problem of integrating

$$\exp\left[-i\int j_\mu(1)A_\mu(1)d\tau_1\right]\cdot\exp\left[i\int(\partial A_\mu/\partial x_\nu)^2d\tau\right](8\pi e^2)^{-1}$$

over all distributions of field $A_\mu(1)$ required in III, Sec. VIII, serves as a further example. Here $y(s)$ is replaced by $A_\mu(1)$, and $B(s)$ by $j_\mu(1)$. The operator $A(t, s)$ becomes $\Box_1^2\delta(2, 1)$, the inverse of which is again not unique. The inverse $N(t, s)$ required in III is $\delta_+(s_{12}^2)$. (The boundary conditions required to define this particular inverse are probably related to the condition that no photons are supplied in the past and none are wanted in the future, so that the inverse must have no positive frequencies for $t\to-\infty$ and no negative ones to $t\to+\infty$.) Thus, $E[\bar{y}(s)]$ becomes the important quantity $-\frac{1}{2}\int\int j_\mu(1)j_\mu(2)\delta_+(s_{12}^2)d\tau_1d\tau_2$, so that Eq. (24-a) gives Eq. (43) or III, Eq. (48), which we had taken such pains in III to derive in a more rigorous manner. In none of these examples do we require J.

A more complicated example is that of the analysis of the operators corresponding to the electron-positron field given in I, Appendix. If electrons obeyed Bose statistics, the commutation rules would have been altered, the net effect being just to change a few signs in the final expressions. Analyzed as an Einstein-Bose field, however, the operators Ψ can be considered as ordinary functions, and the lagrangian technique may be used. The problem then requires gaussian integrals (actually, integrals of exponentials of bilinear expressions, but these are as easy to work out). The y corresponds to Ψ (or Ψ^*), the $A(t, s)$ is related to the Dirac hamiltonian, and its inverse $K_+^{(A)}(2, 1)$ replaces N. The problem of determining C_v corresponds to that of finding $J[A]$. The problem is complicated somewhat by the necessity of keeping the order of the γ_μ-operators correctly.

The relation of problems with operators obeying Fermi-Dirac statistics and those with the same operators obeying Einstein-Bose commutation rules is very close. The results of the former in practical cases,[24] at least, may be obtained from the latter by simply altering some signs. The Einstein-Bose case is very easily analyzed by ordered operator algebra (as in Sec. 7) or by the lagrangian integral methods. The anticommuting operators seem at first sight more complicated; but this they cannot be, as the results are just as simple. It would seem worth while to develop the analysis of anticommuting operators in much more detail than has been given here. Presumably, good use can be made of the similarity to the Einstein-Bose case. The theorems developed in analysis of this problem may conceivably have application in the problem of disentangling Pauli spin operators.

[24] The only known practical case, of course, is the electron-positron field. Here the problem has been completely worked out. I seem to be affected by the disease so prevalent today in theoretical physics, to delight in seeing a very general method of solving a problem, when actually in physics only one example of the type of problem exists and this has already been worked out.

D. Fock's Parameterization of the Dirac Equation

We wish to call attention to an interesting alternative method of parameterizing the Dirac equation, suggested by Fock.[4] It, like that of Sec. 8, would also have permitted us to pass directly to the formulation of electrodynamic problems. It is more readily interpreted than that of Sec. 8.[25]

As a consequence of the Dirac equation $(i\nabla-B)\psi=m\psi$, ψ also satisfies $(i\nabla-B)(i\nabla-B)\psi=m^2\psi$. Expanding the operator, this is equivalent to

$$[(i\partial/\partial x_\mu)-B_\mu]^2\psi-\tfrac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}\psi=m^2\psi, \tag{30-a}$$

with $\sigma_{\mu\nu}=\frac{1}{2}i(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)$. This differs from the Klein-Gordon equation only through the addition of the term $-\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}$, where $F_{\mu\nu}=(\partial B_\nu/\partial x_\mu)-\partial B_\mu/\partial x_\nu$ is the field tensor.

Just as in the Klein-Gordon case, III, Appendix A, this can be converted by the aid of a fifth parameter to Fock's equation,

$$i\partial\phi/\partial u=\tfrac{1}{2}[(i\partial/\partial x_\mu)-B_\mu]^2\phi-\tfrac{1}{4}\sigma_{\mu\nu}F_{\mu\nu}\phi, \tag{31-a}$$

for which the special solution $\phi=\exp(-\frac{1}{2}im^2u)\psi$ leads back to Eq. (30-a). It can then be analyzed by the lagrangian method. The final result is that the amplitude to go from one point to another [see III Eq. (5-a)] is the sum over all trajectories $x_\mu(u)$ of the hyper complex amplitude

$$\exp\left\{-i\int_0^{u_0}[\tfrac{1}{2}(dx_\mu/du)^2+(dx_\mu/du)B_\mu(x)-\tfrac{1}{4}\sigma_{\mu\nu}(u)F_{\mu\nu}(x(u))]du\right\}, \tag{32-a}$$

the order of operation of the $\sigma_{\mu\nu}$ being determined by the parameter u. This, in fact, is the lagrangian formulation of the Dirac equation suggested in C, Sec. XIV.

A rotation (and Lorentz transformation) by angle $\omega_{\mu\nu}$ in the $\mu\nu$-plane, is represented in Dirac theory by the operator $\exp(\frac{1}{4}i\omega_{\mu\nu}\sigma_{\mu\nu})$. (The summation on both μ and ν accounts for a factor 2.) Hence, we can say that Eq. (32-a) means that the amplitude for arrival is $\exp(iS)$, where S is the classical action $-\int[\frac{1}{2}(dx_\mu/du)^2+B_\mu dx_\mu/du]du$, but the orientation represented by the hypercomplex amplitude has rotated at each point in its path at an angular velocity (per du) equal to the field strength at that point. (Angular velocity in four dimensions is an antisymmetric tensor of second rank, as is the field strength.)

Since the potentials appear in exponential form, this may be directly connected to the form representing the action of virtual photons. The result is a set of rules like that for the Klein-Gordon case, Sec. IX, but with an additional coupling $F_{\mu\nu}\gamma_\mu\gamma_\nu$. They may be shown to be algebraically equivalent to the rules usually given for the Dirac equation, but are somewhat more complicated and not very interesting. There are some properties of the Dirac electron, however, which are more obvious in this formulation than in the usual one, and these we will discuss.

It is apparent from Eq. (32-a) that in the classical limit the trajectory is that of minimum S and therefore satisfies

$$d^2x_\mu/du^2=(dx_\nu/du)F_{\nu\mu}. \tag{33-a}$$

(Hence, $(dx_\mu/du)^2=(ds/du)^2$ is a constant of the motion where s is the proper time, and the minimum action S is $-\frac{1}{2}(ds/du)^2u$ plus a term independent of u. Since this is to vary as $-\frac{1}{2}m^2u$, we find $ds=mdu$.) As $\hbar\to0$, the magnetic moment approaches zero and does not affect the trajectory. But since the intrinsic spin angular momentum also goes to zero, the rate of precession of spin has a classical limit. For completeness we should also give the equation

[25] Y. Nambu, Prog. Theor. Phys. (Japan) **5**, 82, 1950.

of motion of the spin axis (just as the spot on a billiard cue ball has a motion, although it does not affect the trajectory of the ball). From Eq. (32-a) we see the spin axis precesses at angular velocity $F_{\mu\nu}$ (per du). (These are well-known results of the WKB approximation method when applied to the Dirac equation.) Since Eq. (33-a) says only that dx_μ/du precesses at the same angular velocity, we can summarize the classical equations of motion, and of spin precession for a Dirac electron as: *The velocity vector and spin plane are fixed in a four-dimensional coordinate system turning at each instant at an angular velocity per unit proper time equal to e/m times the field strength acting on the electron at that instant.* (For example, for a slowly moving electron in a magnetic field **B** the velocity vector revolves about the magnetic field as an axis at angular velocity $\omega=(e/m)\mathbf{B}$, the cyclotron frequency. The spin does likewise precessing therefore at the same frequency, which is twice the Larmor frequency.)

I have expended considerable effort to obtain an equally simple word description of the quantum mechanics of the Dirac equation. Very many modes of description have been found, but none are thoroughly satisfactory. For example, that of Eq. (32-a) is incomplete, even aside from the geometrical mysteries involved in the superposition of hypercomplex numbers. For in (32-a) the field enters in two *apparently unrelated* ways, once into defining S and again in the rotation rate. In the classical limit both effects of the field can be neatly stated in one principle. What makes things particularly simple in quantum mechanics if, for a diffusing wave, a rotation at rate $F_{\mu\nu}$ is accompanied by a phase shift equal to the line integral of A_μ?[26]

In the case that the fields $F_{\mu\nu}$ are constant in space and time, the operator factor $\exp(\frac{1}{4}iu\sigma_{\mu\nu}F_{\mu\nu})$ is independent of the trajectory and factors out of Eq. (32-a). The remaining path integral is gaussian and can be carried out exactly (Appendix C), giving the results of Fock[4] and Nambu.[25]

If the operator on the right-hand side of Eq. (31-a) is considered as a type of hamiltonian, the rate of change with u of all the relevant physical quantities (given by the commutator with this operator) are very easily interpreted by classical analogy.

There are, of course, twice as many solutions of Eq. (30-a) as solutions of the Dirac equation (50). (The others correspond to Eq. (50) with negative m.) If χ is a solution of Eq. (30-a), the projected part $\psi=(2m)^{-1}(i\nabla-\boldsymbol{B}+m)\chi$ solves the Dirac equation (50). Projection operators must still be used, therefore, in calculating matrix elements if Eq. (30-a) in perturbation is used instead of the Dirac equation.[27]

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[26] If the $\frac{1}{4}\sigma_{\mu\nu}F_{\mu\nu}$ term is considered to have a coefficient α analogous to a kind of anomalous magnetic moment, difficulties arise in the resulting theory unless $\alpha=1$ or $\alpha=0$. Thus, the real part of L, in the amplitude for a vacuum to remain a vacuum, $C_v=\exp(-L)$, should always be positive if the theory is to be easily interpreted (see I, Sec. V). For general α, it seems that the real part of L is positive for some processes (or potentials), negative for others. It is always positive only if $\alpha=1$. But for $\alpha=0$ it is always negative, so we can reinterpret the theory in this case as referring to Bose particles, in which case C_v should be $\exp(+L)$ (I, Sec. V). For $\alpha=0$, Eq. (30-a) becomes the Klein-Gordon equation, of course.

[27] A convenient way to make the correspondence of solutions χ of Eq. (30-a) and ψ of the Dirac equation unique is to assume χ is also an eigenfunction of γ_5, that is, $\gamma_5\chi=i\chi$. This is possible, as γ_5 commutes with the operators of Eq. (30-a). Then, for each ψ, the corresponding χ is $\chi=(1-i\gamma_5)\psi$.