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Quantum Electrodynamics with Nonvanishing Photon Mass

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A theory of neutral particles with spin one and arbitrary mass κ in interaction with electrons is developed in such a form that all observable results go over continuously into the corresponding results of ordinary quantum electrodynamics as $\kappa\rightarrow 0$.

I. INTRODUCTION

In most present field theories the particle mass enters as an arbitrary continuous parameter whose numerical value is only empirically determined. The theory of the quantized maxwell field is the only exception to this rule. The vanishing of the photon mass is a necessary consequence of the theoretical requirement of gauge invariance, and the quantized maxwell field $A_\mu(x)$ cannot be regarded in a straightforward manner as a limiting case of a quantized Proca field $\mathcal{A}_\mu(x)$. The two fields satisfy, respectively, the commutation rules[1]

$$[A_\mu(x), A_\nu(x')] = -ig_{\mu\nu}D(x-x') \tag{1}$$

and

$$[\mathcal{A}_\mu(x), \mathcal{A}_\nu(x')] = -i\left(g_{\mu\nu}-\frac{1}{\kappa^2}\partial_\mu\partial_\nu\right)D(x-x'), \tag{2}$$

where D depends[2] continuously on κ. Equation (2) becomes singular as $\kappa\rightarrow 0$. This is related to the fact that $\mathcal{A}_\mu(x)$ satisfies by definition the operator identity

$$\partial_\nu\mathcal{A}^\nu = 0, \tag{3}$$

while no such identity can be imposed on the maxwell field; but a subsidiary condition on the state functionals yields

$$\langle\partial_\nu A^\nu\rangle = 0. \tag{4}$$

Furthermore, the hamiltonian density $-j^\mu(x)\mathcal{A}_\mu(x)$, which would be analogous to the interaction of the maxwell potential with the current, does not satisfy the integrability condition for the Schwinger-Tomanaga equation, since[3]

$$[\mathcal{A}_0(x), \mathcal{A}_k(x')]_{t=t'} = \frac{i}{\kappa^2}\frac{\partial}{\partial x^k}\delta(x-x'). \tag{5}$$

In view of this situation and the common emphasis on the requirement of gauge invariance in quantum electrodynamics, it may be of interest to construct a theory of neutral particles with spin one and arbitrary mass which gives the same observable results as ordinary quantum electrodynamics with any desired accuracy, provided the photon mass is sufficiently small.[4] To achieve this the theory must satisfy the following requirements:

1. *The field equations for the expectation values go over continuously into Maxwell's equation.*
2. *The theory must account for the fact that no longitudinal or scalar photons are observed.*

II. THE FREE PHOTON FIELD

Let $A_\mu(x)$ be a vector field satisfying the Klein-Gordon equation

$$(\Box-\kappa^2)A_\mu(x) = 0, \tag{6}$$

and the commutation rules

$$[A_\mu(x), A_\nu(x')] = -ig_{\mu\nu}D(x-x'), \tag{7}$$

where $D(x-x')$ is the same function as in (2) satisfying (6). $A_\mu(x)$ is not restricted by any operator identity.[5]

It is well known that such a field describes particles with both spin one and spin zero. Their contributions to A_μ can be separated by defining a scalar field $B(x)$ by

$$B(x) = (1/\kappa)\partial_\mu A^\mu, \tag{8}$$

and a vector field $\mathcal{A}_\mu(x)$ by

$$A_\mu(x) = \mathcal{A}_\mu(x) + (1/\kappa)\partial_\mu B. \tag{9}$$

As a consequence of Eqs. (7)–(9), $\mathcal{A}_\mu(x)$ satisfies the commutation rules (2) and the identity (3). In addition, we have

$$[\mathcal{A}_\mu(x), B(x')] = 0 \tag{10}$$

and

$$[B(x), B(x')] = iD(x-x'). \tag{11}$$

If $\mathcal{A}_\mu(x)$ is a hermitian operator, its positive and negative frequency[6] parts $\mathcal{A}_\mu^{(+)}$ and $\mathcal{A}_\mu^{(-)}$ contain annihilation and creation operators of the spin-one photons, respectively. For hermitian $B(x)$, however, the roles of $B^{(+)}$ and $B^{(-)}$ are interchanged because of the sign on the right-hand side of (11). $B^{(+)}$ contains the

[1] $\hbar=c=1$.

[2] For the definition of the D-function see, for instance, W. Pauli, Revs. Modern Phys. **13**, 211 (1941).

[3] This has been pointed out by F. J. Belinfante, Phys. Rev. **76**, 66 (1949), who satisfies the integrability condition by adding a surface dependent term to the interaction. The theory becomes thereby considerably more involved than quantum electrodynamics.

[4] See F. J. Belinfante, Phys. Rev. **75**, 1321(A) (1949), **76**, 66 (1949).

[5] Our formalism is somewhat similar to the vector meson theory proposed by E. C. G. Stueckelberg, Helv. Phys. Acta **11**, 225 (1938). However, our subsidiary condition (24) or (35) is different and does not require the introduction of an additional scalar field.

[6] For the definition of the positive and negative frequency parts of any field operator see J. Schwinger, Phys. Rev. **75**, 651 (1949), Eqs. (1.19) and (1.16).

creation operators, $B^{(-)}$ the annihilation operators. In the S-matrix formalism, where one expands into a sum of "ordered" terms in which all positive frequency parts stand to the right of all negative frequency parts, it may be desirable to have $A_\mu^{(+)}$ contain annihilation operators only. Since the expectation values of $A_\mu(x)$ must be real, this cannot be achieved if they are defined in the usual way. There is, however, a possibility of defining the expectation value of any operator F with the help of an indefinite metric η in Hilbert space:[7]

$$\langle F\rangle=(\Psi,\,\eta F\Psi),\tag{12}$$

where η is an hermitian operator satisfying $\eta^2=1$. If η commutes with $\mathcal{A}_\mu$ and anticommutes with B, all expectation values are real if $\mathcal{A}_\mu$ is hermitian and B is antihermitian. For antihermitian B, $B^{(+)}$ contains the annihilation operators. The existence of such an η can be proven by pointing out an explicit matrix representation:

$$(N'|\eta|N)=(N'|1|N)(-1)^{N_s},\tag{13}$$

where N_s is the total number of scalar photons present and N and N' stand for the complete set of all photon numbers. As far as the scope of the present paper is concerned, we shall see that the definitions of the expectation values with or without indefinite metric in Hilbert space are equally satisfactory. In order to make the energy positive definite we must require the absence of scalar photons, which is expressed by the subsidiary condition

$$B^{(-)}(x)\Psi=0\tag{14a}$$

or

$$B^{(+)}(x)\Psi=0,\tag{14b}$$

in the two alternative cases.

III. THE FIELD EQUATIONS FOR THE EXPECTATION VALUES

If the field is interacting with a current $j^\mu(x)$, the state functional $\Psi(\tau)$ in the interaction representation[8] satisfies the Schrödinger equation

$$i\partial\Psi(\tau)/\partial\tau=H(\tau)\Psi(\tau),\tag{15}$$

with

$$H(\tau)=-\int_{\sigma(\tau)}d\sigma j^\mu(x)A_\mu.\tag{16}$$

Consequently, the expectation values satisfy the equations

$$(\Box-\kappa^2)\langle A_\mu\rangle=-\langle j_\mu\rangle,\tag{17}$$

and

$$\partial\langle j^\mu\rangle/\partial x^\mu=\langle\partial_\mu j^\mu\rangle=0,\tag{18}$$

the equation for the field strengths

$$F_{\mu\nu}=(\partial\langle A_\nu\rangle/\partial x^\mu)-\partial\langle A_\mu\rangle/\partial x^\nu$$

go over into Maxwell's equations as $\kappa\to 0$ if we require in addition

$$\partial\langle A_\mu\rangle/\partial x^\mu=0.\tag{19}$$

Since $[H,\,A_\mu]=0$ if x lies on the surface $\sigma(\tau)$, we have

$$\partial\langle A^\mu\rangle/\partial x^\mu=\langle\partial^\mu A_\mu\rangle.\tag{20}$$

From (17), (18), and (20) follows

$$(\Box-\kappa^2)\langle\partial^\mu A_\mu\rangle=0.\tag{21}$$

Therefore, the identity,

$$\langle\partial^\mu A_\mu\rangle=0,\tag{22}$$

is equivalent to the initial condition

$$\langle\partial^\mu A_\mu\rangle_{\tau=\tau_0}=0,\tag{23}$$

$$\left(\frac{\partial}{\partial\tau}\langle\partial^\mu A_\mu\rangle\right)_{\tau=\tau_0}=\langle\partial\partial^\mu A_\mu-n^\nu j_\nu\rangle=0,\tag{24}$$

which in turn is equivalent to

$$\langle\Omega(x,\,\tau_0)\rangle_{\tau_0}=0\quad\text{for all } x,\tag{25}$$

where the expectation value in (25) is taken with $\Psi(\tau_0)$ and $\Omega(x,\tau_0)$ is defined by

$$\Omega(x,\,\tau_0)=\partial^\mu A_\mu-\int_{\sigma(\tau_0)}d\sigma' n^\nu j_\nu(x')D(x-x').\tag{26}$$

The proof of this equivalence is the same as that for the equivalence of (9) and (16) in A. That (25) can always be satisfied is easily seen with the help of a canonical transformation

$$\Phi(\tau)=e^{-i\Sigma(\tau)}\Psi(\tau),\tag{27}$$

where

$$\Sigma(\tau)=-(1/\kappa)\int_{\sigma(\tau)}d\sigma' j^\mu(x')n_\mu B(x').\tag{28}$$

In the new representation Ω becomes

$$\begin{aligned}\Omega'&=\exp[-i\Sigma(\tau_0)]\Omega(x,\,\tau_0)\,\exp[i\Sigma(\tau_0)]\\&=\Omega(x,\,\tau_0)-i[\Sigma(\tau_0),\,\Omega(x,\,\tau_0)]=\partial^\nu A_\nu.\end{aligned}\tag{29}$$

The condition

$$(\Phi(\tau_0),\,B(x)\Phi(\tau_0))=0,\tag{30a}$$

or for all x

$$(\Phi(\tau_0),\,\eta B(x)\Phi(\tau_0))=0,\tag{30b}$$

can always be satisfied by requiring

$$B^{(-)}(x)\Phi(\tau_0)=0,\tag{31a}$$

or

$$B^{(+)}(x)\Phi(\tau_0)=0,\tag{31b}$$

which means absence of scalar photons. The corresponding Ψ satisfying (25) is obtained by the transformation (27). $\Phi(\tau)$ satisfies the Schrödinger equation

$$i\partial\Phi/\partial\tau=G(\tau)\Phi,\tag{32}$$

[7] This procedure is analogous to the one introduced into ordinary quantum electrodynamics by S. N. Gupta, Proc. Phys. Soc. (London) **63**, 681 (1950), and K. Bleuler, Helv. Phys. Acta **23**, 567 (1950), in order to secure a normalizable vacuum state.

[8] For definition of the notation see F. Coester and J. M. Jauch, Phys. Rev. **78**, 149 (1950), quoted in the following as A.

where

$$G(\tau)=-\int d\sigma j^{\mu}\mathcal{A}_{\mu}-\frac{1}{2\kappa^2}\int_{\sigma(\tau)}d\sigma\int_{\sigma(\tau)}d\sigma' j^{\nu}(x)\times j^{\mu}(x')n_{\mu}\partial_{\nu}D(x-x'). \quad (33)$$

The scalar field has been eliminated from the hamiltonian.[9]

IV. EVALUATION OF THE S-MATRIX

In order to state the subsidiary condition for the initial state $\Psi(-\infty)$ in the S-matrix formalism, we notice that[10]

$$\lim_{\tau_0\to-\infty}\Omega(x,\tau_0)=\partial^{\nu}A_{\nu}. \quad (34)$$

The subsidiary condition is therefore

$$B^{(-)}\Psi(-\infty)=0, \quad (35a)$$

if expectation values are defined in the usual way, or

$$B^{(+)}\Psi(-\infty)=0, \quad (35b)$$

if they are defined with an indefinite metric η. However, B can be eliminated from the S-matrix without using (35). The nth-order term in the S-matrix

$$S^{(n)}=(i^n/n!)\int dx_n\cdots\int dx_1 P(j^{\mu_n}(x_n)\cdots j^{\mu_1}(x_1))\times P(A_{\mu_n}(x_n)\cdots A_{\mu_1}(x_1)) \quad (36)$$

is expanded into a sum of ordered products in the manner described in A, Sec. V. In the ordered products of the vector-potentials we replace A_{μ} by the right-hand side of (8) and notice that all terms containing B vanish. This follows from the proof of Eq. (82) of A without the use of (35). If (35) holds for the initial state, it will also hold for the final state.

In order to investigate the behavior of the longitudinal photons, we define a longitudinal field $\Lambda(x)$ by

$$\Lambda(x)=\frac{\partial\partial^{\nu}-\kappa^2 n^{\nu}}{\kappa[-(\kappa^2+\partial^2)]^{\frac{1}{2}}}A_{\nu}(x). \quad (37)$$

This Λ commutes with B and satisfies the commutation rules,

$$[\Lambda(x),\Lambda(x')]=-iD(x-x'). \quad (38)$$

It describes, therefore, creation and annihilation of longitudinal photons in the usual way. The transverse field $\mathfrak{A}_{\mu}$ satisfying $n^{\mu}\mathfrak{A}_{\mu}=0$ and $\partial^{\mu}\mathfrak{A}_{\mu}=0$ is defined by

$$\mathcal{A}_{\mu}(x)=\mathfrak{A}_{\mu}(x)+\frac{\partial\partial_{\mu}-\kappa^2 n_{\mu}}{\kappa[-(\kappa^2+\partial^2)]^{\frac{1}{2}}}\Lambda(x). \quad (39)$$

Terms proportional to $\partial_{\mu}\Lambda$ do not contribute anything in the S-matrix for the same reason which made it possible to eliminate the scalar field. We may therefore replace finally A_{μ} in the ordered products in the S-matrix by

$$\mathfrak{A}_{\mu}-\frac{\kappa(n_{\mu}+\partial^{-1}\partial_{\mu})}{[-(\kappa^2+\partial^2)]^{\frac{1}{2}}}\Lambda.$$

The probability for the emission of a longitudinal photon of momentum k will therefore be $\kappa^2/(\kappa^2+k^2)$ times the probability for the emission of a corresponding transverse photon. For sufficiently small but non-vanishing photon mass one cannot expect to observe any longitudinal photons. The part of S which contains the transverse field $\mathfrak{A}_{\mu}(x)$ goes over continuously into the corresponding expressions of ordinary quantum electrodynamics only as $\kappa\to 0$.

We have shown that it possible to construct a non-gauge invariant quantum electrodynamics with non-vanishing photon mass in agreement with observation. The requirement of gauge invariance has, of course, a strong aesthetic appeal, but it is not warranted by observations alone.

[9] The details of the calculation are similar to those of the derivation of (34) in A.

[10] See A, Eq. (69).