The Range and Straggling of High Energy Electrons

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Simple expressions are derived for the range and straggling of individual electrons. Large energy losses due to bremsstrahlung make the average range smaller than would be calculated from the average energy loss. Effects of multiple scattering are included. The results are in good agreement with numerical calculations by the Monte Carlo method.

THE electron range is a somewhat indefinite concept: at low energies, because of the large multiple scattering, electrons diffuse through matter, and at high energies the initial electron is soon obscured by an accompanying shower of electrons. The propagation of electrons at low energies, less than 10 Mev, has been treated by Bothe[1] and by Fowler.[2] The emphasis in this paper is on high energy; and the problem is considered because of its bearing on calculations concerning shower production.

Let us first neglect the effects of multiple scattering which, at high energy, are important only near the end of the electrons' range and which can be corrected for later. The simple methods of calculating range and straggling which are useful for protons are vitiated by the occurrence of large energy fluctuations corresponding to the emission of high energy photons. Bethe and Heitler[3,4] give the solution to the problem of the fluctuations of energy of electrons which have traversed a given thickness of matter. They approximate the radiation spectrum by

$$\sigma(k)dk=\frac{dkdt}{E\ln[E/(E-k)]},\tag{1}$$

where $\sigma(k)$ is the probability of an electron of energy E radiating a photon of energy k in passing through a distance dt measured in shower units of length,[5] i.e., radiation lengths divided by ln2; and they find that the probability of an electron of initial energy E_0 still having energy greater than E after traversing a distance t is

$$W(y,t)=(t-1,y)!/(t-1)!\tag{2}$$

in terms of the incomplete gamma-function, $(t-1, y)!$, where $y=\ln(E_0/E)$.

The following approximation for (2) was guessed by the author:

$$W(y,t)=1-\int_0^t e^{-y}y^x/x!dx;\tag{3}$$

it is accurate for y and t large compared to unity. The probability, wdt, that an electron's energy falls below E between t and $t+dt$ is found from (3) by differentiation;[6] i.e.,

$$w(y,t)dt=e^{-y}y_t dt/t!\tag{4}$$

For large values of y and t this can be further approximated by the gaussian form,

$$w(y,t)dt=(2\pi y)^{-\frac{1}{2}}\exp[-(t-y)^2/2y]dt.\tag{5}$$

From this the mean range r will be

$$r=y_{\max}\tag{6}$$

if we can define a value of $y_{\max}$ corresponding to the loss of all the energy by radiation only, when the particle has stopped. To do this the loss of energy by ionization must be included. Expressing (6) in terms of energy, and differentiating the mean range with respect to the initial energy, gives[7] for the average radia-

TABLE I. Monte Carlo calculations of electron range.*

E (Mev)	n	r	s/r	ϵ
10	466	0.85	0.38	0.02
20	461	1.12	0.43	0.02
50	346	1.75	0.42	0.04
100	292	2.45	0.36	0.05
200	95	2.56	0.43	0.11
500	100	3.04	0.46	0.14
1000	100	3.72	0.40	0.15

* E = initial electron energy, n = number of Monte Carlo trials, r = mean range in radiation lengths, s/r = fractional rms straggling, ϵ = statistical error of Monte Carlo mean range determination.

[1] W. Bothe, *Handbuch der Physik* **2212**, 1 (1933).

[2] Fowler, Lauritsen, and Lauritsen, Revs. Modern Phys. **20**, 236 (1948); see also J. Steinberger, Phys. Rev. **75**, 1136 (1949).

[3] H. A. Bethe and W. Heitler, Proc. Roy. Soc. (London) **146**, 84 (1934). W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1944), second edition, p. 224. Heitler also calculates the range of electrons, p. 223, but with considerably different results because of his neglect of fluctuations which shorten the range considerably.

[4] L. Eyges, Phys. Rev. **76**, 264 (1949). Eyges has extended the calculation to more refined approximations of the radiation spectrum.

[5] Shower units of length and energy are used in all equations in this paper, but results and numbers mentioned are given in radiation lengths and Mev.

[6] Equation (4) may be derived directly from Eq. (2) in the following way. Using

$$\int_0^y x^t e^{-x}dx=t!-e^{-y}\sum_0^t t!/\mu!y^\mu,$$

which can be derived by successive partial integrations (for integral t), one obtains equation (4) by taking the difference between t and $(t-1)$.

[7] If we integrate (1) over k to find the average loss of energy, we would get the usual expression $-dE/dt=E\ln2$, remembering that t is in shower units. The difference comes about because fluctuations have been included in arriving at (7).

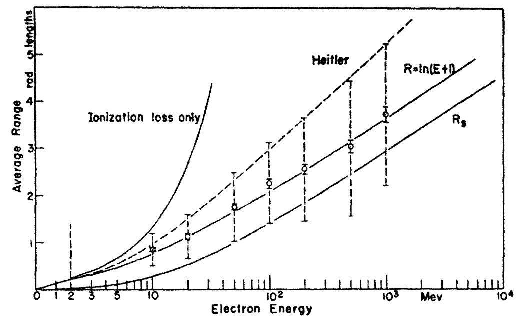

FIG. 1. The average range of electrons in lead is plotted against the energy in Mev. The dashed line curve is taken from W. Heitler, *The Quantum Theory of Radiation.* The full line marked $R=\ln(E+1)$ is given by that expression if the range is measured in shower units of length, i.e., (r.l./ln2) and the initial energy in units of critical energy times ln2, i.e., $(E_{\mathrm{Mev}}/\beta\ \ln 2)$. The circles indicate the Monte Carlo results: the short bars represent the statistical accuracy of the mean, and the long dashed vertical bars represent the mean square deviation of a single electron track from the mean range. The curve marked R_s shows the mean range as corrected for multiple scattering.

tion loss on traveling a distance dt

$$-dE/dt=E. \tag{7}$$

Let the energy be measured in units equal to ln2 times the ionization loss in radiation lengths, i.e., $E=E_{\mathrm{Mev}}/\beta\ \ln 2$. Then ionization loss can be added to (7), i.e.,

$$-dE/dt=E+1, \tag{8}$$

and integrating this over the energy gives for the mean range corresponding to the stopping of electrons,

$$r=\log(E_0+1). \tag{9}$$

If s is defined as the root-mean-square deviation of a track length from the mean range, it is evident from (5) that for high energies where ionization can be neglected,

$$s^2=y_{\max}=r. \tag{10}$$

Thus, surprisingly, enough the percentage straggling decreases with increasing energy. At lower energies ionization loss becomes important. Now the average energy loss by ionization for the electrons traveling a distance r is just r in the above energy units; hence a fraction of the range r/E_0 can be ascribed to ionization loss and the straggling of this part will be negligible compared to the straggling due to the part of the range corresponding to radiation loss. The simplest procedure for correcting the straggling as calculated from (5), then, will be to reduce the straggling by the fraction, r/E_0, i.e.,

$$s/r=(1-r/E_0)r^{-\frac{1}{2}}. \tag{11}$$

The above expressions for the range and straggling were checked for Pb in the energy interval from 10 to 1000 Mev using the Monte Carlo device already described.[8] Exactly the same procedure was used as for showers except that only the initial electron was followed. The results are summarized in Table I which shows the number of electrons followed in each case, the mean range, the rms straggling, and the standard error of the mean. In Figs. 1 and 4 these values are compared with the theoretical expressions (9) and (11). It can be seen that the agreement is well within the accuracy of the Monte Carlo method, about five percent. In Fig. 2 the empirical range distributions found for a few low energies are compared to the gaussian expression (5): the agreement is satisfactory.

That the rough theory outlined above compares so well with the Monte Carlo calculations is probably fortuitous: it is better to regard the expressions derived as semi-empirical formulas summarizing the Monte Carlo calculations.

Multiple scattering can now be included. Inasmuch as the scattering is appreciable only near the end of the track, we will make the approximation that the electrons proceed in the original direction until they reach an energy at which the calculated rms angle of multiple scattering has attained such a large value that the electrons thereafter diffuse in a random manner. The average straight distance will then be the mean range, and the random motion will contribute only to the straggling.

The mean square angle of multiple scattering $\langle\theta^2\rangle_{\mathrm{Av}}$ of an electron of initial energy E_0 is readily calculated

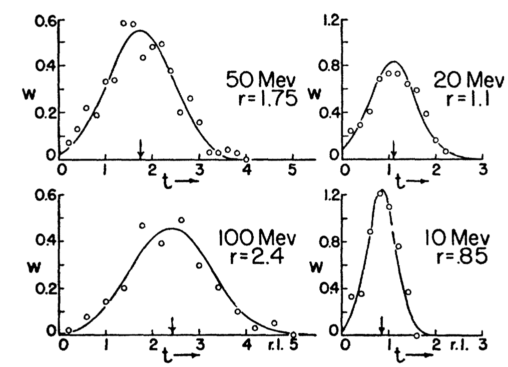

FIG. 2. Monte Carlo results for the probability w of an electron stopping between t and $t+dt$. The curves drawn through the points are theoretical and are given by expression (5) of the text. The arrow indicates the mean range.

using $\langle d\theta^2\rangle_{\mathrm{Av}}=(K^2/E^2)dt$,[9] where the constant K^2 has the value $21^2/\beta^2\ \ln 2$ for shower units (β is the ionization energy lost by an electron in passing through a radiation length, i.e., the critical energy), and (8) to give[10]

$$\langle\theta^2\rangle_{\mathrm{Av}}=\int_E^{E_0}\frac{K^2dE}{E^2(E+1)}=K^2\left[\frac{1}{E}-\frac{1}{E_0}-\ln\frac{E_0(E+1)}{E(E_0+1)}\right]. \tag{12}$$

[8] R. R. Wilson, Phys. Rev. **79**, 204 (1950).

[9] E. J. Williams, Proc. Roy. Soc. (London) **169**, 531 (1939); B. Rossi and K. Greisen, Revs. Modern Phys. **13**, 263 (1941).

[10] For the case of large E_0, (12) reduces to

$$\langle\theta^2\rangle_{\mathrm{Av}}=21^2\left(\frac{1}{E}-\ln\frac{E+1}{E}\right)\Big/\beta^2\ln 2,$$

which can be compared to the careful numerical computations of J. Roberg and L. W. Nordheim, Phys. Rev. **75**, 444 (1949), for the mean square angle of electrons in showers. The agreement is surprisingly good, especially at low energies.

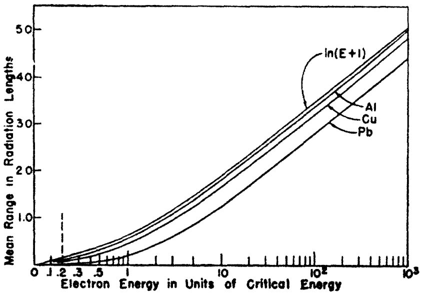

FIG. 3. Mean range in radiation lengths is plotted as a function of energy in units of the critical energy. The upper curve neglects multiple scattering and is given by Eq. (9) of the text (note the units). Multiple scattering has been included in the curves marked Al, Cu, and Pb.

We want to set this characteristic angle equal to $\langle\theta_r^2\rangle_{Av}$, a value which if reached corresponds to random motion, and then to solve for E_r, the energy at which the motion of the electron becomes random. In the Appendix it is shown that $\langle\theta_r^2\rangle_{Av}$ is 2. Hence E_r is given by,

$$E_r^{-1}=2\beta^2 21^{-2}\ln 2 + E_0^{-1}+\ln[E_0(E_r+1)/E_r(E_0+1)]. \quad (14)$$

Values of E_r were found from this expression[11] and the range r_r corresponding to E_r given by (9) is subtracted from the total range corresponding to E_0 to give the mean range, R, including multiple scattering. This is plotted in Fig. 3 for various elements.

The Monte Carlo method was used to check this result empirically. Electrons of 50-Mev initial energy were followed in lead as before but the angle of scattering was also determined at each interval by a chance method similar to that used for finding the radiation. A three-dimensional protractor was constructed[12] to keep track of the angle of the electron direction with respect to its initial direction. At each interval the forward component of the distance moved by the electron was added to give finally the total forward range of each electron followed. One thousand electrons[13] were followed and the mean range was observed to be reduced by 0.65 radiation length in exact agreement with that predicted by the above calculation. A similar calculation at 1000 Mev gave similar agreement.

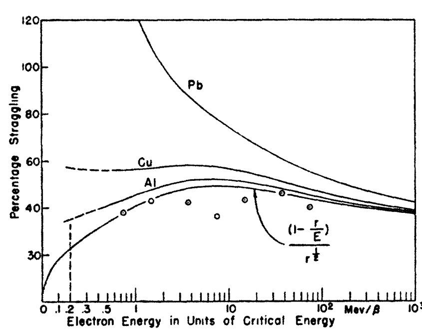

FIG. 4. The percentage root mean square straggling from the mean range. The lower curve is given by the indicated formula and neglects multiple scattering. The circles indicate Monte Carlo calculations which also neglect multiple scattering. Multiple scattering has been included in the curves marked Al, Cu, and Pb.

The straggling can also be considered in two parts: firstly, the variation in range produced by photon emission as the electron is slowed down to E_r, and then secondly the random motion due to scattering which takes place until the electron comes to rest.

The straggling produced while the path is straight, S_s^2, can be calculated from our previous result by considering the straggling of the total path length as being compounded also of two parts, i.e.,

$$s^2=S_s^2+s_r^2, \quad (13)$$

where s_r^2 is the straggling of the electrons of initial energy E_r as given by (12), i.e., without considering the multiple scattering. The quantities s_r^2 and s^2 can be calculated from (11), and then S_s^2 can be obtained from (13).

The straggling, S_r, due to random motion over a total path length r_r will be given by the usual expression for random walk:

$$S_r^2=\frac{1}{3}\int_0^{r_r}\lambda_{tr}dt, \quad (14)$$

if we can define an equivalent mean free path, λ_{tr}, that would characterize the diffusion of electrons by multiple scattering—evidently the equivalent of a transport mean free path. In the appendix, it is shown that

$$\lambda_{tr}=2E^2/K^2. \quad (15)$$

Then from (13) we get

$$S_r^2=\frac{2}{3K^2}\int_0^{r_r}E^2dt =\frac{2\beta^2\ln 2}{3\times 21^2}\int_0^{E_r}E^2dE/(E+1),$$

and integrating,

$$S_r^2=1.0\times 10^{-3}\beta^2[\log(E_r+1)-E_r+E_r^2/2]. \quad (16)$$

The total straggling including multiple scattering effects is $S^2=S_s^2+S_r^2$, and the fractional total straggling S/R calculated from (13) and (15) is plotted as a function of E in Fig. 4 for various elements.

[11] E_r is given approximately by the formula: $E_r=E_r'[1-\exp(-E_0/E_r')]$, where $E_r'=(10/\beta)^{\frac{3}{4}}$, if the critical energy β is in Mev.

[12] Miss Leonilda Altman, who operated the Monte Carlo machine, also constructed an electronic device which was equivalent to the protractor but more accurate and convenient. The details will be published elsewhere by her.

[13] The graphs of electron energy against distance already obtained in preparing Fig. 1 were used again for this calculation. Each electron was followed and the angle at each interval was determined from a family of curves drawn on the chance cylinder used previously. Each electron graph was used successively five times to give five different angle histories for each energy history.

The straggling produced while the electron has energy below E_r is small, i.e., both S_r^2 and s_r^2 are small compared to S_s^2; hence to about ten percent $S^2=s^2$, so Eq. (11) can be used to obtain the total straggling directly.

A useful concept in calculating shower phenomena is the "pair range," i.e., the average distance traveled by the electrons resulting from pair production. Because all energies up to the photon energy are possible in pair production, the number-distance curve looks more like a simple exponential than the integrated gaussians which characterize monoenergetic electrons. This is true only at not too high energies.

If we assume an exponential distribution, then the number of electrons (apart from secondaries) at a distance t from the place of pair production is

$$n=2\exp(-t/R_\pi) \tag{17}$$

which defines R_π, the pair range. Now if we consider the energy lost by ionization, we can write

$$\int_0^\infty n\beta dt=2\int_0^w P(E)R(E)\beta dE, \tag{18}$$

where $P(E)$ is the probability of one member of the pair receiving an energy between E and $E+dE$. This was assumed to be a constant, and using expression (9) to determine $R(E)$ and (17) for n, we get upon integrating both sides of (18)

$$R_\pi=(1+1/W)\ln(W+1)-1, \tag{19}$$

where W is the initiating photon energy. This relation neglects multiple scattering, which can be best included by subtracting the random part of the range corresponding to E_r given by (14).[11] At high energy, E_r is independent of the initial electron energy and the problem is simple. At low energies, an appropriate average must be taken. The normalizing factor in (17) must now be increased so that the total energy loss will still come out right. How this should be done depends exactly on the problem being considered, so no details will be given here. At high energy, $E>100$, an integrated gaussian form must be used, but (19) is still correct for the average.

APPENDIX

Let us calculate the mean square multiple scattering angle of an electron after going through dt in terms of a fictitious cross section $\sigma(\theta)$, i.e.,

$$\begin{aligned}\langle d\theta^2\rangle_{\text{Av}}&=\int\theta^2\sigma(\theta)d\Omega dt\\&=2\int\sigma(\theta)(1-\cos\theta)d\Omega dt\\&=2dt/\lambda_{tr},\end{aligned}$$

by definition of the transport mean free path.[14] We can equate the $\langle d\theta^2\rangle_{\text{Av}}$ so obtained to the usual expression,[9] i.e., $\langle d\theta^2\rangle_{\text{Av}}=K^2dt/E^2$. Solving for λ_{tr} we get

$$\lambda_{tr}=2E^2/K^2. \tag{15}$$

For the moment let us assume that the energy remains constant, then the characteristic random angle is given by

$$\langle\theta_r^2\rangle_{\text{Av}}=\int_0^{\lambda_{tr}}K^2dt/E^2=K^2\lambda_{tr}/E^2$$

and using (15) for λ_{tr}

$$\langle\theta_r^2\rangle_{\text{Av}}=2.$$

[14] I am indebted to Dr. R. P. Feynman for a discussion on this point.