it is to abandon the simplified concepts and use a general method as introduced by us in VI. We are not certain if Furry wants his results to apply to the scattering of very hard quanta; if this is the case then there exist basic differences with the content of VII.

We want to express the hope that later investigations of a deeper-reaching mathematical nature will lead not only to a formulation as given here, but also to a computation of the differences in the two schemes and thereby present opportunities for an experimental decision between the two forms of the theory, which in our opinion will doubtless show the validity of scheme II.

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Reversibility of Quantum Electrodynamics

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The aim of this paper is to re-establish the reversibility of classical electrodynamics in terms of the "expectation values" given by quantum electrodynamics. The reversibility requirement combined with the charge conjugation necessitates that charged fields should obey certain types of statistics. However, the reversibility requirement as such does not determine the statistics, showing that it is the requirement of charge-invariance that has the power to determine the statistics of charged fields. A new interpretation will be given to the old problem concerning the conflict of electromagnetic reversibility *versus* "retarded" potential. Four different kinds of tensors, four different kinds of spinors (pseudospinors), bi-spinors (eight-component spinors) and bi-tensors are introduced as useful representation vectors of the entire congruent group including spatial and temporal inversions.

I. INTRODUCTION

IN connection with the proof that phenomenological irreversibility originates essentially from the process of observation, it seemed to the present author to be of importance to ascertain the complete time-reversibility of quantum mechanics. In an earlier paper,[1] reversibility of the Dirac equation was demonstrated in its one-particle interpretation, and the behavior of the spin, electric moment and magnetic moment of the electron in the "reversed" motion was examined in detail. Then, the reversibility of quantum electrodynamics was proved in the frame-work of Dirac's many-time theory.[2] It was thereafter noticed that the same method can be applied to the theory in which the electron field also is quantized if a correct treatment of charge conjugation is introduced.[3] In all these considerations, it was observed that commutation relations of the field quantities, under certain assumptions, played important roles in the proof of reversibility.

Now that many authors are interested in the problem of necessary general forms of commutation relations, it may be of some interest to publish a summary of the results hitherto obtained by the author, clarifying the relationship between reversibility and commutation relations. In the meantime, Schwinger is reported to have used in his lectures a similar consideration to deduce the commutation relations from the requirement of reversibility.[4] However, since his method as well as his conclusion seems to be at variance with those of the present author, perhaps they may justifiably be presented here.

Against a formal requirement of reversibility objections are often raised to the effect that we can never reverse the direction of time in our actual experience. However, we can formulate the "reversibility" in such a manner that it does not involve any hypothetical inversion of time.

The reversibility of classical point mechanics can be expressed in the following way. Let us call two states of a mechanical system mutually reversed states if particles have the same positions and the opposite velocities. Then the reversibility of mechanics means that, if a mechanical system, which was in the state S_1 at the initial instant ($t=0$), finds itself in the state S_2 at the final instant ($t=t_1$), then the fundamental laws allow for another solution representing the similar system which was in the reversed state of S_2 at the initial instant ($t=0$) and which finds itself in the reversed state of S_1 at the final instant ($t=t_1$).

To extend this notion of reversibility to electrodynamics, we need only to add to the definition of reversed states the condition that the electric field has the same value and the magnetic field has the same absolute value but opposite sign. Then the above statement of reversibility holds again in the Maxwellian theory.

It is to be noted that this concept of reversibility does not invoke any fictional time-reversal. Also, as far as

[1] S. Watanabe, *Le Deuxième Théorème de la Thermodynamique et la Mécanique Ondulatoire* (Hermann et Cie, Paris, 1935).
[2] S. Watanabe, Sci. Pap. Inst. Phys. Chem. Research (Tokyo) 31, 109 (1937).
[3] Unpublished.
[4] Lectures by J. Schwinger, notes taken by M. L. Goldberger.

Newton's laws and the Maxwell equations are correct, this reversibility is guaranteed by their mathematical structure.

Now the Maxwell equations have, other than their Lorentz invariance, a trivial invariance under charge conjugation. Namely, the simultaneous change of signs of the electromagnetic field quantities and change of signs of the sources leave the equations unchanged. In virtue of this charge invariance, we can modify the definition of reversed states in the following way. In the reversed states, particles of opposite charges are performing the reversed motion, the magnetic field has the same value and the electric field has the same absolute value but opposite sign. Then the reversibility with this modified definition holds also insofar as the Maxwell equations are true. Physically, this modified standpoint corresponds to a combination of charge conjugation and reversal of motion.

Now we require that quantum electrodynamics should provide this classical reversibility in the "expectation values" of the physical quantities concerned. More precisely, two state functions Ψ and Ψ' are said to represent mutually reversed states if the expectation values of the physical quantities in the states Ψ and Ψ' satisfy the classical definitions of reversed states. Then our requirement is that, if a physical system which was in the state Ψ_1 at the initial instant ($t=0$) develops with time according to the Schroedinger equation and becomes Ψ_2 at the final instant ($t=t_1$), then the similar system which is in the reversed state Ψ'_2 of Ψ_2 at the initial instant ($t=0$) should become the reversed state Ψ'_1 of Ψ at the final instant ($t=t_1$).

It will be shown that this condition can be restated as follows: the probability of finding the physical system which was in the state Ψ_A at the initial instant in the state Ψ_B at the final instant is equal to the probability of finding the system which was in the reversed state of Ψ_B at the initial instant in the reversed state of Ψ_A at the final instant, where Ψ_A and Ψ_B are arbitrary states.

Here again, attention is drawn to the fact that our conception of reversibility does not imply any hypothetical inversion of time.

Our analysis will show that if we use the "modified" definition of reversed states, the reversibility requirement in quantum electrodynamics can be satisfied only if the charged field quantities are assumed to obey certain types of commutation rules. But if we translate the original definition of reversed states in the quantum theory, the reversibility requirement does not determine the commutation rules. This shows, in its physical implications, that it is not reversibility but charge conjugation that has the power of determining the commutation relations of the charged field quantities.

It is interesting to notice that if we consider the reversibility from a purely formalistic point of view, i.e., if we search for the simplest transformation that keeps the mathematical expressions covariant for the time-reversal, we hit upon a transformation which corresponds to the "modified" definition of reversibility. This, of course, does not preclude the original definition of reversed states from being mathematically formulated.

One may raise an objection to the theory already sketched and contend that the classical electrodynamics is not reversible, because we have always to choose only the "retarded" solution out of the possible solutions. To answer to this question, we shall show, in Sec. IX, that the reversible quantum electrodynamics indeed gives one-half the retarded potential plus one-half the advanced potential; however, that in spite of this formal symmetry of past and future, this advanced potential refers to the mathematically constructed future and can be transformed into the retarded potential, thus leading to a full retarded potential. This expression in terms of full retarded potential is the only permissible one, if we admit as a postulate that any physical law should be formulated in the form of a prediction based on a past observation.

As a matter of fact, in classical electrodynamics, the dismissal of advanced potential cannot be logically reconciled with the basic reversibility except by resorting to a statistical assumption of some kind.[5] In contrast to this, in quantum electrodynamics, the reduction of advanced potential to the retarded potential, and therefore the elimination of the former, can be done on a very general basis, see Sec. IX.

This situation presents an instructive parallelism to the problem of entropy increase. In classical physics, decrease of entropy can happen, seldom as it may be, while, in quantum physics, increase of (microscopically defined) entropy by the act of observation is definitive.

Our understanding of these problems can then be summarized in the statement that the fundamental laws *per se* are completely reversible, while the irreversible phenomena appear as the result of the particular nature of our human cognition. In classical physics, this conflict between basic reversibility and phenomenological irreversibility cannot easily be clarified, because the physical quantities (and the state) are, in this theory, taken for "reality" rather than for "potentiality," i.e., mathematical instruments to correlate one observation to another observation. Indeed, reversibility in quantum physics pertains to this "potentiality."

Section II of this paper will give a c-number lagrangian formalism of electromagnetic interaction, which is completely covariant for any congruent transformation including time-reversal. This c-number theory will also help understand the mathematical gist of the time-reversible q-number theory, which is developed in the following sections. In the q-number theory of reversibility, a unitary operator R, introduced previously by the author under the name of reversion operator,[2] will play a central role. This paper is essentially a study of the nature of this reversion operator.

[5] See for instance, J. A. Wheeler and R. P. Feynman, Revs. Modern Phys. **17**, 157 (1945).

The customary tensor- and spinor-analysis are very inadequate for a problem like the present one. The mathematical instrumentalities introduced in this paper are believed to prove useful for many other problems involving spatial and temporal inversions. For this reason, they are explained in some detail in the Appendix. Those who want to use spinors in a 5-dimensional space will also find our definition of spinors particularly convenient.

II. COVARIANT FORMULATION OF ELECTROMAGNETIC INTERACTION

We consider the electromagnetic field a^μ interacting with the electric current generated by a spinor field ψ and a complex scalar (or pseudoscalar) field u. The ordinary lagrangian density $\mathfrak{L}$ can be written in the form:

$$\mathfrak{L}=\mathfrak{L}_1+\mathfrak{L}_2+\mathfrak{L}_3+\mathfrak{L}_{12}+\mathfrak{L}_{13}, \tag{2.1}$$

with

$$\left.\begin{aligned}\mathfrak{L}_1(a)&=-(1/2)(\partial a_\mu/\partial x_\nu)(\partial a^\mu/\partial x^\nu),\\ \mathfrak{L}_2(u)&=-(\partial\bar{u}/\partial x_\nu)(\partial u/\partial x^\nu)-\kappa^2\bar{u}u,\\ \mathfrak{L}_3(\psi)&=(1/2i)\bar{\psi}[(\partial/\partial x_\mu)E_\mu\\ &\qquad-(\overleftarrow{\partial/\partial x_\mu})E_\mu+2imE_5]\psi,\end{aligned}\right\} \tag{2.2}$$

$$\left.\begin{aligned}&\mathfrak{L}_{12}(a,u)=ea^\mu i_\mu;\\ &\qquad i_\mu(u)=i[(\partial\bar{u}/\partial x^\mu)u-\bar{u}(\partial u/\partial x^\mu)],\\ &\mathfrak{L}_{13}(a,\psi)=ea^\mu s_\mu;\quad s_\mu(\psi)=-\bar{\psi}E_\mu\psi\end{aligned}\right\}. \tag{2.3}$$

The second order (in e) interaction term in $\mathfrak{L}_{12}$ is dropped, for inclusion of such term does not affect our discussion in the following.

This lagrangian has, of course, a wrong property for transformations of the classes $\mathfrak{B}$ and $\mathfrak{D}$ ($\sigma_t=-1$). (See Appendix.) For $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are regular scalars whatever kind of tensors a^μ and u may be, whereas $\mathfrak{L}_3$ is a second kind pseudoscalar. $\mathfrak{L}_{12}$ and $\mathfrak{L}_{13}$ are necessarily of different kinds from each other, for s_μ is a second kind pseudovector while i_μ is a regular vector.

All the terms (2.2) (2.3) can be brought to any one "kind" of scalar by the method developed in the Appendix whatever kinds of property we may assign to a^μ and u. We shall, however, discuss only two of the various possibilities:

(I) $\mathfrak{L}$: regular scalar; a^μ: regular vector

(II) $\mathfrak{L}$: regular scalar; a^μ: 2nd kind pseudovector.

The method consists in introducing two tensors or two spinors to represent one field. These two tensors or two spinors are not only transformed according to the ordinary transformation rules but also interchanged whenever we perform a transformation with $\sigma_t=-1$. Corresponding to a^μ, u and ψ, we introduce, respectively, (a^μ, b^μ), (u, v) and (ψ, φ), which constitute two bi-tensors and a bi-spinor:

$$A^\mu=\begin{pmatrix}a^\mu\\ b^\mu\end{pmatrix};\quad U=\begin{pmatrix}u\\ v\end{pmatrix};\quad \bar{U}=\begin{pmatrix}\bar{u}\\ \bar{v}\end{pmatrix};\quad X=\begin{pmatrix}\psi\\ \varphi\end{pmatrix};\quad \bar{X}=\begin{pmatrix}\bar{\psi}\\ \bar{\varphi}\end{pmatrix}. \tag{2.4}$$

And to represent the current we introduce two bi-tensors by

$$I^\mu=\begin{pmatrix}i^\mu(u)\\ i^\mu(v)\end{pmatrix};\quad S^\mu=\begin{pmatrix}s^\mu(\psi)\\ s^\mu(\varphi)\end{pmatrix}. \tag{2.5}$$

In virtue of the theorem given in connection with (A.60), I^μ is a regular bi-vector whatever property u (and v) may have. S^μ is a second kind pseudo-bi-vector according to (A.61).

Case (I): We can then using (A.47), (A.53), and (A.55) easily build a regular scalar lagrangian density by replacing (2.2) and (2.3) by

$$\left.\begin{aligned}2\mathfrak{L}'_1&=-(1/2)(\partial A_\mu/\partial x_\nu)(\partial A^\mu/\partial x^\nu)\\ &\qquad=\mathfrak{L}_1(a)+\mathfrak{L}_1(b)\\ 2\mathfrak{L}'_2&=-(\partial\bar{U}/\partial x_\mu)(\partial U/\partial x^\mu)-\kappa^2\bar{U}U\\ &\qquad=\mathfrak{L}_2(u)+\mathfrak{L}_2(v)\\ 2\mathfrak{L}'_3&=(1/2i)\bar{X}[(\partial/\partial x_\mu)\mathrm{E}_\mu\\ &\qquad-(\overleftarrow{\partial/\partial x_\mu})\mathrm{E}_\mu+2im\mathrm{E}_5]\Pi X\\ &\qquad=\mathfrak{L}_3(\psi)-\mathfrak{L}_3(\varphi)\end{aligned}\right\} \tag{2.6}$$

$$\left.\begin{aligned}2\mathfrak{L}'_{12}&=eA^\mu I_\mu=ea^\mu i_\mu(u)+eb^\mu i_\mu(v)\\ 2\mathfrak{L}'_{13}&=eA^\mu\Pi S_\mu=ea^\mu s_\mu(\psi)-eb^\mu s_\mu(\varphi)\end{aligned}\right\}. \tag{2.7}$$

In expressions (2.6) and (2.7) the terms in a^μ, u, $\bar{u}$, ψ, $\bar{\psi}$, and the terms in b^μ, v, $\bar{v}$, φ, $\bar{\varphi}$ should be written in such a way that corresponding factors are placed in the same order in both. This is essential for the later reinterpretation in q-number theory.

Case (II): $\mathfrak{L}'_1$, $\mathfrak{L}'_2$ and $\mathfrak{L}'_3$ in (2.6) need not be changed. (2.7) should however be changed into

$$\left.\begin{aligned}2\mathfrak{L}'_{12}&=eA^\mu\Pi I_\mu=ea^\mu i_\mu(u)-eb^\mu i_\mu(v)\\ 2\mathfrak{L}'_{13}&=eA^\mu S_\mu=ea^\mu s_\mu(\psi)+eb^\mu s_\mu(\varphi)\end{aligned}\right\}. \tag{2.8}$$

Thus we have succeeded in writing the lagrangian in a covariant form for the entire congruent group in the c-number theory.

Although the quantum-theoretical consideration of reversibility is being developed in the later sections, the mathematical core of the results obtained there will now be given in anticipation.

In the q-number theory all the field quantities are, of course, to be regarded as matrices, and the requirement of reversibility is then expressed by certain relations involving these matrices and their transposes. In terms of bi-tensors and bi-spinors, these relations will imply certain connections between their first and second parts involving the operation of transposition.

The operation of transposition is of course not commutable with the ordinary unitary transformations. But, for a reason which will be explained later, we need not here specify the representation which is to be used.

It should be borne in mind that the second half of a bi-tensor or of a bi-spinor must belong to the same transformation rules as its first half, except for their interchange. Therefore, we can expect without a detailed analysis that the aforementioned connections between the first halves and the second halves will turn out to be certain combinations of the following possibilities (see (A.32)):

$$\left.\begin{array}{l} b^\mu = a^{T\mu}, \\ v=u^T, \quad \bar{v}=\bar{u}^T \quad \text{or} \quad v=\bar{u}^T, \quad \bar{v}=u^T, \\ \psi=\psi^T, \quad \bar{\varphi}=\bar{\psi}^T \quad \text{or} \quad \varphi=\bar{\psi}^T K, \quad \bar{\varphi}=\psi K^{-1}. \end{array}\right\} \quad (2.9)$$

Case (I): In this case, the conclusions drawn from the requirement of reversibility in q-number theory can be summarized in the following two items:

(a) $b^\mu = a^{T\mu};\ v=u^T;\ \bar{v}=\bar{u}^T;\ \varphi=\psi^T;\ \bar{\varphi}=\bar{\psi}^T,$ (2.10)

(b) $\mathfrak{L}'=\mathfrak{L}^{T\prime},$ (2.11)

where $\mathfrak{L}'$ is the sum of all the $\mathfrak{L}'$'s of (2.6) and (2.7). The symbol of equality in (2.11) is to be understood in the sense of "equal except for an additional c-number."

If relations (2.10) are substituted in (2.6) and (2.7), condition (2.11) lets us draw conclusions as to the commutation relations for the u-field and the ψ-field. The condition that $\mathfrak{L}'_{12}=\mathfrak{L}^{T\prime}_{12}$ and $\mathfrak{L}'_{13}=\mathfrak{L}^{T\prime}_{13}$ with (2.10) evidently requires:

$$\bar{\psi}E_\mu\psi = -\psi E^T{}_\mu\bar{\psi};$$

$$(\partial\bar{u}/\partial x^\mu)u-\bar{u}(\partial u/\partial x^\mu) = u(\partial\bar{u}/\partial x^\mu)-(\partial u/\partial x^\mu)\bar{u}, \quad (2.12)$$

which can be true only when

$$\begin{aligned} \psi\bar{\psi}+\bar{\psi}\psi &= c\text{-number}, \\ u\bar{u}-\bar{u}u &= c\text{-number}. \end{aligned} \quad (2.13)$$

It can easily be seen that (2.13) guarantees the conditions $\mathfrak{L}'_2=\mathfrak{L}^{T\prime}_2$ and $\mathfrak{L}'_3=\mathfrak{L}^{T\prime}_3$ in (2.6).

The foregoing method applied to the electromagnetic field a^μ entails no conclusion as to its commutation rule. For (2.10) for a^μ automatically satisfies $\mathfrak{L}'_1=\mathfrak{L}^{T\prime}_1$ with (2.6). Indeed, if a^μ obeyed the Fermi statistics, $\mathfrak{L}_1$ would become essentially a c-number, so that such an assumption can hardly become a subject of the present consideration.

Case (II): Reversibility requirement here takes the form:

(a) $b^\mu=a^{T\mu};\ v=\bar{u}^T;\ \bar{v}=u^T;\ \varphi=\bar{\psi}^T K;\ \bar{\varphi}=\psi^T K^{-1},$ (2.14)

(b) $\mathfrak{L}'=\mathfrak{L}^{T\prime},$ (2.15)

with (2.6) and (2.8).

It can be verified that (2.14) substituted in (2.6) and (2.8) automatically satisfies (2.15), so that no conclusion as to the commutation rules can be drawn.

It is interesting to notice that we in reality do not need all of the conditions enumerated in (2.10) or (2.14) in order to obtain the foregoing results. In fact, we need only $b^\mu=a^{T\mu}$ and $\mathfrak{L}'=\mathfrak{L}^{T\prime}$, if we knew that the choice of v and φ should be made out of the possibilities indicated in (2.9).

III. REVERSIBILITY OF CLASSICAL ELECTRODYNAMICS

The reversibility of point mechanics can be stated in the following manner. Two states $Z_{(1)}$ and $Z_{(2)}$ are called "reversed" states of each other if

$$x^a{}_{(1)}=x^a{}_{(2)}, \quad p^a{}_{(1)}=-p^a{}_{(2)} \quad (a=1,2,3); \quad (3.1)$$
$$p^0{}_{(1)}=p^0{}_{(2)}; \quad E_{(1)}=E_{(2)},$$

where x^μ, $p^\mu(\mu=1,2,3,0)$ and E stand, respectively, for the positions of the particles, their momenta and the total energy of the system, and the subscripts (1) and (2) refer to the two states. This situation (3.1) will be written symbolically as

$$Z_{(1)}\overset{\text{rev}}{\sim}Z_{(2)}. \quad (3.2)$$

The statement that the mechanical law is reversible means that if $Z_{(1)}(t)$ is a solution then it allows for a second solution $Z_{(2)}(t)$ such that, if at a certain instant τ

$$Z_{(1)}(\tau)\overset{\text{rev}}{\sim}Z_{(2)}(\tau), \quad (3.3)$$

then

$$Z_{(1)}(\tau+t)\overset{\text{rev}}{\sim}Z_{(2)}(\tau-t) \quad (3.4)$$

for any value of t. In virtue of the displacement group in time that the mechanical law allows for, we can write, instead of (3.3) and (3.4),

$$Z_{(1)}(t)\overset{\text{rev}}{\sim}Z_{(2)}(-t). \quad (3.5)$$

This is true not only in the absence of external forces, but also in the cases in which the external forces for $Z_{(1)}$ at t are the same as those for $Z_{(2)}$ at $-t$. Frictional forces which change their signs in the reversed motion invalidates the reversibility.

There are now two "standpoints" by which the concept of reversibility can be adapted to electrodynamics.

According to one standpoint, the reversed states $Z_{(1)}$ and $Z_{(2)}$ are defined by

$$\left.\begin{array}{l} x^a{}_{(1)}=x^a{}_{(2)}, \quad p^a{}_{(1)}=-p^a{}_{(2)}, \quad (a=1,2,3); \\ \qquad\qquad\qquad\qquad p^0{}_{(1)}=p^0{}_{(2)} \\ i^a{}_{(1)}=-i^a{}_{(2)}; \quad i^0{}_{(1)}=i^0{}_{(2)}; \quad a^a{}_{(1)}=-a^a{}_{(2)}; \\ \qquad\qquad a^0{}_{(1)}=a^0{}_{(2)}; \quad E_{(1)}=E_{(2)} \end{array}\right\}, \quad (3.6)$$

where i^μ and a^μ are, respectively, the current and the potential vectors. This standpoint corresponds to the "original" standpoint in our "Introduction" and will be hereafter referred to as standpoint (II).

Then it can be proved that the classical electrodynamics guarantees the reversibility which can be written as (3.5). This is because we can regard the field quantities a^μ, $f^{\mu\nu}$, the current i^μ and the four-velocity v^μ as pseudo-tensors of the second kind in the Maxwell equations:

$$\left.\begin{aligned} \Box a^\mu(x) = -i^\mu(x) = -\sum ev^\mu(x')\delta(x-x'); \\ \partial a^\mu/\partial x^\mu = 0 \\ f^{\mu\nu} = \partial a^\nu/\partial x_\mu - \partial a^\mu/\partial x_\nu; \quad m\dot{v}^\mu = ef^{\mu\nu}v_\nu \end{aligned}\right\}. \quad (3.7)$$

(the proper time is a second kind pseudoscalar).

We now notice that, in these equations, we can alternatively regard a^μ, $f^{\mu\nu}$, i^μ as regular tensors. This entails, since the velocities must be second kind pseudovectors, that e should behave as a second kind pseudoscalar. This means that, in the reversed state, particles of the opposite charges are performing the reversed motion. In symbols:

$$\left.\begin{aligned} x^a{}_{(1)}(\pm) = x^a{}_{(2)}(\mp); \quad p^a{}_{(1)}(\pm) = -p^a{}_{(2)}(\mp); \\ p^0{}_{(1)}(\pm) = p^0{}_{(2)}(\mp) \\ i^a{}_{(1)} = i^a{}_{(2)}; \quad i^0{}_{(1)} = -i^0{}_{(2)}; \quad a^a{}_{(1)} = a^a{}_{(2)}; \\ a^0{}_{(1)} = -a^0{}_{(2)}; \quad E_{(1)} = E_{(2)} \end{aligned}\right\}. \quad (3.8)$$

This modified definition of the reversed state will be referred to as (I) in the following. It is obvious that this standpoint (I) is nothing but the combination of the reversal of motion in its proper sense and the reversal of charge. The latter means the change of signs of the charges and the field quantities, for which the equations (3.7) remain invariant. If the electrodynamics is reversible in standpoint (II), it is also reversible in standpoint (I), and vice versa, provided the charge invariance is guaranteed.

In both standpoints, electromagnetic waves in the reversed states propagate in opposite directions, with the same direction of linear polarization. There is however a phase difference of 180 degrees between (I) and (II). It is also clear in both standpoints that, if in $Z_{(1)}$ a light pulse is emitted at instant t, the corresponding light pulse is absorbed at instant $-t$ in $Z_{(2)}$.

IV. REVERSION OPERATOR

In the following sections, use will mainly be made of the interaction picture, in which the state function Ψ changes with time according to the Schroedinger equation with the interaction hamiltonian:

$$d\Psi/dt = -i\mathbf{H}\Psi, \quad (4.1)$$

while all the physical quantities Q changes with time according to the Heisenberg equation with the non-interacting hamiltonian:

$$dQ/dt = i(\mathbf{H}_0 Q - Q\mathbf{H}_0). \quad (4.2)$$

The values of Ψ at t_1 and at t_2 are related to each other by

$$\Psi(t_1) = U(t_1, t_2)\Psi(t_2), \quad (4.3)$$

where

$$\left.\begin{aligned} \partial U(t_1, t_2)/\partial t_1 = -i\mathbf{H}(t_1)U(t_1, t_2); \\ \partial U(t_1, t_2)/\partial t_2 = +iU(t_1, t_2)\mathbf{H}(t_2) \end{aligned}\right. \quad (4.4)$$

with

$$U(t_1, t_2) = 1 \quad \text{for} \quad t_1 = t_2. \quad (4.5)$$

The transformation function $U(t_1, t_2)$ has the properties:

$$\bar{U}(t_1, t_2) = U^{-1}(t_1, t_2) = U(t_2, t_1). \quad (4.6)$$

We now proceed to define the reversed states in the quantum-mechanical language. $\Psi_{(1)}(t_1)$ and $\Psi_{(2)}(t_2)$ are said to be reversed states of each other if the expectation values of physical quantities for $\Psi_{(1)}(t_1)$ and those for $\Psi_{(2)}(t_2)$ are related by the conditions which characterize the reversed states in classical electrodynamics, i.e., relations (3.8) in standpoint (I) and relation (3.6) in standpoint (II). These conditions will then take the general form:

$$(\Psi_{(1)}(t_1), P(t_1)\Psi_{(1)}(t_1)) = (\Psi_{(2)}(t_2), Q(t_2)\Psi_{(2)}(t_2)). \quad (4.7)$$

For instance, in standpoint (I), $P = a^0$, $Q = -a^0$; $P = N_\pm(\mathbf{k})$, $Q = N_\mp(-\mathbf{k})$; etc., where $N_+(\mathbf{k})$, for example, means the number of positively charged particles of a certain field having momentum $\mathbf{k}$.[6] There are quantities in the quantum theory which do not have direct counterparts in the classical theory. The physical meaning of such a quantity, however, always enables us to deduce its transformation property from the known transformation properties of the physically related quantities.

If relation (4.7) is satisfied for all the relevant physical quantities, we say that $\Psi_{(1)}(t_1)$ and $\Psi_{(2)}(t_2)$ are reversed states of each other and write

$$\Psi_{(1)}(t_1) \overset{\text{rev}}{\sim} \Psi_{(2)}(t_2). \quad (4.8)$$

Now the requirement of reversibility demands that, if $\Psi_{(1)}(t)$ is a solution of the Schroedinger equation, it should allow for a second solution $\Psi_{(2)}(t)$ such that for any value of t,

$$\Psi_{(1)}(t) \overset{\text{rev}}{\sim} \Psi_{(2)}(-t). \quad (4.9)$$

(4.9) is a summarized expression for

$$\left.\begin{aligned} d\Psi_{(1)}(t)/dt = -i\mathbf{H}(t)\Psi_{(1)}(t); \\ d\Psi_{(2)}(t)/dt = -i\mathbf{H}(t)\Psi_{(2)}(t), \end{aligned}\right. \quad (4.10)$$

and

$$(\Psi_{(1)}(t), P(t)\Psi_{(1)}(t)) = (\Psi_{(2)}(-t), Q(-t)\Psi_{(2)}(-t)). \quad (4.11)$$

[6] Classical electrodynamics is a mixture of particle picture (for charge fields) and wave picture (for electromagnetic field). We can reformulate all the reversibility requirements consistently in terms of wave picture. In this point of view, requirements regarding particle numbers such as the one indicated in the text can be dispensed with, if requirements are fulfilled regarding momentum density, energy density, current density, charge density, spin density, and electromagnetic moment density.

If $\Psi_{(1)}(t)$ and $\Psi_{(2)}(t)$ are normalized they can be linked by a yet to be determined unitary operator R:

$$\Psi_{(2)}(-t)=\Psi^*_{(1)}(t)R, \tag{4.12}$$

with

$$\tilde{R}=R^{-1}. \tag{4.13}$$

The star on Ψ means its complex conjugate. The "reversion operator" R is to be determined by (4.11).

Substitution of (4.12) in (4.11) gives, because of the hermiticity of P and Q,

$$(\Psi_{(1)}(t),\, P(t)\Psi_{(1)}(t)) = (\Psi_{(1)}(t),\, RQ^T(-t)R^{-1}\Psi_{(1)}(t)), \tag{4.14}$$

which must hold for any value of t. Since the choice of $\Psi_{(1)}$ is independent of (P, Q), it follows from (4.14) that

$$Q(-t)=(R^{-1}P(t)R)^T. \tag{4.15}$$

As for the Schroedinger Eqs. (4.10), they can be rewritten as

$$d\Psi^*_{(1)}(t)/dt=i\Psi^*_{(1)}(t)\mathbf{H}(t);\quad d\Psi_{(2)}(-t)/dt=i\Psi_{(2)}(-t)\mathbf{H}^T(-t), \tag{4.16}$$

which cannot be made compatible with each other by (4.12) unless R is independent of time and

$$\mathbf{H}(-t)=(R^{-1}\mathbf{H}(t)R)^T. \tag{4.17}$$

This last relation is nothing but a special case of (4.15) for $\mathbf{H}(t)$.

Therefore, the reversibility requirement is equivalent to the requirement of existence of a time-independent unitary operator R which is defined by (4.15).

Under a time-independent unitary transformation which transforms hermitian operators (and ordinary unitary operators) Q into $T^{-1}QT$ and state functions Ψ into $T^{-1}\Psi$, the reversion operator will, according to its definition, be transformed into

$$R\rightarrow T^{-1}RT^{T-1}=T^{-1}RT^*, \tag{4.18}$$

which preserves (4.13). This particular transformation warrants invariant meanings to the operation of transposition and to the operation of taking complex conjugates of state functions, which have been involved in the foregoing formulas.

Solutions of (4.2) can be expressed in terms of a time-dependent unitary transformation:

$$Q(t)=U_0^{-1}(t)Q(0)U_0(t), \tag{4.19}$$

with

$$dU_0(t)/dt=-i\mathbf{H}_0U_0(t);\quad U_0(0)=1. \tag{4.20}$$

The defining relation for R (4.15) then takes the form:

$$Q^T(0)=(U^T{}_0^{-1}(-t)R(t)U_0^{-1}(t))\times P(0)(U_0(t)R(t)U_0{}^T(-t)), \tag{4.21}$$

showing

$$R(t)=U_0^{-1}(t)R(0)U^T{}_0^{-1}(-t), \tag{4.22}$$

from which follows

$$dR(t)/dt=i(\mathbf{H}_0R(t)-R(t)\mathbf{H}^T{}_0), \tag{4.23}$$

because $\mathbf{H}_0$ is independent of time. The time-independence of R implies

$$\mathbf{H}_0R-R\mathbf{H}^T{}_0=0. \tag{4.24}$$

This is only a special case of (4.15) for $\mathbf{H}_0$.

Hence, if the reversion operator defined by (4.15) exists, it automatically satisfies the requirement that it should be time-independent.

(4.24) gives an interesting information: The time-independent R commutes with $\mathbf{H}_0$ in the particular representation in which $\mathbf{H}_0$ is diagonal. Since $\mathbf{H}_0$ is a degenerate operator, this does not imply that R is diagonal in this representation.

Since P and Q must be interchangeable in (4.15) and since we have not specified the sign of t, we can write

$$P(t)=(R^{-1}Q(-t)R)^T=R^TR^{-1}P(t)RR^{T-1}, \tag{4.25}$$

which implies that R^TR^{-1} commutes with P. If we were allowed to make an assumption that any arbitrary hermitian operator P should have its associated operator Q satisfying (4.7), then (4.25) would mean that R^TR^{-1} should be a c-number, i.e.,

$$R^T=cR,\quad \text{or}\quad R^T=\pm R. \tag{4.26}$$

It is interesting to note that the symmetry or antisymmetry of R is preserved under the transformation (4.18). However, such an assumption is obviously too hasty a generalization. In fact, we shall encounter in a later section a concrete example of R which without being symmetrical or antisymmetrical warrants the symmetry of relevant physical quantities with respect to t and $-t$ as expressed by the commutability relation (4.25). Physically, however, R and R^T have the same meaning.

Application of (4.3) to (4.12) yields

$$U^{T-1}(t,-t)=R^{-1}U(t,-t)R. \tag{4.27}$$

If the reversion operator exists, we can demonstrate the following statement with the help of (4.27): The probability of finding a physical system, which was in state Ω at $-t$, in state Θ at t is equal to the probability of finding a system, which was in the reversed state of Θ at $-t$, in the reversed state of Ω at t. This can be regarded as an alternative expression of the "reversibility." The first probability is given by

$$|(\Theta,\, U(t,-t)\Omega)|^2, \tag{4.28}$$

and the second by

$$|((\Omega^*R),\, U(t,-t)(\Theta^*R))|^2. \tag{4.29}$$

The latter is, in virtue of (4.6), equal to

$$|(\Omega,\, RU^{T-1}(t,-t)R^{-1}\Theta)^*|^2, \tag{4.30}$$

which coincides with (4.28) on account of (4.27).

Finally, the method of Sec. II can be linked to the consideration of this section in the following manner. The lagrangian density, on account of its relation to the hamiltonian density, must have the same sign in the reversed state:

$$\mathfrak{L}(t)=(R^{-1}\mathfrak{L}(-t)R)^T \quad (4.31)$$

or

$$\mathfrak{L}'(t)=\mathfrak{L}^{T\prime}(t)$$

with

$$\mathfrak{L}'(t)=\mathfrak{L}(t)+R^{-1}\mathfrak{L}(-t)R. \quad (4.32)$$

Relations (2.10) or (2.14) give the field quantities transformed by the reversion operator, from which the factors due to the congruent transformation (time-reversal) are dropped. In the following sections, we shall investigate the effect of the reversion operator on the field quantities, determining this operator by (4.15), in which P and Q should be taken in accordance with (3.6) or (3.8). The lagrangian (2.1) will be assumed.

V. CHARGED SCALAR FIELD

In the three sections that follow, standpoint (I) will be adopted, while Sec. VIII will be reserved for the discussion in standpoint (II).

In the case of a scalar or a pseudoscalar field, we shall have to consider the current i^μ and the noninteracting hamiltonian density H_0 as the physical quantities under the time-reversal:

$$\left.\begin{aligned} &i^\mu=i[(\partial\bar{u}/\partial x_\mu)u-\bar{u}(\partial u/\partial x_\mu)],\\ &H_0=(\partial\bar{u}/\partial t)(\partial u/\partial t)+(\partial\bar{u}/\partial x_a)(\partial u/\partial x^a)+\kappa^2\bar{u}u.\\ &\qquad\qquad\qquad\qquad (a=1,2,3)\end{aligned}\right\}. \quad (5.1)$$

The requirement of reversibility is now expressed as the requirement of existence of R such that

$$\begin{aligned} i^{Ta}(-t)=R^{-1}i^a(t)R;\quad i^{T0}(-t)=-R^{-1}i^0(t)R;\\ H^T{}_0(-t)=R^{-1}H_0(t)R.\end{aligned} \quad (5.2)$$

We need not consider the interaction hamiltonian because the correct transformation of the current guarantees the correct transformation of the interaction hamiltonian provided the electromagnetic field satisfies the correct transformation. Regarding the particle numbers, see (5.9).

We shall first show that the assumption of Bose statistics permits the existence of R, and then show that the Fermi statistics is incompatible with the existence of R.

It is easy to prove that conditions (5.2) are satisfied by

$$\begin{aligned} R^{-1}u(x,t)R=\pm u^T(x,-t);\\ R^{-1}\bar{u}(x,t)R=\pm\bar{u}^T(x,-t).\end{aligned} \quad (5.3)$$

For this proof we need once change the order of u and $\bar{u}$ using the Bose assumption. Hence the above statement is true "except for an additional c-number." On this point, see Sec. VIII. Since choice of the sign in (5.3), which refers to the "kind" of the u-field, does not essentially affect the argument, the positive sign will be adopted.

It should be noted that the two relations of (5.3) are just hermitian-conjugate of each other, provided the unitarity of R (4.13) and the hermitian-conjugate relation between u and $\bar{u}$. The transpose of a relation in (5.3) becomes identical with the original relation with t and $-t$ interchanged if $R^T=\pm R$. Hence, we can expect here a symmetrical or antisymmetrical R.

It can easily be ascertained that the transformation (5.3) leaves the commutation relation:

$$u(x,t)\bar{u}(x',t')-\bar{u}(x',t')u(x,t)=-iD_\kappa(x-x',t-t') \quad (5.4)$$

unchanged, because the D_κ-function is an odd function of $t-t'$.

We now want to pass to the particle picture by the expansion:

$$\left.\begin{aligned} u(x,t)=\sum[1/\surd(2V|k^0|)]\{u_+(\mathbf{k})\\ \times\exp(i\mathbf{kx}-i|k^0|t)+\bar{u}_-(\mathbf{k})\\ \times\exp(-i\mathbf{kx}+i|k^0|t)\}\\ \bar{u}(x,t)=\sum[1/\surd(2V|k^0|)]\{\bar{u}_+(\mathbf{k})\\ \times\exp(-i\mathbf{kx}+i|k^0|t)+u_-(\mathbf{k})\\ \times\exp(i\mathbf{kx}-i|k^0|t)\}\end{aligned}\right\}, \quad (5.5)$$

with

$$\mathbf{k}^2+\kappa^2=(k^0)^2. \quad (5.6)$$

Then

$$N_+(\mathbf{k})=\bar{u}_+(\mathbf{k})u_+(\mathbf{k});\quad N_-(\mathbf{k})=\bar{u}_-(\mathbf{k})u_-(\mathbf{k}), \quad (5.7)$$

bear the usual meanings. The transformation (5.3) applied to (5.5) now take the form:

$$R^{-1}u_\pm(\mathbf{k})R=\bar{u}^T{}_\mp(-\mathbf{k});\quad R^{-1}\bar{u}_\pm(\mathbf{k})R=u^T{}_\mp(-\mathbf{k}) \quad (5.8)$$

from which follows:

$$\begin{aligned} (R^{-1}N_+(\mathbf{k})R)^T=N_-(-\mathbf{k});\\ (R^{-1}N_-(\mathbf{k})R)^T=N_+(-\mathbf{k}).\end{aligned} \quad (5.9)$$

This is exactly what was expected in the discussion given below (4.7). (5.9) shows that R commutes with $\mathbf{H}_0$ in the representation in which the particle numbers are diagonal. See (4.24). In the N-representation, (5.8) can be written as

$$R^{-1}u_\pm(\mathbf{k})R=u_\mp(-\mathbf{k}). \quad (5.10)$$

The reversion operator can now be written explicitly as a matrix in this representation:

$$\left.\begin{aligned}&(\cdots N'_+(\mathbf{k}_i),\, N'_+(-\mathbf{k}_i)\cdots N'_-(\mathbf{k}_i),\\&N'_-(-\mathbf{k}_i)\cdots|R|\cdots N''_-(-\mathbf{k}_i),\\&N''_-(\mathbf{k}_i)\cdots N''_+(-\mathbf{k}_i)N''_+(\mathbf{k}_i)\cdots)\\&=\Pi\delta(N'_+(\mathbf{k}_i),\, N''_-(-\mathbf{k}_i))\\&\times\delta(N'_+(-\mathbf{k}_i),\, N''_-(\mathbf{k}_i))\\&\times\delta(N'_-(\mathbf{k}_i),\, N''_+(-\mathbf{k}_i))\\&\times\delta(N'_-(-\mathbf{k}_i),\, N''_+(\mathbf{k}_i)).\end{aligned}\right\}\quad(5.11)$$

Let us now proceed to examine the assumption of Fermi statistics for the u-field. In this case, we realize that there are two and only two possible ways to fulfill (5.2), *viz.*

$$\begin{aligned}&R^{-1}u(x,t)R=\mp(1/c)u^T(x,-t);\\&R^{-1}\bar{u}(x,t)R=\pm c\bar{u}^T(x,-t),\end{aligned}\quad(5.12)$$

$$\begin{aligned}&R^{-1}u(x,t)R=\mp(1/c)\bar{u}^T(x,-t);\\&R^{-1}\bar{u}(x,t)R=\pm cu^T(x,-t).\end{aligned}\quad(5.13)$$

(N.B. Double sign is inverted in two relations of each group.) c is arbitrary. (5.12) requires, in the process of proof, the change of order of u and $\bar{u}$ according to the Fermi-statistics, while (5.13) does not.

It can now easily be seen that either choice, (5.12) or (5.13), destroys the hermitian relationship between u and $\bar{u}$, if R is to satisfy (4.13). Therefore, there cannot exist a reversion operator in the case of Fermi statistics. This is what was to be demonstrated.

The incompatibility of the hermitian relationship between u and $\bar{u}$ and the two relations in (5.12) or (5.13) leads us to attempt a formalism in which $\bar{u}$ is not the hermitian conjugate of u. Such a formalism has recently been discussed also by Pauli.[7] To avoid confusion, $\bar{u}$ in the foregoing will be written $\bar{v}$ in the following.

The first thing to notice is that either transformation (5.12) or (5.13) leaves the following Fermi type commutation relation unchanged:

$$u(x,t)\bar{v}(x',t')+\bar{v}(x',t')u(x,t)=-iD_\kappa(x-x',t-t').\quad(5.14)$$

Instead of (5.5), u and $\bar{v}$ will here be developed as

$$\left.\begin{aligned}u(x,t)&=\sum[1/\surd(2V|k^0|)]\\&\quad\times\{u_+(\mathbf{k})\exp(i\mathbf{kx}-i|k^0|t)+\bar{u}_-(\mathbf{k})\\&\qquad\times\exp(-i\mathbf{kx}+i|k^0|t)\}\\\bar{v}(x,t)&=\sum[1/\surd(2V|k^0|)]\\&\quad\times\{\bar{u}_+(\mathbf{k})\exp(-i\mathbf{kx}+i|k^0|t)-u_-(\mathbf{k})\\&\qquad\times\exp(i\mathbf{kx}-i|k^0|t)\}\end{aligned}\right\},\quad(5.15)$$

[7] W. Pauli, Prog. Theor. Phys. 5, 526 (1950).

while (5.7) retains its meaning. Of the two possibilities (5.12) and (5.13), the first one will give (adopting the upper sign and $c=1$)

$$\left.\begin{aligned}&R^{-1}u_+(\mathbf{k})R=-\bar{u}^T_-(-\mathbf{k});\\&R^{-1}u_-(\mathbf{k})R=-\bar{u}^T_+(-\mathbf{k})\\&R^{-1}\bar{u}_+(\mathbf{k})R=-u^T_-(-\mathbf{k});\\&R^{-1}\bar{u}_-(\mathbf{k})R=-u^T_+(-\mathbf{k})\end{aligned}\right\},\quad(5.16)$$

which does not contradict with the hermitian relationship between $u(k)$ and $\bar{u}(k)$, assuming (4.13). (5.16) evidently gives the desired transformation for the particle numbers:

$$(R^{-1}N_\pm(\mathbf{k})R)^T=N_\mp(-\mathbf{k}).\quad(5.17)$$

VI. CHARGED SPINOR FIELD

According to the standpoint (I), the quantities which should change their signs in the reversed state are the charge s^0, the electric moment ϵ^a and the spin σ^a, and those which should keep their signs are the current s^a, the magnetic moment μ^a and the noninteracting energy H_0:

$$s^{T0}(-t)=-R^{-1}s^0(t)R;\quad \epsilon^{Ta}(-t)=-R^{-1}\epsilon^a(t)R;\quad \sigma^{Ta}=-R^{-1}\sigma^a(t)R,\quad(6.1)$$

$$s^{Ta}(-t)=R^{-1}s^a(t)R;\quad \mu^{Ta}(-t)=R^{-1}\mu^a(t)R;\quad H^T{}_0(-t)=R^{-1}H_0(t)R,\quad(6.2)$$

where

$$\left.\begin{aligned}&s^\mu\approx\bar{\psi}E^\mu\psi;\quad \epsilon^a\approx\bar{\psi}E^bE^c\psi;\quad \mu^a\approx\bar{\psi}E^0E^a\psi;\quad \sigma^a\approx\bar{\psi}E^5E^a\psi\\&H_0\approx\bar{\psi}[E^a(\partial/\partial x^a)-E^a(\overleftarrow{\partial/\partial x^a})+2imE^5]\psi\end{aligned}\right\}.\quad(6.3)$$

In (6.3) all the numerical factors are dropped for the sake of simplicity. In the expression of the electric moment ϵ^a, (a, b, c) must be an even permutation of a given order, say (1, 2, 3). Requirements regarding the particle numbers will be considered later.

We shall first show that the reversion operator exists if the Fermi statistics is assumed for the spinor field.

Under the Fermi assumption, it is easy to prove that all the foregoing requirements can be satisfied, except for an additional c-number (see Sec. VIII), by

$$\begin{aligned}&R^{-1}\psi(t)R=cE^0\psi^T(-t);\\&R^{-1}\bar{\psi}(t)R=(1/c)\bar{\psi}^T(-t)E_0,\end{aligned}\quad(6.4)$$

where c is arbitrary. In the process of proof, we must once invert the order of ψ and $\bar{\psi}$ according to the Fermi statistics. In order that all the physical quantities be hermitian, it is necessary that ψ and $\psi^*=\bar{\psi}J$ [see (A.23)] be hermitian conjugate to each other. For this reason, the two relations in (6.4) can be compatible with each other only if $c^*c=1$ or $c=e^{i\theta}$, in virtue of

(4.13). To fix our further calculation, we shall adopt $\theta=\pi/2$:

$$R^{-1}\psi(t)R=iE^0\psi^T(-t);\quad R^{-1}\bar{\psi}(t)R=-i\bar{\psi}^T(-t)E_0. \tag{6.5}$$

By adopting this value for c, we can expect to obtain a symmetrical or antisymmetrical R. Indeed, the transposes of relations (6.5) are identical with the original relations (with t and $-t$ interchanged) if $R^T=\pm R$.

If we adopt $\theta=0$ or $\theta=\pi$, (6.4) coincides with (A.48).

It can also be shown that transformation (6.4) transforms the usual commutation relation:

$$\psi_r(x,t)\bar{\psi}_s(x',t')+\bar{\psi}_s(x',t')\psi_r(x,t)=[E^\mu(\partial/\partial x^\mu)+imE^5]_{rs}D_m(x-x',t-t') \tag{6.6}$$

back into itself.

We shall now examine the effect of R on the particle numbers. In the interaction picture, ψ and $\bar{\psi}$ obey

$$[E^\mu(\partial/\partial x^\mu)+imE^5]\psi=0;\quad \bar{\psi}[E^\mu(\overleftarrow{\partial/\partial x^\mu})-imE^5]=0. \tag{6.7}$$

If we, according to (A.32), decompose ψ and $\bar{\psi}$ into charge conjugate waves by

$$\psi=\psi_++\bar{\psi}_-K;\quad \bar{\psi}=\bar{\psi}_++\psi_-K^{-1}, \tag{6.8}$$

then each of ψ_+ and ψ_- will again obey the first equation of (6.7), and each of $\bar{\psi}_+$ and $\bar{\psi}_-$ will obey the second equation of (6.7).

We now expand ψ and $\bar{\psi}$ into their fourier components by

$$\left.\begin{aligned}\psi(t)&=(1/\sqrt{V})\textstyle\sum_{\mathbf{k}}\sum_\rho[g_+{}^\rho(\mathbf{k})\alpha^\rho(\mathbf{k})\\&\quad\times\exp(i\mathbf{kx}-i|k^0|t)+\bar{g}_-{}^\rho(\mathbf{k})\bar{\alpha}^\rho(\mathbf{k})K\\&\quad\times\exp(-i\mathbf{kx}+i|k^0|t)]\\ \bar{\psi}(t)&=(1/\sqrt{V})\textstyle\sum_{\mathbf{k}}\sum_\rho[\bar{g}_+{}^\rho(\mathbf{k})\bar{\alpha}^\rho(\mathbf{k})\\&\quad\times\exp(-i\mathbf{kx}+i|k^0|t)+g_-{}^\rho(\mathbf{k})\alpha^\rho(\mathbf{k})K^{-1}\\&\quad\times\exp(i\mathbf{kx}-i|k^0|t)]\end{aligned}\right\}, \tag{6.9}$$

where the index $\rho=1,2$ corresponds to the spin freedom. g and $\bar{g}$ are supposed to satisfy

$$\bar{g}_+{}^\rho(\mathbf{k})g_+{}^\sigma(\mathbf{k})+g_+{}^\sigma(\mathbf{k})\bar{g}_+{}^\rho(\mathbf{k})=\delta_{\rho\sigma}\text{ etc.}, \tag{6.10}$$

and the c-number spinors α and $\bar{\alpha}$ are solutions of

$$(k_aE^a-|k^0|E^0+mE^5)\alpha^\rho(\mathbf{k})=0,\quad \bar{\alpha}^\rho(\mathbf{k})(-k_aE^a+|k^0|E^0-mE^5)=0. \tag{6.11}$$

They should be orthogonal and normalized so that

$$\bar{\alpha}^\rho(\mathbf{k})E^0\alpha^\sigma(\mathbf{k})=-\delta_{\rho\sigma};\quad \bar{\alpha}^\rho(\mathbf{k})\alpha^\sigma(\mathbf{k})=0. \tag{6.12}$$

This normalization is certainly possible in the hermitian system (A.19). Then $N_\pm{}^\rho(\mathbf{k})=\bar{g}_\pm{}^\rho(\mathbf{k})g_\pm{}^\rho(\mathbf{k})$ represents the electron numbers of the specified kind.

From (6.11), we see that $\alpha^\rho(\mathbf{k})$ and $E^0K\bar{\alpha}^\rho(-\mathbf{k})$ obey the same equation. And moreover they belong to the opposite spin if ρ has the same value. We therefore establish the correspondence between α^1 and α^2 by

$$\alpha^1(\mathbf{k})=E^0K\bar{\alpha}^2(-\mathbf{k});\quad \alpha^2(\mathbf{k})=-E^0K\bar{\alpha}^1(-\mathbf{k}). \tag{6.13}$$

Theorem (A.17) proves that the two relations (6.13) are identical. The orthogonality of α^1 and α^2 is guaranteed by the fact that α^1 and α^2 can be considered as eigenfunctions belonging to different eigenvalues of a spin operator, say, σ_3:

$$E_5E_3\alpha^\rho(\mathbf{k})=\pm E^0\alpha^\rho(\mathbf{k}). \tag{6.14}$$

Now we construct, remembering (6.13), the right sides of (6.5):

$$\left.\begin{aligned}&iE^0\psi^T(-t)\\&\quad=(1/\sqrt{V})\textstyle\sum_{\mathbf{k}}\sum_\rho[\{g^T{}_+{}^1(-\mathbf{k})(i\bar{\alpha}^2(\mathbf{k})K)\\&\qquad+g^T{}_+{}^2(-\mathbf{k})(-i\bar{\alpha}^1(\mathbf{k})K)\}\\&\qquad\times\exp(-i\mathbf{kx}+i|k^0|t)\\&\qquad+\{\bar{g}^T{}_-{}^1(-\mathbf{k})(i\alpha^2(\mathbf{k}))\\&\qquad+\bar{g}^T{}_-{}^2(-\mathbf{k})(-i\alpha^1(\mathbf{k}))\}\\&\qquad\times\exp(i\mathbf{kx}+i|k^0|t)]\\&-i\bar{\psi}^T(-t)E_0\\&\quad=(1/\sqrt{V})\textstyle\sum_{\mathbf{k}}\sum_\rho[\{\bar{g}^T{}_+{}^1(-\mathbf{k})(-i\alpha^2(\mathbf{k})K^{-1})\\&\qquad+\bar{g}^T{}_+{}^2(-\mathbf{k})(i\alpha(\mathbf{k})K^{-1})\}\exp(i\mathbf{kx}-i|k^0|t)\\&\qquad+\{g^T{}_-{}^1(-\mathbf{k})(-i\bar{\alpha}^2(\mathbf{k}))\\&\qquad+g^T{}_-{}^2(-\mathbf{k})(i\alpha^1(\mathbf{k})\}\exp(-i\mathbf{kx}+i|k^0|t)]\end{aligned}\right\}, \tag{6.15}$$

which yields, by comparison with (6.9),

$$\left.\begin{aligned}R^{-1}g_\pm{}^1(\mathbf{k})R&=\mp i\bar{g}^T{}_\mp{}^2(-\mathbf{k});\\R^{-1}g_\pm{}^2(\mathbf{k})R&=\pm i\bar{g}^T{}_\mp{}^1(-\mathbf{k})\\R^{-1}\bar{g}_\pm{}^1(\mathbf{k})R&=\pm ig^T{}_\mp{}^2(-\mathbf{k});\\R^{-1}\bar{g}_\pm{}^2(\mathbf{k})R&=\mp ig^T{}_\mp{}^1(-\mathbf{k})\end{aligned}\right\}. \tag{6.16}$$

This entails the desired transformation rule for the particle numbers:

$$(R^{-1}N_\pm{}^1(\mathbf{k})R)^T=N_\mp{}^2(-\mathbf{k});\quad (R^{-1}N_\pm{}^2(\mathbf{k})R)^T=N_\mp{}^1(-\mathbf{k}). \tag{6.17}$$

The matrix elements of R can now be obtained. For simplicity, we consider here only those matrix elements which affect $N_+{}^1(\mathbf{k})$ and $N_-{}^2(-\mathbf{k})$. Assigning to the matrices $g_+{}^1$ and $g_-{}^2$ the values:

$$g_+{}^1=\begin{pmatrix}0&1\\0&0\end{pmatrix};\quad \bar{g}_+{}^1=\begin{pmatrix}0&0\\1&0\end{pmatrix};\quad g_-{}^2=\begin{pmatrix}0&i\\0&0\end{pmatrix};\quad \bar{g}_-{}^2=\begin{pmatrix}0&0\\-i&0\end{pmatrix}, \tag{6.18}$$

we obtain

$$(N'_+{}^1(\mathbf{k}), N'_-{}^2(-\mathbf{k})|R|N''_+{}^1(\mathbf{k}), N''_-{}^2(-\mathbf{k}))$$
$$=(1-2N'_+{}^1(\mathbf{k}))(1-2N'_-{}^2(-\mathbf{k}))$$
$$\times\delta(N'_+{}^1(\mathbf{k}), N''_-{}^2(-\mathbf{k}))$$
$$\times\delta(N'_-{}^2(-\mathbf{k}), N''_+{}^2(\mathbf{k})), \quad (6.19)$$

which obviously satisfies (6.16). This R is symmetric.[8]

Now we want to demonstrate that the Bose statistics is incompatible with the reversibility requirement in standpoint (I). Let us first consider only two of the requirements:

$$R^{-1}s^\mu(t)R=-s^T{}_\mu(-t);\ R^{-1}\mu^{\mu\nu}(t)R=-\mu^T{}_{\mu\nu}(-t), \quad (6.20)$$

with

$$\mu^{\mu\nu}\approx\bar{\psi}E^\mu E^\nu\psi. \quad (6.21)$$

By a simple inspection, we learn that, under the Bose assumption, there are two and only two possibilities to satisfy (6.20), *viz.*,

$$R^{-1}\psi(t)R=V^{-1}E^0\psi^T(-t);\quad R^{-1}\bar{\psi}(t)R=-\bar{\psi}^T(-t)E_0U, \quad (6.22)$$

and

$$R^{-1}\psi(t)R=V^{-1}E_0K\bar{\psi}^T(-t);\quad R^{-1}\bar{\psi}(t)R=\psi^T(-t)K^{-1}E^0U, \quad (6.23)$$

where, in both cases, the yet undetermined nonsingular matrices U and V must satisfy

$$UE^\mu V^{-1}=E^\mu;\quad UE^\mu E^\nu V^{-1}=E^\mu E^\nu, \quad (6.24)$$

to meet the requirements (6.20). Combination of the two relations in (6.24) gives

$$UV^{-1}=U^{-1}V=1 \quad\text{or}\quad U=V. \quad (6.25)$$

Then the first relation of (6.24) becomes an expression to the effect that U commutes with all the four basic E^μ. Therefore, due to Theorem (4), Appendix,

$$U=V=c. \quad (6.26)$$

(6.23) satisfies the requirements without using the Bose assumption, while (6.22) satisfies them only by once inverting the order of ψ and $\bar{\psi}$ according to the Bose statistics, therefore "except for an additional c-number."

The first possibility (6.22) becomes, on account of (6.26),

$$R^{-1}\psi(t)R=(1/c)E^0\psi^T(-t);\quad R^{-1}\bar{\psi}(t)R=-c\bar{\psi}^T(-t)E_0. \quad (6.27)$$

Incompatibility of the two relations of (6.27) is obvious. Indeed, the hermitian conjugate of one relation of (6.27) contradicts the other, assuming of course the hermitian conjugate relation between ψ and ψ^* and the unitarity of R (4.13).

In consideration of (6.26), the second possibility (6.23) takes the form:

$$R^{-1}\psi(t)R=(1/c)E_0K\bar{\psi}^T(-t);\quad R^{-1}\bar{\psi}(t)R=\psi^T(-t)K^{-1}E^0. \quad (6.28)$$

This, however, gives the wrong sign for the requirement:

$$R^{-1}H_0(t)R=H^T{}_0(-t). \quad (6.29)$$

Thus, we conclude that, as far as we take the standpoint (I), the Fermi statistics is allowed for the charged spinor field but not the Bose statistics, in order to fulfill the reversibility requirement.

It suggests itself that we could save (6.27), and through it the Bose statistics, by abandoning the hermitian relationship between ψ and ψ^*.

VII. ELECTROMAGNETIC FIELD

According to standpoint (I), the requirements regarding the electromagnetic field can be written as

$$a^{Tb}(-t)=R^{-1}a^b(t)R \quad (b=1,2,3);\quad a^0(-t)=-R^{-1}a^0(t)R, \quad (7.1)$$

and

$$H^T{}_0(-t)=R^{-1}H_0(t)R, \quad (7.2)$$

with

$$H_0=(\partial a_\mu/\partial t)(\partial a^\mu/\partial t)+(\partial a_\mu/\partial x_a)(\partial a^\mu/\partial x^a). \quad (7.3)$$

It is evident that the transformation (7.1) automatically, i.e., without any specific assumption as to the statistics, satisfies (7.2). It is also obvious that (7.1) is self-consistent, in consideration of (4.13). The matrix R in this case will be symmetric or antisymmetric, for the transposes of the relations in (7.1) are identical with themselves, if $R^T=\pm R$.

The transformation (7.1) leaves invariant the customary commutation relation:

$$a^\mu(x,t)a^\nu(x',t')-a^\nu(x',t')a^\mu(x,t)=-ig^{\mu\nu}D_0(x-x',t-t'). \quad (7.4)$$

However, it leaves the Fermi type commutation relations also invariant:

$$a^\mu(x,t)a^\nu(x',t')+a^\nu(x',t')a^\mu(x,t)=g^{\mu\nu}F_0(x-x',t-t'), \quad (7.5)$$

where F_0, in order to be consistent with the left side, must remain unchanged for the interchange of all the four x^μ with $x^{\mu\prime}$ and also for the interchange of x^a with $x^{a\prime}$ $(a=1,2,3)$ and for the interchange of t with t' separately. $D_0^{(1)}$ is such a function.

As is well known, no self-consistent particle picture can be made on the Fermi assumption for the electromagnetic field,[9] but this is due to the difficulties which

[8] After having thus determined R, we should restore Jordan-Wigner's signum operators $\Pi(1-2N)$, which have provisorily been omitted in the g's and f's. The effect of R on them is just altering the ordering of the oscillators for the description of the reversed state.

[9] W. Pauli, Revs. Modern Phys. 13, 203 (1941).

are irrelevant to the reversibility requirements. The requirement that the photons in the reversed states should be traveling in the opposite momenta may be replaced by the requirement that the momentum density (Poynting vector) should be reversed in the reversed motion. It goes without saying that the transformation (7.1) fulfills this requirement irrespective of the statistics adopted.

Thus the reversibility requirement does not determine the commutation relations of the electromagnetic field.

Finally, the effect of R on the quantities in the particle picture will be given, assuming the Bose statistics. With the help of the expansion

$$a^{\mu}(x)=\sum[1/(2V|k^0|)^{\frac{1}{2}}][a^{\mu}(\mathbf{k})\exp(i\mathbf{kx}-i|k^0|t) + \bar{a}^{\mu}(\mathbf{k})\exp(-i\mathbf{kx}+i|k^0|t)]; \quad (k^0)^2=\mathbf{k}^2, \quad (7.6)$$

with

$$a^{\mu}(\mathbf{k})\bar{a}^{\nu}(\mathbf{k}')-\bar{a}^{\nu}(\mathbf{k}')a^{\mu}(\mathbf{k})=g^{\mu\nu}\delta(\mathbf{k},\mathbf{k}'), \quad (7.7)$$

the transformation (7.1) will be transcribed as

$$R^{-1}a^{\mu}(\mathbf{k})R=\bar{a}^{T}{}_{\mu}(-\mathbf{k}); \quad R^{-1}\bar{a}^{\mu}(\mathbf{k})R=a^{T}{}_{\mu}(-\mathbf{k}). \quad (7.8)$$

which naturally satisfies

$$(R^{-1}N^{\mu}(\mathbf{k})R)^{T}=N^{\mu}(-\mathbf{k}). \quad (7.9)$$

The matrix elements of R can readily be obtained from (7.8).

In the foregoing sections, we have not considered the interaction hamiltonian density H. This is because the requirement regarding H is automatically satisfied if the current and the electromagnetic potentials satisfy the reversibility requirements separately.

It was noticed at the end of Sec. III that emission of a photon in a state should correspond to absorption of a corresponding photon in the reversed state. We can reproduce this fact, in a probabilistic language, using R as defined above (7.8). Similar considerations apply also with regard to the emission and absorption of charged particles using (5.8) and (6.16).

VIII. CHARGE CONJUGATION AND STANDPOINT (II)

Two states $\Psi_{(1)}$ and $\Psi_{(2)}$ will be called "charge-conjugate" states of each other if the expectation values of the "electromagnetic" quantities have the same absolute values but opposite signs and those of the "mechanical" quantities have the same values; this situation will symbolically be written as

$$\Psi_{(1)}\overset{\text{cha}}{\sim}\Psi_{(2)}. \quad (8.1)$$

By electromagnetic quantities are meant: electromagnetic field, current, electromagnetic moment; and by mechanical quantities are meant: momentum, energy, spin.

Then the "charge-invariance" requires

$$\Psi_{(1)}(t)\overset{\text{cha}}{\sim}\Psi_{(2)}(t), \quad (8.2)$$

where $\Psi_{(1)}(t)$ and $\Psi_{(2)}(t)$ should obey the same Schroedinger equation. The requirement of charge-invariance can be replaced by the requirement of existence of a time-independent unitary operator C such that

$$\Psi_{(1)}(t)=C\Psi_{(2)}(t). \quad (8.3)$$

Then the argument runs parallel to our argument regarding the reversion operator, culminating in certain conditions about the commutation relations of charged field quantities. This conclusion is exactly the same as the one which we have drawn from our reversibility requirement in standpoint (I). Determination of commutation relations by the charge-invariance was previously discussed by Pauli and Belinfante.[10]

This situation raises a suspicion that the determination of statistics by the reversibility requirement was rather illusory and that the reversibility requirement may fail to determine the statistics if we take standpoint (II) of the reversed motion. That, in fact, this is the case will be shown in this section.

Before passing to the discussion of standpoint (II), a few words may be spent regarding the reservation we have always made in the preceding sections to the effect: "proven except for an additional c-number." This clause can be dropped if, after having determined R, modify the definitions of physical quantities $P(t)$ by the prescription:

$$P(t)\rightarrow\tfrac{1}{2}(P(t)+RQ^{T}(-t)R^{-1}), \quad (8.4)$$

where Q is associated to P in the sense of (4.7). This modification is nothing but a generalization of the well-known Heisenberg prescription which can be derived from a consideration of charge symmetry.[9]

Let us first consider the scalar field. In standpoint (II), the first two relations of (5.2) change their signs while the third relation of (5.2) remains unchanged. Before discussing our main problem, we want to show that the Bose statistics is compatible also with standpoint (II).

We see that all the requirements in standpoint (II) can be satisfied by

$$\begin{aligned} R^{-1}u(x,t)R&=\pm\bar{u}^{T}(x,-t);\\ R^{-1}\bar{u}(x,t)R&=\pm u^{T}(x,-t). \end{aligned} \quad (8.5)$$

Or in the particle picture (for the upper sign of (8.5))

$$\begin{aligned} R^{-1}u_{\pm}(\mathbf{k})R=\bar{u}^{T}{}_{\pm}(-\mathbf{k}); \quad R^{-1}\bar{u}_{\pm}(\mathbf{k})R=u^{T}{}_{\pm}(-\mathbf{k}),\\ (R^{-1}N_{\pm}(\mathbf{k})R)^{T}=N_{\pm}(-\mathbf{k}). \end{aligned} \quad (8.6)$$

This transformation stands all the tests which are imposed upon the reversion operator. In particular, it is free from the self-contradiction involved in the transformation (5.13).

Now it is essential to notice that the transformation (8.5) satisfies the reversibility requirements without any change of order of u and $\bar{u}$ based on a specific statistics. Therefore, (8.5) is valid also in the case of

[10] W. Pauli and F. J. Belinfante, Physica 7, 177 (1940).

Fermi statistics for the u-field. We can also confirm that (8.5) leaves the Fermi type commutation relation unchanged:

$$u(x,t)\bar{u}(x',t')+\bar{u}(x',t')u(x,t)=F_\kappa(x-x',t-t'), \quad (8.7)$$

where F_κ has a similar property to the F_0 of (7.5).

Because of the well-known difficulties,[9] which are irrelevant to the reversibility consideration, we cannot construct a consistent particle picture in the Fermi case. However, (8.5) gives the correct transformation to the momentum density, irrespective of the statistics.

We thus conclude that the reversibility requirement is ineffective in determining the statistics of the charged scalar (pseudoscalar) field in standpoint (II).

Let us now examine the spinor field in standpoint (II), assuming first the Fermi statistics. The requirements in standpoint (II) are

$$s^{T\alpha}(-t)=-R^{-1}s^\alpha(t)R;\quad \mu^{T\alpha}(-t)=-R^{-1}\mu^\alpha(t)R;\quad \sigma^{T\alpha}(-t)=-R^{-1}\sigma^\alpha(t)R, \quad (8.8)$$

$$s^{T0}(-t)=R^{-1}s^0(t)R;\quad \epsilon^{T\alpha}(-t)=R^{-1}\epsilon^\alpha(t)R;\quad H^T{}_0(-t)=R^{-1}H_0(t)R, \quad (8.9)$$

which take the place of (6.1) and (6.2). All these requirements can be satisfied by

$$R^{-1}\psi(t)R=E_0K\bar{\psi}^T(-t);\quad R^{-1}\bar{\psi}(t)R=-K^{-1}E^0\psi^T(-t). \quad (8.10)$$

That these two relations do not contradict each other can be shown with the help of (4.13) and (A.17). We can also verify that the transformation (8.10) leaves the commutation relations (6.6) unchanged.

Passing to the particle picture by the instrumentality of the expansion (6.9), we obtain here, instead of (6.16),

$$\left.\begin{aligned} R^{-1}g_\pm{}^1(\mathbf{k})R&=-\bar{g}^T{}_\pm{}^2(-\mathbf{k});\\ R^{-1}g_\pm{}^2(\mathbf{k})R&=\bar{g}^T{}_\pm{}^1(-\mathbf{k})\\ R^{-1}\bar{g}_\pm{}^1(\mathbf{k})R&=-g^T{}_\pm{}^2(-\mathbf{k});\\ R^{-1}\bar{g}_\pm{}^2(\mathbf{k})R&=g^T{}_\pm{}^1(-\mathbf{k})\end{aligned}\right\}, \quad (8.11)$$

which guarantee

$$(R^{-1}N_\pm{}^1(\mathbf{k})R)^T=N_\pm{}^2(-\mathbf{k});\quad (R^{-1}N_\pm{}^2(\mathbf{k})R)^T=N_\pm{}^1(-\mathbf{k}), \quad (8.12)$$

as is required in standpoint (II).

Assuming

$$g=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \bar{g}=\begin{pmatrix}0&0\\1&0\end{pmatrix}, \quad (8.13)$$

we can write the matrix elements of R as

$$\begin{aligned}(N'_+{}^1(\mathbf{k}),N'_+{}^2(-\mathbf{k})|R|N''_+{}^1(\mathbf{k}),N''_+{}^2(-\mathbf{k}))\\ =i(1-2N'_+{}^1(\mathbf{k}))\delta(N'_+{}^1(\mathbf{k}),N''_+{}^2(-\mathbf{k}))\\ \times\delta(N'_+{}^2(-\mathbf{k}),N''_+{}^1(\mathbf{k})),\end{aligned} \quad (8.14)$$

for the part of R which affects $N_+{}^1(\mathbf{k})$ and $N_+{}^2(-\mathbf{k})$.

The matrix as given in (8.14) is unitary, but neither symmetric nor antisymmetric. Nevertheless, this R warrants the complete symmetry with respect to t and $-t$. The matrix R^TR^{-1} which was in question in relation to (4.25) becomes here

$$\begin{aligned}(N'_+{}^1(\mathbf{k}),N'_+{}^2(-\mathbf{k})|R^TR^{-1}|N''_+{}^1(\mathbf{k}),N''_+{}^2(-\mathbf{k}))\\ =(1-2N'_+{}^2(-\mathbf{k}))(1-2N'_+{}^1(\mathbf{k}))\\ \times\delta(N'_+{}^1(\mathbf{k}),N''_+{}^1(\mathbf{k}))\\ \times\delta(N'_+{}^2(-\mathbf{k}),N''_+{}^2(-\mathbf{k})).\end{aligned} \quad (8.15)$$

This R^TR^{-1}, without being a c-number, commutes with all the physical quantities bilinear in ψ and $\bar{\psi}$.

Now it is of importance to note that the transformation (8.10) satisfies (8.8) and (8.9) without changing the order of ψ and $\bar{\psi}$ using a statistical assumption. Therefore it applies also to the Bose statistics. It can readily be seen that (8.10) leaves the Bose type commutation relation unchanged:

$$\begin{aligned}\psi_r(x,t)\bar{\psi}_s(x',t')-\bar{\psi}_s(x',t')\psi_r(x,t)\\ =i[E^\mu(\partial/\partial x^\mu)+imE^5]_{rs}F_m(x-x',t-t').\end{aligned} \quad (8.16)$$

Of course, the difficulties[9] that are irrelevant to the reversibility consideration prevent us to formulate a consistent particle theory on the Bose assumption. However, in place of the requirement about particle momentum, the transformation (8.10) satisfies the requirement about the momentum density.

Thus we come to the conclusion that the reversibility requirement as such does not determine the statistics of the charged spinor field.

It is evident that, although the modification of definition (8.4) is not necessary in standpoint (II), it can equally well be used.

IX. RETARDED AND ADVANCED POTENTIALS

The notion of retarded potential is apparently an "irreversible" one. We shall show how this irreversible action can be derived from the completely reversible theoretical scheme. This was one of the main problems which motivated an earlier paper of the author.[2]

As far as we do not perform an observation, i.e., the development of the state function by the Schroedinger equation is concerned, the distinction between the retarded and advanced actions is merely verbal. For exchange of a photon between two electrons can be interpreted either as a retarded or as an advanced action, according as on which electron our attention is placed. This fact, self-evident in quantum theory, can be translated in classical language only as a specific hypothesis such as Tetrode's.[5] Moreover, in the reversed motion, emission and absorption, therefore retarded and advanced actions, exchange their roles. This complete symmetry takes an explicit form if we calculate the interaction potential between two electrons by the second-order perturbation theory.

From the interaction picture in which we have the

Schroedinger equation

$$d\Psi(t)/dt=-i\mathbf{H}(t)\Psi(t);\quad \mathbf{H}(t)=\int(dx)^3H(x,t);\qquad H=ea^\mu\bar{\psi}E_\mu\psi. \tag{9.1}$$

we pass, following Schwinger,[11] to the second-order perturbation picture by the unitary transformation:

$$V(t)=\exp(-iW(t));\qquad V^{-1}(t)=\bar{V}(t)=\exp(+iW(t)), \tag{9.2}$$

with

$$W(t)=\frac{1}{2}\int_{-\infty}^{\infty}\epsilon(t-t')\mathbf{H}(t')dt', \tag{9.3}$$

where

$$\epsilon(t)=1 \quad\text{for}\quad t>0;\quad \epsilon(t)=-1\quad\text{for}\quad t<0. \tag{9.4}$$

The transformed wave function:

$$\Psi'(t)=V^{-1}(t)\Psi(t) \tag{9.5}$$

obeys, in the e^2-approximation,

$$d\Psi'(t)/dt=-i\mathbf{H}'(t)\Psi'(t), \tag{9.6}$$

where

$$\begin{aligned}\mathbf{H}'(t)&=\tfrac{1}{2}i(W(t)\mathbf{H}(t)-\mathbf{H}(t)W(t))\\&=-\tfrac{1}{4}ie^2\int(dx)^3\int(dx')^3\\&\quad\times\int_{-\infty}^{\infty}dt'\epsilon(t-t')B(x,t,x',t'),\end{aligned} \tag{9.7}$$

with

$$B=(1/e^2)[H(x,t)H(x',t')-H(x',t')H(x,t)]. \tag{9.8}$$

The reversion operator is transformed by (9.2) into

$$R\rightarrow R'=\bar{V}(t)RV^{T-1}(-t). \tag{9.9}$$

However, on account of the supposed property of R that

$$R^{-1}\mathbf{H}(t)R=\mathbf{H}^T(-t), \tag{9.10}$$

we obtain

$$R^{-1}W(t)R=-W^T(-t),$$

therefore

$$R'=R. \tag{9.11}$$

We can ascertain that the second-order energy retains its sign by reversion:

$$R^{-1}\mathbf{H}'(t)R=\mathbf{H}^{T\prime}(-t). \tag{9.12}$$

It goes without saying that the reversibility we have discussed in the preceding sections is a "rigorous" reversibility, independent of approximation by perturbation. (9.12) is only one aspect of the general reversibility.

We now express B (9.8) in terms of the absorption and emission operators of positrons, negatrons and photons. Then, by Tati-Tomonaga's prescription,[12] we bring all the emission operators to the left of the corresponding absorption operators, using appropriate commutation relations. The application of the Tati-Tomonaga prescription effaces all the differences between P and its modified definition (8.4). From among the terms thus obtained, we retain only those terms which contains two ψ's and two $\bar{\psi}$'s and no photon operators, and then we drop those terms which involve pair-creation and pair-annihilation. We are then left with

$$\begin{aligned}B=iD_0(x-x',t-t')[&\bar{\psi}_{+l}\bar{\psi}'_{+r}\psi_{+m}\psi'_{+s}E_{\mu lm}E^\mu{}_{rs}\\&+\bar{\psi}_{-m}\bar{\psi}'_{-s}\psi_{-l}\psi'_{-r}E_{\mu ml}E^\mu{}_{sr}\\&-\bar{\psi}_{+l}\bar{\psi}'_{-s}\psi_{+m}\psi'_{-r}E_{\mu lm}E^\mu{}_{sr}\\&-\bar{\psi}_{-m}\bar{\psi}'_{+r}\psi_{-l}\psi'_{+s}E_{\mu ml}E^\mu{}_{rs}],\end{aligned} \tag{9.13}$$

where the ψ's with primes refer to (x',t'). The first two terms in (9.13) represent repulsion between like electrons, while the last two represent attraction between unlike electrons. Taking the first one term as a representative, we rewrite it as follows:

$$\begin{aligned}B=(iD_0/2)\{&-s_{+\mu}(x,t)s_+{}^\mu(x',t')\\&-s_{+\mu}(x',t')s_+{}^\mu(x,t)\\&+\bar{\psi}_+E_\mu[E_\lambda(\partial/\partial x_\lambda)+imE_5]E^\mu\psi'_+D_m{}^+\\&+\bar{\psi}'_+E_\mu[E_\lambda(\partial/\partial x_\lambda)-imE_5]E^\mu\psi_+D_m{}^-\}.\end{aligned} \tag{9.14}$$

The last two terms of this expression obviously has the effect of subtracting the corresponding self-energy. Taking the first two terms, we execute the integration over t' by Nambu's method.[13] The result it

$$\mathbf{H}'_{\text{pot}}(t)=(\tfrac{1}{2})(U_{\text{ret}}(t)+U_{\text{adv}}(t)) \tag{9.15}$$

with

$$\begin{aligned}U_{\text{ret}}=-(e^2/16\pi)\int(dx)^3\int(dx')^3(1/r)\\\times[s_{+\mu}(x,t)s_+{}^\mu(x',t-r)\\+s_{+\mu}(x',t-r)s_+{}^\mu(x,t)]\end{aligned} \tag{9.16}$$

$$\begin{aligned}U_{\text{adv}}=-(e^2/16\pi)\int(dx)^3\int(dx')^3(1/r)\\\times[s_{+\mu}(x,t)s_+{}^\mu(x',t+r)\\+s_{+\mu}(x',t+r)s_+{}^\mu(x,t)]\end{aligned} \tag{9.17}$$

U_{ret} and U_{adv}, respectively, represent obviously the retarded and the advanced potentials. We can check

$$R^{-1}\mathbf{H}'_{\text{pot}}(t)R=\mathbf{H}^{T\prime}{}_{\text{pot}}(-t). \tag{9.18}$$

[11] J. Schwinger, Phys. Rev. **74**, 1439 (1948); **75**, 651 (1949); **76**, 790 (1949).

[12] T. Tati and S. Tomonaga, Prog. Theor. Phys. **3**, 391 (1948).

[13] Y. Nambu, Prog. Theor. Phys. **5**, 614 (1950).

The expression (9.17) is apparently contradictory to the customary classical "dictum" which precludes the use of advanced potential.

To understand the deep-lying difference between the classical theory and the quantum theory, we must recall the meaning of "future" in both theories. In classical physics, the future state as calculated by mathematical continuation is to become "reality" at the future instant in question. In quantum physics, the actual state at a future instant may be different from the mathematical prolongation of the present, because an observation made between the present and the future instant in question may abruptly change the state.

Any physical law becomes significant for empirical cognition when it is so formulated as to give a prediction (deterministic or statistic) based on the past observation. In fact, we can bring the formula (9.15) to a form which conforms to this criterion, because the physical quantities referring to the future in (9.17) are determined by definite differential equations and are to be considered as operative on the mathematically prolonged future state. This reduction to the past is particularly easy when we can assume "absence of any real first order process between the remote past and the remote future (again mathematical future)." See also Nambu.[13] For it is an established fact that Schwinger's second-order picture, under this assumption, becomes mathematically identical with Tomonaga's second-order picture,[12] which is obtained by replacing W (9.3) by

$$W(t)=\int_{-\infty}^{t} dt'\mathbf{H}(t'). \tag{9.19}$$

Now a similar calculation in Tomanaga's picture gives a full retarded potential. Therefore, (9.15) is, under such assumption, mathematically identical to the full retarded potential.

Of course, we can also bring (9.15) to the expression of full advanced potential. This is mathematically equally correct, but physically useless because it refers to the mathematical future.

A more general reduction of Schwinger's reversible picture to Tomanaga's formally irreversible picture may be done in the following manner. We notice that Schwinger's second order energy is not "standardized," in the sense that it does not become identically zero at any instant between $-\infty$ and $+\infty$, unless the aforesaid assumption is made. If we standardize it with reference to the remote past, i.e., if we make the difference between its expectation value at t and its expectation value at $t=-\infty$, we obtain, to the e^2-approximation, exactly the same expression as the Tomonaga picture, thus leading to the full retarded potential. Naturally, if we standardize it with reference to the remote future, we obtain the pure advanced potential. This again can be of little significance to physical experience, for it refers to the mathematically concocted future.

We thus see not only that we can deduce the retarded potential from the quantum electrodynamics without interfering with its basic reversibility, but also that the expression of potential energy in terms of retarded action is the only legitimate expression conformable to the nature of human cognition.

The author is indebted to the members of the theoretical seminar of the University of Michigan for their critical discussions of the problem of reversibility and, in particular, to Dr. K. M. Case for the enlightening exposition of his own interpretation of Schwinger's contention about reversibility. The author is also happy to express appreciation for the constant interest and suggestions that Dr. K. Husimi has given to the author's investigation of reversibility since the time he took up the problem some seventeen years ago.

APPENDIX

(a) Four Kinds of Tensors

It is customary to define spinors by their transformation properties for infinitesimal rotations and then to assign to them, in a more or less *ad hoc* fashion, properties for reflections in such a way that they do not contradict the original definition. It will then be simpler and more consistent to define the spinors from the outset by their properties for reflections and then to deduce from them the properties for rotations using the fact that any rotation can be decomposed into a series of reflections.[14–16]

We consider the entire group of congruent linear transformations:

$$x^{\mu\prime}=a^{\mu}{}_{\nu}x^{\nu} \quad (\mu,\nu=1,2,3,0), \tag{A.1}$$

which do not change the value of

$$D^2=(x^1)^2+(x^2)^2+(x^3)^2-(x^0)^2=x_{\mu}x^{\mu}=g_{\mu\nu}x^{\mu}x^{\nu}=g^{\mu\nu}x_{\mu}x_{\nu}. \tag{A.2}$$

We shall consistently adhere to the real time coordinate $x^0=-x_0=t$, $(c=1)$.

We define three signum functions σ, σ_t, and σ_s by

$$\sigma=\partial(x^{1\prime},x^{2\prime},x^{3\prime},x^{0\prime})/\partial(x^1,x^2,x^3,x^0), \tag{A.3}$$

$$\sigma_t=\frac{\partial x^{0\prime}}{\partial x^0}\bigg/\left|\frac{\partial x^{0\prime}}{\partial x^0}\right|, \tag{A.4}$$

$$\sigma_s=\frac{\partial(x^{1\prime},x^{2\prime},x^{3\prime})}{\partial(x^1,x^2,x^3)}\bigg/\left|\frac{\partial(x^{1\prime},x^{2\prime},x^{3\prime})}{\partial(x^1,x^2,x^3)}\right|. \tag{A.5}$$

Any congruent transformation can be decomposed into a series of reflections with respect to planes passing through the origin. According as the normal n^{μ} to a

[14] E. Cartan, *La Théorie des Spineurs* (Hermann et Cie, Paris, 1938).
[15] S. Watanabe, Sci. Pap. Inst. Phys. Chem. Research (Tokyo) 39, 157 (1941).
[16] S. Watanabe, *Classical Mechanics of Fields* (Kawade-Shobo, Tokyo, 1948). (In Japanese.)

TABLE I. Classification of congruent transformations. For the definitions of σ, σ_t and σ_s, see (A.3), (A.4), and (A.5).

	σ	σ_t	σ_s	
$\mathfrak{A}$	+	+	+	Proper rotations
$\mathfrak{B}$	+	−	−	Improper rotations
$\mathfrak{C}$	−	+	−	Proper inversions
$\mathfrak{D}$	−	−	+	Improper inversions

plane of reflection is time-like or space-like, i.e.,

$$n_\mu n^\mu = -1 \quad \text{or} \quad +1, \tag{A.6}$$

we speak of a temporal or spatial reflection.

The total number ν of reflections of both kinds, the number ν_t of temporal reflections and the number ν_s of spatial reflections in the decomposition of a transformation are connected with the values of the signum functions by

$$\sigma=(-1)^\nu; \quad \sigma_t=(-1)^{\nu_t}; \quad \sigma_s=(-1)^{\nu_s} \tag{A.7}$$

from which follows

$$(\nu=\nu_t+\nu_s): \quad \sigma=\sigma_t\sigma_s. \tag{A.8}$$

The values of σ, σ_t and σ_s have invariant geometrical meanings independent of the coordinates used and of the way of decomposition.

The entire congruent group (A.1) can be divided into four classes: $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{D}$, whose definitions are given in Table I.

It can be easily seen that $\mathfrak{A}$, $\mathfrak{A}+\mathfrak{B}$, $\mathfrak{A}+\mathfrak{C}$ and $\mathfrak{A}+\mathfrak{D}$ are invariant partial groups of the whole group. σ, σ_t and σ_s are faithful representations of the factor groups engendered, respectively, by the invariant partial groups $\mathfrak{A}+\mathfrak{B}$, $\mathfrak{A}+\mathfrak{C}$ and $\mathfrak{A}+\mathfrak{D}$.

The transformation rule of a "regular" tensor which is transformed according to

$$t'^{\mu\nu\cdots} = a^\mu{}_\kappa a^\nu{}_\lambda \cdots t^{\kappa\lambda\cdots} \tag{A.9}$$

will be written for brevity

$$t'=At. \quad \text{(regular)}. \tag{A.10}$$

It is clear from the foregoing remark that, if A is a faithful representation of the entire group (A.1), then each of σA, $\sigma_t A$ and $\sigma_s A$ is also a faithful representation of the same order. Three kinds of tensors which are transformed according to

$$\begin{aligned} t'&=\sigma At, \quad \text{(1st kind)} \\ t'&=\sigma_t At, \quad \text{(2nd kind)} \\ t'&=\sigma_s At, \quad \text{(3rd kind)} \end{aligned} \tag{A.11}$$

were introduced by the present author under the names, respectively, of a first kind, a second kind and a third kind pseudotensors.[15]

If the improper rotations and improper inversions are disregarded, the second kind pseudotensors become identical with the regular tensors and the third kind pseudotensors with the first kind. The pseudotensors in the ordinary terminology correspond to the latter.

(b) Spinors Defined by Reflections

Four basic matrices E_1, E_2, E_3, E_0 are defined by

$$\tfrac{1}{2}(E_\mu E_\nu + E_\nu E_\mu) = g_{\mu\nu}. \tag{A.12}$$

The basic matrices with superscripts are related to those with subscripts by

$$E^\mu = g^{\mu\nu} E_\nu. \tag{A.13}$$

For convenience in later use, we define $E_5 = E^5$ by

$$E_5 = -iE_1E_2E_3E_0. \tag{A.14}$$

Then (A.12) can be generalized to five dimensions with $g_{55}=+1$ and $g_{5\mu}=0$ ($\mu=1, 2, 3, 0$).

For our analysis, the following theorems are essential:[17]

(1) For any set of E_μ there exists a matrix J such that

$$J^{-1}E_iJ=-\bar{E}_i; \quad \bar{J}=-J. \quad (i=1, 2, 3, 0, 5), \tag{A.15}$$

where the bar on a matrix means its hermitian conjugate.

(2) For any set of E_μ there exists a matrix K such that

$$K^{-1}E_\mu K=-E^T{}_\mu; \quad K^{-1}E_5K=E^T{}_5; \quad K^T=-K, \tag{A.16}$$

and

$$\bar{K}=J^TK^{-1}J, \tag{A.17}$$

where the symbol T on a matrix means its transpose.

(3) There exist such sets of E_μ that

$$E_i=\bar{E}^i. \tag{A.18}$$

Such sets of E_μ will be called hermitian systems for E_1, E_2, E_3, (iE_0) and E_5 are hermitian. In a hermitian system

$$J=E^0; \quad \bar{K}=K^{-1}=-K^*, \tag{A.19}$$

where the star is used in the sense of complex conjugate.

(4) Any matrix that commutes with all the four basic matrices (of any set) is the unity matrix multiplied by a number.

We now define a matrix N corresponding to the reflection with respect to a normal n^μ by

$$N=n_\mu E^\mu \quad (N^2=n_\mu n^\mu). \tag{A.20}$$

And corresponding to a transformation which can be decomposed into a series of reflections $N_1, N_2, \cdots N_\nu$, we introduce a matrix S:

$$S=N_\nu N_{\nu-1}\cdots N_1; \quad S^{-1}=\sigma_t N_1N_2\cdots N_\nu, \tag{A.21}$$

where each N is defined by (A.20).

It can be shown that S is a two-valued faithful representation of the entire group (A.1). The two valuedness comes from the fact that a plane does not determine the sign of its normal.

[17] W. Pauli, Ann. Inst. Henri Poincaré 6, 137 (1936).

The spinor ξ is defined as the representation vector of S:

$$\xi'=S\xi. \tag{A.22}$$

Then its spinor conjugate:

$$\bar{\xi}=\xi^* J^{-1} \tag{A.23}$$

transforms according to

$$\bar{\xi}'=\bar{\xi}\sigma_s S^{-1}. \tag{A.24}$$

Definition (A.21) leads to the relation:

$$E_\mu a^\mu{}_\nu=\sigma S E_\nu S^{-1}. \tag{A.25}$$

Comparison of our definition of S with the ordinary transformation rule for spinors establishes the correspondence between the E-system and the γ-system:

$$E_\mu=i\gamma_5\gamma_\mu;\quad E_5=\gamma_5, \tag{A.26}$$

where

$$\gamma_0=-\gamma^0=i\gamma_4. \tag{A.27}$$

A hermitian system of E_μ (A.18) corresponds to a hermitian γ-system. The ordinary conjugate spinor $\xi^\dagger=\xi^*\gamma_4$ is connected to our conjugate spinor $\bar{\xi}$ in the hermitian system by

$$\xi^\dagger=-\bar{\xi}E_5. \tag{A.28}$$

The transformation rule (A.25) is rewritten in the γ-system in the form:

$$\gamma_\mu a^\mu{}_\nu=S\gamma_\nu S^{-1}, \tag{A.29}$$

which coincides with the ordinarily assumed rule, but here we have no ambiguity as to the sign in case of reflections.

The transformation properties of tensorial quantities built with the spinors are tabulated in Table II.

One of the convenient features of the present method consists in that, in case of reversal of a axis, the spinor is simply multiplied by the E_μ corresponding to that axis. See (A.20). For instance, for the time-inversion, we have

$$\xi\rightarrow\pm E^0\xi;\quad \bar{\xi}\rightarrow\pm\bar{\xi}E_0, \tag{A.30}$$

where the same sign should be adopted in both relations.

It is of importance to note the effect of K (A.16) on the transformation matrix (A.21):

$$KS^TK^{-1}=\sigma_s S^{-1}, \tag{A.31}$$

which is true as far as 4-dimensional congruent transformations are concerned. If two spinors ψ_1 and ψ_2 are connected by

$$\psi_1=-K\bar{\psi}_2=\bar{\psi}_2K;\quad \bar{\psi}_1=-K^{-1}\psi_2=\psi_2K^{-1}, \tag{A.32}$$

this connection, in virtue of (A.31), is kept invariant for any transformation provided that ψ_1 and ψ_2 obey the same transformation rule (A.22). Conversely, there is no other matrix than K that enjoys this property. The compatibility of the two relations (A.32) is proved by (A.17).

Table II. Transformation properties of tensors built with spinors of the same "kind."

$\bar{\xi}\xi$	$\xi^\dagger\gamma_5\xi$	3rd kind pseudoscalar
$\bar{\xi}E_5\xi$	$\xi^\dagger\xi$	2nd kind pseudoscalar
$\bar{\xi}E_\mu\xi$	$\xi^\dagger\gamma_\mu\xi$	2nd kind pseudovector
$\bar{\xi}E_5E_\mu\xi$	$\xi^\dagger\gamma_5\gamma_\mu\xi$	3rd kind pseudovector
$\bar{\xi}E_\mu E_\nu\xi$	$\xi^\dagger\gamma_5\gamma_\mu\gamma_\nu\xi$	3rd kind pseudotensor
$\bar{\xi}E_5E_\mu E_\nu\xi$	$\xi^\dagger\gamma_\mu\gamma_\nu\xi$	2nd kind pseudotensor

In place of (A.26), an alternative correspondence between the E-system and the γ-system can be established by

$$E_\mu=\gamma'_\mu;\quad E_5=\gamma'_5;\quad \bar{\xi}=-i\xi^\dagger \tag{A.33}$$

which leads to

$$\gamma'_\mu a^\mu{}_\nu=\sigma S\gamma_\nu S^{-1}. \tag{A.34}$$

The use of this γ'-system instead of the γ-system only interchanges the second kind and the third kind of the resulting tensors. Other than this modification, this choice of correspondence produces nothing essentially new; we shall adhere to the previous choice (A.26) in this work.

(c) Pseudospinors, Bi-Spinors, Bi-Tensors

In view of the fact that regular tensors and first kind pseudotensors cannot be built by the foregoing method, one may be tempted to introduce "pseudospinors" by

$$\eta'=\sigma S\eta;\quad \zeta'=\sigma_t S\zeta;\quad \omega'=\sigma_s S\omega. \tag{A.35}$$

But obviously, the tensors built with these pseudospinors have the same transformation properties as those indicated in Table II. Different kinds of tensors, of course, can be obtained if one of ξ-spinor and η-spinor and one of ζ-spinor and ω-spinor are combined to build tensors. Table III gives the case of mixed products of a ξ-spinor and a ζ-spinor.

The method of pseudospinors is mathematically not elegant because S has by definition an indefinite sign. Combined use of two kinds of spinors is equivalent to employing eight components. If, from the outset, eight components are to be used, we can devise a more compact formalism, which also will prove to be useful for our problem.

We introduce signum matrices Σ, Σ_t and Σ_s by

$$\left.\begin{aligned} \Sigma=\begin{pmatrix}0&1\\1&0\end{pmatrix}^{\nu};\quad \Sigma^t=\begin{pmatrix}0&1\\1&0\end{pmatrix}^{\nu_t};\\ \Sigma_s=\begin{pmatrix}0&1\\1&0\end{pmatrix}^{\nu_s}\end{aligned}\right\} \tag{A.36}$$

which form groups, respectively, isomorphic to the groups formed by σ, σ_t and σ_s. To avoid repetition, we shall discuss only Σ_t in this Appendix.

TABLE III. Transformation properties of tensors built with a ξ-spinor and a ζ-spinor.

$\bar{\zeta}\xi$	$\zeta^\dagger\gamma_5\xi$	1st kind pseudoscalar
$\bar{\zeta}E_5\xi$	$\zeta^\dagger\xi$	regular scalar
$\bar{\zeta}E_\mu\xi$	$\zeta^\dagger\gamma_\mu\xi$	regular vector
$\bar{\zeta}E_5E_\mu\xi$	$\zeta^\dagger\gamma_5\gamma_\mu\xi$	1st kind pseudovector
$\bar{\zeta}E_\mu E_\nu\xi$	$\zeta^\dagger\gamma_5\gamma_\mu\gamma_\nu\xi$	1st kind pseudotensor
$\bar{\zeta}E_5E_\mu E_\nu\xi$	$\zeta^\dagger\gamma_\mu\gamma_\nu\xi$	regular tensor

We now regard each row and each column of the matrix (A.36) as corresponding to four spinorial components. Correspondingly, the basic matrices E_μ will be replaced by eight-eight matrices $\mathbf{E}_\mu$ defined by

$$\mathbf{E}_\mu=\begin{pmatrix}E_\mu & 0\\ 0 & E_\mu\end{pmatrix}, \tag{A.37}$$

which satisfy the definition (A.12).

We next introduce a matrix Π which is defined by

$$\Pi=\Pi^{-1}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{A.38}$$

This matrix Π as well as Σ_t commutes with $\mathbf{E}_\mu$:

$$\Pi^{-1}\mathbf{E}_\mu\Pi=\mathbf{E}_\mu;\quad \Sigma_t^{-1}\mathbf{E}_\mu\Sigma_t=\mathbf{E}_\mu;\quad (\Sigma_t^{-1}=\Sigma_t). \tag{A.39}$$

The commutation rule between Σ_t and Π has an important property

$$\Sigma_t^{-1}\Pi\Sigma_t=\sigma_t\Pi. \tag{A.40}$$

The transformation matrix S will be replaced by

$$\mathbf{S}=\mathbf{N}_\nu\mathbf{N}_{\nu-1}\cdots\mathbf{N}_1, \tag{A.41}$$

where each $\mathbf{N}$ is defined by $\mathbf{N}=n^\mu\mathbf{E}_\mu$.

Corresponding to σ_tS we introduce

$$\Sigma_t\mathbf{S}. \tag{A.42}$$

That $\Sigma_t\mathbf{S}$ is a representation of the group is guaranteed by the commutability of Σ_t with $\mathbf{E}_\mu$, and therefore with $\mathbf{S}$.

TABLE IV. Transformation properties of tensors built with bi-spinors.

$\bar{X}X$	$X^\dagger\Gamma_5X$	3rd kind pseudoscalar
$\bar{X}\Pi X$	$X^\dagger\Gamma_5\Pi X$	1st kind pseudoscalar
$\bar{X}\mathbf{E}_5X$	$X^\dagger X$	2nd kind pseudoscalar
$\bar{X}\mathbf{E}_5\Pi X$	$X^\dagger\Pi X$	regular scalar
$\bar{X}\mathbf{E}_\mu X$	$X^\dagger\Gamma_\mu X$	2nd kind pseudovector
$\bar{X}\mathbf{E}_\mu\Pi X$	$X^\dagger\Gamma_\mu\Pi X$	regular vector
$\bar{X}\mathbf{E}_5\mathbf{E}_\mu X$	$X^\dagger\Gamma_5\Gamma_\mu X$	3rd kind pseudovector
$\bar{X}\mathbf{E}_5\mathbf{E}_\mu\Pi X$	$X^\dagger\Gamma_5\Gamma_\mu\Pi X$	1st kind pseudovector
$\bar{X}\mathbf{E}_\mu\mathbf{E}_\nu X$	$X^\dagger\Gamma_5\Gamma_\mu\Gamma_\nu X$	3rd kind pseudotensor
$\bar{X}\mathbf{E}_\mu\mathbf{E}_\nu\Pi X$	$X^\dagger\Gamma_5\Gamma_\mu\Gamma_\nu\Pi X$	1st kind pseudotensor
$\bar{X}\mathbf{E}_5\mathbf{E}_\mu\mathbf{E}_\nu X$	$X^\dagger\Gamma_\mu\Gamma_\nu X$	2nd kind pseudotensor
$\bar{X}\mathbf{E}_5\mathbf{E}_\mu\mathbf{E}_\nu\Pi X$	$X^\dagger\Gamma_\mu\Gamma_\nu\Pi X$	regular tensor

Now an eight-component spinor or bi-spinor X can be defined as the representation vector of $\Sigma_t\mathbf{S}$:

$$X'=\Sigma_t\mathbf{S}X. \tag{A.43}$$

The spinor conjugate to X will be given by

$$\bar{X}=X^*\mathbf{J}^{-1};\quad \mathbf{J}=\begin{pmatrix}J & 0\\ 0 & J\end{pmatrix}. \tag{A.44}$$

Then $\bar{X}$ is obviously transformed according to

$$\bar{X}'=\bar{X}\sigma_s\mathbf{S}^{-1}\Sigma_t, \tag{A.45}$$

Tensors formed with a Π between $\bar{X}$ and X have the transformation properties different from those of the tensors without Π by the factor σ_t due to (A.40). Table IV shows that all the four kinds of tensors can be formed with the help of a bi-spinor.

$X^\dagger$ and Γ_i in Table IV are natural extensions of $\xi^\dagger$ and γ_i to the eight-component case and will hardly require any explanation.

Corresponding to the two columns and two rows of the matrices, we shall write the eight components of X in two parts:

$$X=\begin{pmatrix}\psi\\ \varphi\end{pmatrix}, \tag{A.46}$$

where each of ψ and φ has four components. For the present, we do not assume any relation between ψ and φ.

A tensor or pseudotensor constructed without the help of Π is just the sum of the corresponding two terms constructed respectively with ψ and φ. A tensor or pseudotensor constructed with the help of Π is on the contrary the difference between the corresponding term in ψ and the corresponding term in φ. For instance:

$$\begin{aligned}\bar{X}\mathbf{E}_\mu X&=\bar{\psi}E_\mu\psi+\bar{\varphi}E_\mu\varphi,\\ \bar{X}\mathbf{E}_\mu\Pi X&=\bar{\psi}E_\mu\psi-\bar{\varphi}E_\mu\varphi.\end{aligned} \tag{A.47}$$

The transformation rule given in (A.43) means that, for the transformations of the classes $\mathfrak{A}$ and $\mathfrak{C}$, ψ and φ are separately transformed as ξ-spinors, and that, for the transformations of the classes $\mathfrak{B}$ and $\mathfrak{D}$, ψ and φ are interchanged besides their transformations as ξ-spinors. In particular, for the time reversal, ψ, φ, $\bar{\psi}$ and $\bar{\varphi}$ are transformed as

$$\begin{aligned}&\psi\rightarrow\pm E^0\varphi;\quad \bar{\psi}\rightarrow\pm\bar{\varphi}E_0,\\ &\varphi\rightarrow\pm E^0\psi;\quad \bar{\varphi}\rightarrow\pm\bar{\psi}E_0,\end{aligned} \tag{A.48}$$

where either the upper or the lower sign should be adopted throughout.

The method which has been used to define the bi-spinor can also be applied to tensors with some advantage in our discussion. If B represents any one of A, σA, σ_tA and σ_sA, the matrix:

$$\mathbf{B}=\begin{pmatrix}B & 0\\ 0 & B\end{pmatrix}, \tag{A.49}$$

as well as $\Sigma_t\mathbf{B}$, is a faithful representation of the group. The bi-tensor (and pseudo-bi-tensor) is defined by

$$T'=\Sigma_t\mathbf{B}T, \tag{A.50}$$

with

$$T=\begin{pmatrix} t \\ u \end{pmatrix}. \tag{A.51}$$

Taking two such bi-tensors T_1 and T_2:

$$T_1=\begin{pmatrix} t_1 \\ u_1 \end{pmatrix}; \quad T_2=\begin{pmatrix} t_2 \\ u_2 \end{pmatrix}, \tag{A.52}$$

which can be of different ranks and different "kinds," we construct a product tensor:

$$T_1T_2=t_1t_2+u_1u_2, \tag{A.53}$$

which is the result of a contraction with regard to the index specifying the two parts of bi-tensors. Then T_1T_2 will transform by

$$(T_1T_2)'=B_1B_2(T_1T_2), \tag{A.54}$$

where B_1 operates on t_1 and u_1, while B_2 operates on t_2 and u_2.

If we construct

$$T_1\Pi T_2=t_1t_2-u_1u_2, \tag{A.55}$$

then the transformation rule of this quantity differs from that of (A.54) by the factor σ_t:

$$(T_1\Pi T_2)'=\sigma_tB_1B_2(T_1\Pi T_2). \tag{A.56}$$

In particular, if we take as T_2 a constant regular bi-scalar ($B_2=1$):

$$T_1=\begin{pmatrix} t \\ u \end{pmatrix}; \quad T_2=\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{A.57}$$

then

$$T_1T_2=t+u; \quad (T_1T_2)'=B(T_1T_2), \tag{A.58}$$

and

$$T_1\Pi T_2=t-u; \quad (T_1\Pi T_2)'=\sigma_tB(T_1\Pi T_2), \tag{A.59}$$

writing B for B_1.

When two bi-tensors (A.52) are given, we can construct a third bi-tensor by

$$T_3=\begin{pmatrix} t_1t_2 \\ u_1u_2 \end{pmatrix}, \tag{A.60}$$

which satisfies (A.50) with $B_3=B_1B_2$. The tensor products considered in (A.53) and (A.55) can be derived from this bi-tensorial product (A.60) by the processes given in (A.58) and (A.59), respectively.

A good example of a bi-tensor is given by the term in ψ and the term in φ in the tensors constructed out of bi-spinors. For instance, the first and the second terms of (A.47) form a 2nd kind pseudo-bi-tensor:

$$T_\mu=\begin{pmatrix} \bar{\psi}E_\mu\psi \\ \bar{\varphi}E_\mu\varphi \end{pmatrix}, \tag{A.61}$$

for this T_μ (A.61) obeys the transformation rule (A.50) with

$$B=\sigma_tA. \tag{A.62}$$

$\bar{X}\mathbf{E}_\mu X$ and $\bar{X}\mathbf{E}_\mu\Pi X$ correspond then, respectively, to (A.58) and (A.59).

It goes without saying that, by using Σ, Σ_t, and Σ_s instead of only Σ_t, we can obtain any kind of tensor by (A.53) and (A.55).

(d) Remark for 5-Dimensional Space

All the formulas and statements in this Appendix, except those indicated below, are applicable without modification to the 5-dimensional space whose metric is given by

$$D^2=(x^1)^2+(x^2)^2+(x^3)^2-(x^0)^2+(x^5)^2. \tag{A.63}$$

The only exceptions are (A.29), (A.31) and (A.32). The tensors built with spinors containing E_5 will be incorporated in tensors of rank higher by one.[17] The third rank antisymmetric tensor is naturally complementary to a second rank antisymmetric tensor, and its transformation rule differs from that of this latter by the factor σ. In order to obtain all the four kinds of tensors by mixed products of spinors, we need here all the four kinds of spinors. Correspondingly, we have to use all the three Σ's to obtain all the four kinds of tensors by the method of bi-spinors.

[17] One can for instance easily see that Møller's 5-dimensional theory of mesons has a wrong transformation property whether one adopts (A.26) or (A.33). See S. Watanabe, Proc. Phys. Math. Soc. (Japan) **25**, 561 (1943).