predicts $g_{7/2}$ for the configuration of Cs^{131} but $d_{5/2}$ may also be possible. If the configuration of Cs^{131} is $g_{7/2}$ and enough energy is available, one would expect the transition to $h_{11/2}$ to be more probable than that to $d_{3/2}$. If slightly less than 80 kev is available for the K capture process, the value of log $ft{\sim}5$. This would indicate an allowed transition and would suggest $d_{5/2}$ as the configuration of the ground state of Cs^{131}.

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The Internal Conversion Coefficients. I: The K-Shell*

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The internal conversion coefficients for electric and magnetic multipole radiation have been computed for the K-shell in the relativistic case with the unscreened coulomb field acting on the electron. The numerical results, which are obtained to four-significant-figure accuracy, were computed on the automatic sequence relay calculator (Mark I) and are given here for 12 values of Z in the range $10 \leqslant Z \leqslant 96$ and 6 gamma-ray energies (between 0.3 mc^2 and 5.0 mc^2) for the first five electric and first five magnetic multipoles.

I. INTRODUCTION

THE results of the calculation of the K-shell internal conversion coefficient (defined as the ratio of conversion electrons to quanta) which are presented herein were very briefly described in a previous communication.[1] Subsequent to May, 1949, tables of these coefficients together with an extensive interpolation were circulated privately. Inasmuch as plans for the calculation of the L-shell coefficients were instituted very soon after the completion of the K-shell work, publication of the present material was held up in the hope of presenting all of the numerical results together. This does not seem to be feasible, and this paper is written in the interest of making the K-shell results more readily available.

At the time of completion of this work the only existing accurate calculations of the K-shell internal conversion coefficients were those of Hulme[2] (for electric dipole, $Z=84$), of Taylor and Mott[3] (electric quadrupole, $Z=84$) and of Fisk and Taylor[4] (magnetic dipole, quadrupole, and octupole, $Z=84$). Shortly afterward, Griffith and Stanley[5] made calculations of the K-shell electric dipole coefficients for five values of Z in the range 69–89. Subsequently, the coefficients for electric dipole, quadrupole, and magnetic dipole for Cu, In, Po, and U at low energies were obtained by Reitz.[6] Here screening was taken into account by numerical integration of the Dirac radial equations with a Thomas-Fermi-Dirac potential. In the present work where $k \geqslant 0.3$ (kmc^2 is the gamma-ray energy) no effect of screening is considered. Comparison with Reitz's results where our calculations overlap fully justifies this procedure. Calculations of the L_I-shell coefficients with unscreened wave functions have been carried out by Gellman *et al.*[7] for the same multipoles, and the same Z and k-values as appear in Reitz's work. However, these results are primarily of orientation value, since the neglect of screening cannot be justified in this case. In fact, in the calculation of the L-shell coefficients[8] (including all sub-shells) which are now under way, screening is taken into account in the same manner as was done by Reitz. In addition, low energy K-shell coefficients for all important multipoles and for essentially the same range of values of Z as in the K-shell work are being carried out with screened wave functions in parallel with the L-shell computation.

Until the L-shell and low energy K-shell results become available, it is necessary to supplement the present values of the coefficients with low energy extrapolations based on a comparison of these values and those obtained from the nonrelativistic formulas of Uhlenbeck and Hebb[9] and the essentially nonrelativistic

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TABLE I. Internal conversion coefficients. The power of ten by which the entries should be multiplied is given by the number in parentheses.

Z	k	α_1	α_2	α_3	α_4	α_5	β_1	β_2	β_3	β_4	β_5
10	0.3	9.041 (−4)	8.549 (−3)	6.822 (−2)	5.173 (−1)	3.845 (0)	4.465 (−4)	3.321 (−3)	2.461 (−2)	1.829 (−1)	1.364 (0)
	0.5	1.804 (−4)	1.112 (−3)	5.866 (−3)	2.947 (−2)	1.451 (−1)	1.356 (−4)	6.709 (−4)	3.282 (−3)	1.606 (−2)	7.865 (−2)
	1.0	2.476 (−5)	8.837 (−5)	2.803 (−4)	8.544 (−4)	2.558 (−3)	2.993 (−5)	9.132 (−5)	2.728 (−4)	8.105 (−4)	2.404 (−3)
	1.8	6.361 (−6)	1.521 (−5)	3.360 (−5)	7.221 (−5)	1.531 (−4)	9.379 (−6)	2.051 (−5)	4.379 (−5)	9.259 (−5)	1.949 (−4)
	3.0	2.552 (−6)	4.646 (−6)	8.051 (−6)	1.368 (−5)	2.301 (−5)	3.791 (−6)	6.595 (−6)	1.124 (−5)	1.895 (−5)	3.175 (−5)
	5.0	1.224 (−6)	1.823 (−6)	2.645 (−6)	3.786 (−6)	5.377 (−6)	1.685 (−6)	2.462 (−6)	3.545 (−6)	5.057 (−6)	7.169 (−6)
20	0.3	5.617 (−3)	4.892 (−2)	3.576 (−1)	2.486 (0)	1.700 (1)	4.313 (−3)	3.159 (−2)	2.265 (−1)	1.629 (0)	1.176 (1)
	0.5	1.183 (−3)	6.850 (−3)	3.392 (−2)	1.599 (−1)	7.395 (−1)	1.245 (−3)	6.139 (−3)	2.940 (−2)	1.406 (−1)	6.737 (−1)
	1.0	1.721 (−4)	5.941 (−4)	1.817 (−3)	5.340 (−3)	1.541 (−2)	2.584 (−4)	8.028 (−4)	2.385 (−3)	7.006 (−3)	2.052 (−2)
	1.8	4.522 (−5)	1.084 (−4)	2.371 (−4)	5.020 (−4)	1.047 (−3)	7.685 (−5)	1.743 (−4)	3.755 (−4)	7.935 (−4)	1.662 (−3)
	3.0	1.808 (−5)	3.391 (−5)	5.944 (−5)	1.013 (−4)	1.701 (−4)	2.965 (−5)	5.409 (−5)	9.415 (−5)	1.602 (−4)	2.692 (−4)
	5.0	8.542 (−6)	1.327 (−5)	1.973 (−5)	2.870 (−5)	4.116 (−5)	1.261 (−5)	1.937 (−5)	2.870 (−5)	4.166 (−5)	5.970 (−5)
30	0.3	1.518 (−2)	1.194 (−1)	7.852 (−1)	4.919 (0)	3.042 (1)	1.857 (−2)	1.341 (−1)	9.187 (−1)	6.292 (0)	4.322 (1)
	0.5	3.362 (−3)	1.804 (−2)	8.285 (−2)	3.627 (−1)	1.562 (0)	5.101 (−3)	2.503 (−2)	1.165 (−1)	5.400 (−1)	2.508 (0)
	1.0	5.175 (−4)	1.717 (−3)	5.044 (−3)	1.424 (−2)	3.955 (−2)	9.937 (−4)	3.132 (−3)	9.204 (−3)	2.661 (−2)	7.663 (−2)
	1.8	1.393 (−4)	3.343 (−4)	7.233 (−4)	1.510 (−3)	3.100 (−3)	2.796 (−4)	6.563 (−4)	1.420 (−3)	2.986 (−3)	6.206 (−3)
	3.0	5.552 (−5)	1.075 (−4)	1.907 (−4)	3.263 (−4)	5.476 (−4)	1.024 (−4)	1.964 (−4)	3.477 (−4)	5.949 (−4)	9.999 (−4)
	5.0	2.583 (−5)	4.198 (−5)	6.417 (−5)	9.488 (−5)	1.375 (−4)	4.134 (−5)	6.730 (−5)	1.025 (−4)	1.511 (−4)	2.183 (−4)
40	0.3	2.949 (−2)	2.042 (−1)	1.180 (0)	6.518 (0)	3.569 (1)	5.865 (−2)	4.131 (−1)	2.645 (0)	1.681 (1)	1.070 (2)
	0.5	6.850 (−3)	3.352 (−2)	1.410 (−1)	5.672 (−1)	2.255 (0)	1.544 (−2)	7.465 (−2)	3.329 (−1)	1.474 (0)	6.536 (0)
	1.0	1.116 (−3)	3.538 (−3)	9.954 (−3)	2.698 (−2)	7.211 (−2)	2.834 (−3)	8.975 (−3)	2.585 (−2)	7.302 (−2)	2.052 (−1)
	1.8	3.083 (−4)	7.407 (−4)	1.589 (−3)	3.275 (−3)	6.632 (−3)	7.540 (−4)	1.818 (−3)	3.920 (−3)	8.161 (−3)	1.675 (−3)
	3.0	1.227 (−4)	2.457 (−4)	4.426 (−4)	7.613 (−4)	1.278 (−3)	2.614 (−4)	5.253 (−4)	9.405 (−4)	1.610 (−3)	2.696 (−3)
	5.0	5.621 (−5)	9.603 (−5)	1.513 (−4)	2.277 (−4)	3.332 (−4)	9.938 (−5)	1.723 (−4)	2.689 (−4)	4.009 (−4)	5.820 (−4)
54	0.3	5.640 (−2)	3.082 (−1)	1.414 (0)	6.252 (0)	2.776 (1)	2.317 (−1)	1.520 (0)	8.376 (0)	4.518 (1)	2.429 (2)
	0.5	1.401 (−2)	5.830 (−2)	2.123 (−1)	7.487 (−1)	2.643 (0)	5.831 (−2)	2.672 (−1)	1.086 (0)	4.360 (0)	1.751 (1)
	1.0	2.479 (−3)	7.323 (−3)	1.943 (−2)	5.001 (−2)	1.277 (−1)	9.973 (−3)	3.085 (−2)	8.461 (−2)	2.270 (−1)	6.059 (−1)
	1.8	7.136 (−4)	1.725 (−3)	3.682 (−3)	7.508 (−3)	1.500 (−2)	2.467 (−3)	6.013 (−3)	1.269 (−2)	2.570 (−2)	5.123 (−2)
	3.0	2.848 (−4)	6.026 (−4)	1.115 (−3)	1.938 (−3)	3.259 (−3)	7.906 (−4)	1.666 (−3)	2.984 (−3)	5.057 (−3)	8.345 (−3)
	5.0	1.277 (−4)	2.366 (−4)	3.910 (−4)	6.040 (−4)	8.964 (−4)	2.732 (−4)	5.160 (−4)	8.245 (−4)	1.237 (−3)	1.793 (−3)
64	0.3	7.931 (−2)	3.456 (−1)	1.283 (0)	4.663 (0)	1.735 (1)	5.753 (−1)	3.441 (0)	1.613 (1)	7.280 (1)	3.257 (2)
	0.5	2.072 (−2)	7.485 (−2)	2.435 (−1)	7.808 (−1)	2.547 (0)	1.417 (−1)	6.019 (−1)	2.215 (0)	8.036 (0)	2.914 (1)
	1.0	3.910 (−3)	1.099 (−2)	2.824 (−2)	7.075 (−2)	1.766 (−1)	2.332 (−2)	6.848 (−2)	1.777 (−1)	4.519 (−1)	1.145 (0)
	1.8	1.165 (−3)	2.856 (−3)	6.133 (−3)	1.248 (−2)	2.475 (−2)	5.520 (−3)	1.311 (−2)	2.680 (−2)	5.259 (−2)	1.017 (−1)
	3.0	4.676 (−4)	1.041 (−3)	1.974 (−3)	3.464 (−3)	5.834 (−3)	1.679 (−3)	3.553 (−3)	6.271 (−3)	1.044 (−2)	1.693 (−2)
	5.0	2.069 (−4)	4.113 (−4)	7.067 (−4)	1.112 (−3)	1.666 (−3)	5.401 (−4)	1.064 (−3)	1.706 (−3)	2.547 (−3)	3.664 (−3)
72	0.3	9.882 (−2)	3.425 (−1)	1.037 (0)	3.143 (0)	9.969 (0)	1.186 (0)	6.344 (0)	2.488 (1)	9.204 (1)	3.349 (2)
	0.5	2.699 (−2)	8.579 (−2)	2.555 (−1)	7.651 (−1)	2.361 (0)	2.892 (−1)	1.116 (0)	3.697 (0)	1.206 (1)	3.930 (1)
	1.0	5.384 (−3)	1.468 (−2)	3.718 (−2)	9.186 (−2)	2.258 (−1)	4.650 (−2)	1.268 (−1)	3.098 (−1)	7.451 (−1)	1.789 (0)
	1.8	1.656 (−3)	4.161 (−3)	9.044 (−3)	1.841 (−2)	3.628 (−2)	1.070 (−2)	2.411 (−2)	4.739 (−2)	8.991 (−2)	1.683 (−1)
	3.0	6.703 (−4)	1.572 (−3)	3.054 (−3)	5.400 (−3)	9.087 (−3)	3.138 (−3)	6.456 (−3)	1.113 (−2)	1.814 (−2)	2.880 (−2)
	5.0	2.941 (−4)	6.255 (−4)	1.112 (−3)	1.776 (−3)	2.673 (−3)	9.565 (−4)	1.895 (−3)	3.010 (−3)	4.444 (−3)	6.320 (−3)
78	0.3	1.136 (−1)	3.196 (−1)	8.102 (−1)	2.095 (0)	5.732 (0)	2.075 (0)	9.877 (0)	3.254 (1)	9.856 (1)	2.906 (2)
	0.5	3.215 (−2)	9.242 (−2)	2.594 (−1)	7.417 (−1)	2.190 (0)	5.030 (−1)	1.756 (0)	5.272 (0)	1.557 (1)	4.596 (1)
	1.0	6.712 (−3)	1.810 (−2)	4.575 (−2)	1.121 (−1)	2.716 (−1)	7.989 (−2)	2.005 (−1)	4.630 (−1)	1.058 (0)	2.421 (0)
	1.8	2.120 (−3)	5.499 (−3)	1.211 (−2)	2.463 (−2)	4.819 (−2)	1.808 (−2)	3.812 (−2)	7.212 (−2)	1.325 (−1)	2.409 (−1)
	3.0	8.659 (−4)	2.135 (−3)	4.233 (−3)	7.516 (−3)	1.261 (−2)	5.190 (−3)	1.016 (−2)	1.706 (−2)	2.721 (−2)	4.238 (−2)
	5.0	3.783 (−4)	8.549 (−4)	1.562 (−3)	2.518 (−3)	3.795 (−3)	1.528 (−3)	2.952 (−3)	4.616 (−3)	6.724 (−3)	9.447 (−3)
83	0.5	3.669 (−2)	9.721 (−2)	2.615 (−1)	7.184 (−1)	2.022 (0)	8.171 (−1)	2.556 (0)	6.957 (0)	1.862 (1)	4.979 (1)
	1.0	7.980 (−3)	2.163 (−2)	5.485 (−2)	1.332 (−1)	3.171 (−1)	1.289 (−1)	2.946 (−1)	6.430 (−1)	1.399 (0)	3.054 (0)
	1.8	2.584 (−3)	6.970 (−3)	1.554 (−2)	3.153 (−2)	6.109 (−2)	2.888 (−2)	5.617 (−2)	1.022 (−1)	1.820 (−1)	3.217 (−1)
	3.0	1.065 (−3)	2.768 (−3)	5.584 (−3)	9.931 (−3)	1.659 (−2)	8.179 (−3)	1.497 (−2)	2.441 (−2)	3.809 (−2)	5.821 (−2)
	5.0	4.646 (−4)	1.115 (−3)	2.083 (−3)	3.379 (−3)	5.093 (−3)	2.354 (−3)	4.328 (−3)	6.630 (−3)	9.514 (−3)	1.320 (−2)
88	0.5	4.138 (−2)	1.020 (−1)	2.643 (−1)	6.900 (−1)	1.813 (0)	1.373 (0)	3.726 (0)	9.029 (0)	2.149 (1)	5.111 (1)
	1.0	9.409 (−3)	2.613 (−2)	6.661 (−2)	1.594 (−1)	3.698 (−1)	2.156 (−1)	4.356 (−1)	8.900 (−1)	1.829 (0)	3.784 (0)
	1.8	3.132 (−3)	8.929 (−3)	2.016 (−2)	4.063 (−2)	7.762 (−2)	4.804 (−2)	8.362 (−2)	1.452 (−1)	2.493 (−1)	4.265 (−1)
	3.0	1.306 (−3)	3.625 (−3)	7.432 (−3)	1.321 (−2)	2.189 (−2)	1.349 (−2)	2.234 (−2)	3.512 (−2)	5.340 (−2)	7.982 (−2)
	5.0	5.697 (−4)	1.468 (−3)	2.805 (−3)	4.568 (−3)	6.868 (−3)	3.824 (−3)	6.454 (−3)	9.613 (−3)	1.353 (−2)	1.848 (−2)
92	0.5	4.516 (−2)	1.066 (−1)	2.675 (−1)	6.590 (−1)	1.601 (0)	2.152 (0)	5.057 (0)	1.098 (1)	2.339 (1)	4.976 (1)
	1.0	1.068 (−2)	3.083 (−2)	7.878 (−2)	1.850 (−1)	4.172 (−1)	3.379 (−1)	6.003 (−1)	1.153 (0)	2.250 (0)	4.435 (0)
	1.8	3.640 (−3)	1.102 (−2)	2.507 (−2)	5.008 (−2)	9.423 (−2)	7.514 (−2)	1.162 (−1)	1.929 (−1)	3.204 (−1)	5.323 (−1)
	3.0	1.535 (−3)	4.551 (−3)	9.435 (−3)	1.670 (−2)	2.745 (−2)	2.104 (−2)	3.119 (−2)	4.731 (−2)	7.020 (−2)	1.028 (−1)
	5.0	6.709 (−4)	1.851 (−3)	3.595 (−3)	5.860 (−3)	8.775 (−3)	5.932 (−3)	9.028 (−3)	1.306 (−2)	1.805 (−2)	2.429 (−2)
96	0.5	4.890 (−2)	1.129 (−1)	2.713 (−1)	6.142 (−1)	1.341 (0)	3.529 (0)	6.901 (0)	1.319 (1)	2.466 (1)	4.596 (1)
	1.0	1.206 (−2)	3.706 (−2)	9.438 (−2)	2.156 (−1)	4.684 (−1)	5.547 (−1)	8.356 (−1)	1.496 (0)	2.752 (0)	5.137 (0)
	1.8	4.222 (−3)	1.383 (−2)	3.156 (−2)	6.216 (−2)	1.147 (−1)	1.235 (−1)	1.635 (−1)	2.578 (−1)	4.123 (−1)	6.629 (−1)
	3.0	1.804 (−3)	5.799 (−3)	1.212 (−2)	2.130 (−2)	3.461 (−2)	3.461 (−2)	4.421 (−2)	6.428 (−2)	9.274 (−2)	1.327 (−1)
	5.0	7.913 (−4)	2.370 (−3)	4.664 (−3)	7.589 (−3)	1.130 (−2)	9.769 (−3)	1.286 (−2)	1.796 (−2)	2.427 (−2)	3.211 (−2)

results of Drell.[10] A convenient set of graphs for this purpose has been prepared by Axel and Goodrich.[11] As has been emphasized elsewhere, the direct use of these nonrelativistic formulas leads to considerable error for all but rather small Z and k. The same remark applies to the Born approximation results of Dancoff and Morrison,[9] which applies with sufficient accuracy only for very small Z and large k, or more specifically, for $e^2Z/\hbar v \ll 1$, where v is the velocity of the conversion electron in the continuum. From the results of the Toronto group[7] it also appears that the ratio of K to L conversion coefficients as calculated by Hebb and Nelson[12] (electric) and by Tralli and Lowen[13] (mag-

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netic) is subject to surprisingly large errors because of the approximations made.

II. RESULTS AND DISCUSSION

The parameters chosen for the machine computation, which was performed on the automatic sequence relay calculator (Mark I) at the Computation Laboratory of Harvard University, were: 12 values of Z in the range $10 \leqslant Z \leqslant 96$ and 6 values of k in the range $0.3 \leqslant k \leqslant 5.0$ For $Z \geqslant 83$ it was necessary to drop the value $k=0.3$, and five values in the range $0.5 \leqslant k \leqslant 5.0$ were used. In each case the first five electric and first five magnetic multipoles were computed. The calculations were originally carried out for $Z \geqslant 40$, based on the expectation that the approximative results would be sufficiently accurate for $Z<40$. In this range the values of Z chosen were such as to obtain an approximately uniform scale in Z^3 since one expected, very roughly, a Z^3 dependence of the conversion coefficient. The extension of the computations to $Z<40$ was made at a later stage when the original expectations proved to be too optimistic.

The restriction to $k \geqslant 0.3$ was made for the following reasons. The computations involve the evaluation of a large number of series (hypergeometric functions) which become slowly convergent in this region ($k \approx 0.3$). More rapid convergence could be obtained by transforming to functions of the reciprocal argument on the other side of the radius of convergence. However, in this region of the $k-Z$ plane the effect of screening is important, and it did not appear worth while to invest the necessary effort in such unscreened calculations in view of the plans for making screened calculations. The hypergeometric series which occur [see Appendix, Eq. (28)] were evaluated term by term up to the point where the first term neglected had an absolute value less than 10^{-6}, the first term being equal to unity, of course. The gamma-functions of real and complex argument (see Appendix) were obtained by a Taylor series representation of Sterling's formula wherein the first term neglected made a relative error of less than 10^{-6}. The 680 values of the internal conversion coefficients obtained in this way are accurate to at least four significant figures. While this accuracy is far better than present experimental needs require, it is necessary for the purpose of interpolation to values of the conversion coefficient at $Z-k$ values other than those which appear in the machine calculations. Interpolations were carried out to obtain coefficients for 26 values of Z and 16 values of k representing a total of 4020 values of the conversion coefficients. For this purpose it is convenient to interpolate the ratio of the values given below in the tables to the Dancoff-Morrison Born approximation formulas,[9] since this ratio is much less sensitive with Z and k than are the computed coefficients themselves. The interpolated results checked with other calculations where overlap occurred.[2,3,5]

The analytical basis of the calculation is given in the Appendix. For reasons of space limitation only the machine computed coefficients are given in Table I below. In this table α_l and β_l are the conversion coefficients for electric and magnetic 2^l-pole radiation, respectively. More extensive tables including the interpolated values appear separately.[14] A limited number of interpolated values can be obtained from the curves given in Figs. 1–10, which represents only a part of the numerical results. Although it is not to be expected that many cases will arise which involve multipole orders with $l>5$, reasonably accurate values for $l=6$ can be obtained as follows. While α_l and β_l are sensitively dependent on l, the ratios α_{l+1}/α_l and β_{l+1}/β_l are fairly insensitive. Consequently, a one-step extrapolation gives α_6 and β_6 with an error of about 5 percent. Figure 11 illustrates this extrapolation for electric multipoles.

A considerable amount of experimental data on internal conversion coefficients now exists, and the numerical results given here have been used by many investigators to make assignments of angular momentum and parity to nuclear levels. It would seem certain that the theoretical basis of these calculations is sound, but it is worth while to note that in many cases the assignments made are in agreement with other nuclear spectroscopic data. For example, the assignments based on internal conversion measurements[15] in the decay of Co^{60} and Cs^{134} are in agreement with the results of angular correlation measurements.[16] However, in some instances discrepancies exist. Notable cases are those in which assignments are based on the rough theoretical estimates of radiation lifetime. A compilation of the internal conversion data as well as experimental results obtained by other methods of nuclear spectroscopy is now being made. It seems too early to draw any conclusions from such comparisons, in view of the uncertainty of many of the proposed decay schemes and the possibility of more accurate measurements of internal conversion coefficients. One conclusion which seems to be valid concerns the apparent scarcity of electric cipole lines. Out of a total of 89 cases where assignments can be made, 84 are fairly definitely not electric dipole and in the remaining cases the assignment as electric dipole is not certain by any means.

The authors take pleasure in expressing their appreciation to Professor H. H. Aiken of the Harvard Computation Laboratory and to Dr. A. M. Weinberg of the Oak Ridge National Laboratory through whose efforts the Mark I calculator was made available to us. The assistance of all members of the Oak Ridge Mathematics Panel, led by Dr. A. S. Householder, as well as

[14] M. E. Rose and G. H. Goertzel, AEC report (to be published).
[15] Waggoner, Moon, and Roberts, Phys. Rev. **80**, 420 (1950).
[16] E. L. Brady and M. Deutsch, Phys. Rev. **74**, 1541 (1948); **78**, 558 (1950). F. Metzger and M. Deutsch, Phys. Rev. **78**, 551 (1950).

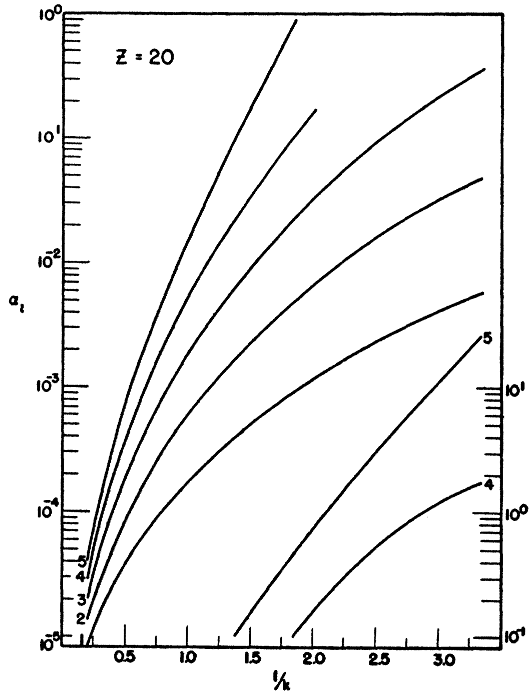

FIG. 1. The electric multipole conversion coefficients (K-shell) for $Z=20$ as a function of $1/k=(\text{gamma-energy}/mc^2)^{-1}$. The numbers attached to the curves give the value of l. The ordinate scale at the right refers to the low energy portion of the curves for 2^4 and 2^5 poles.

that rendered by Drs. N. Tralli, S. D. Drell, and G. B. Arfken was invaluable.

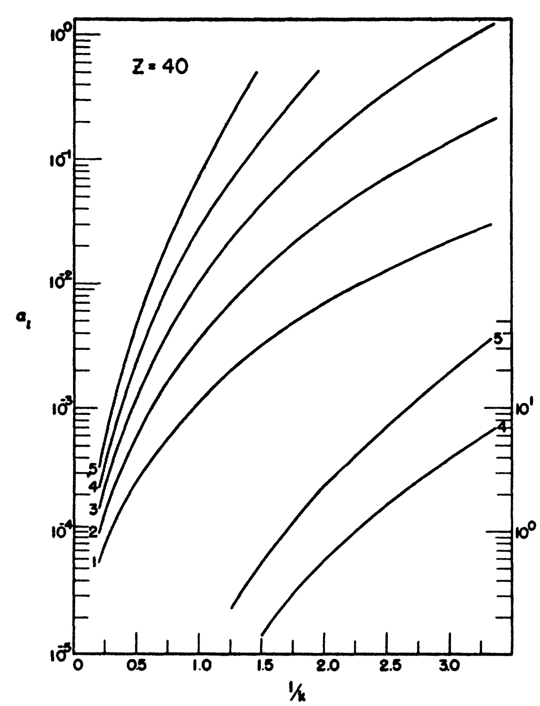

FIG. 2. Same as Fig. 1 for $Z=40$.

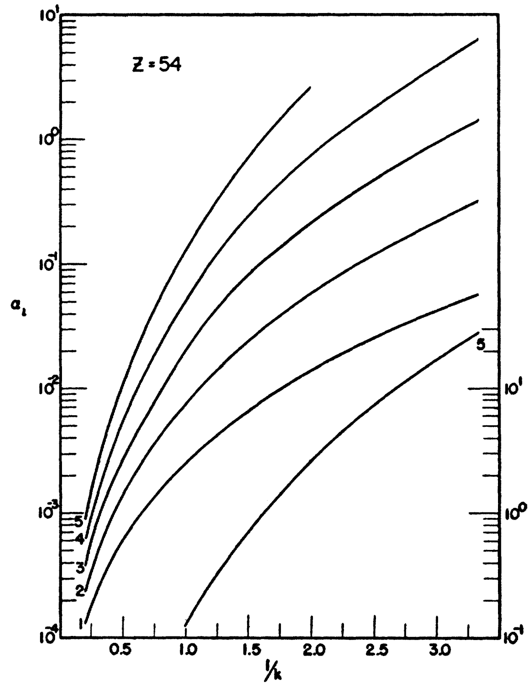

FIG. 3. Same as Fig. 1 for $Z=54$.

APPENDIX

The number of conversion electrons per unit time is found by the usual perturbation procedure to be

$$N_e=2\pi\alpha n \sum |(\psi_f|\phi+\boldsymbol{\alpha}\cdot\mathbf{A}|\psi_i)|^2, \tag{1}$$

where ψ_f, the final state wave function, is normalized

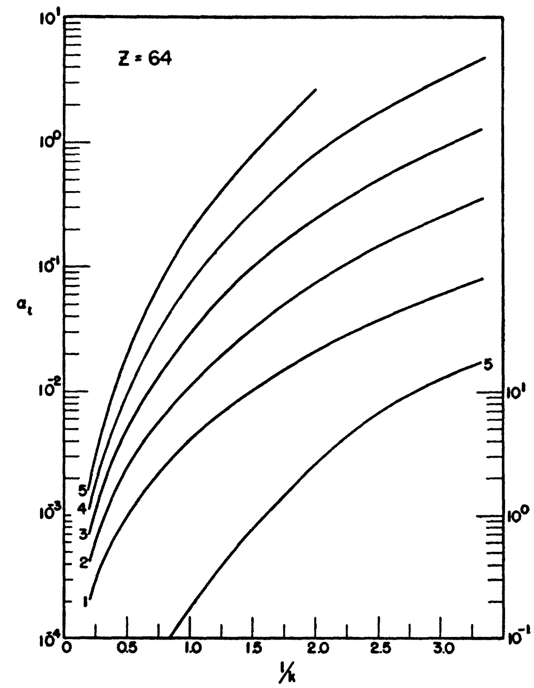

FIG. 4. Same as Fig. 1 for $Z=64$.

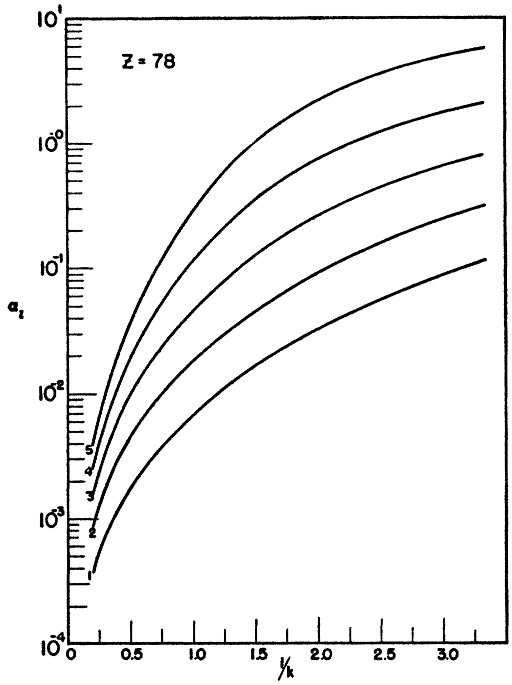

FIG. 5. Same as Fig. 1 for $Z=78$.

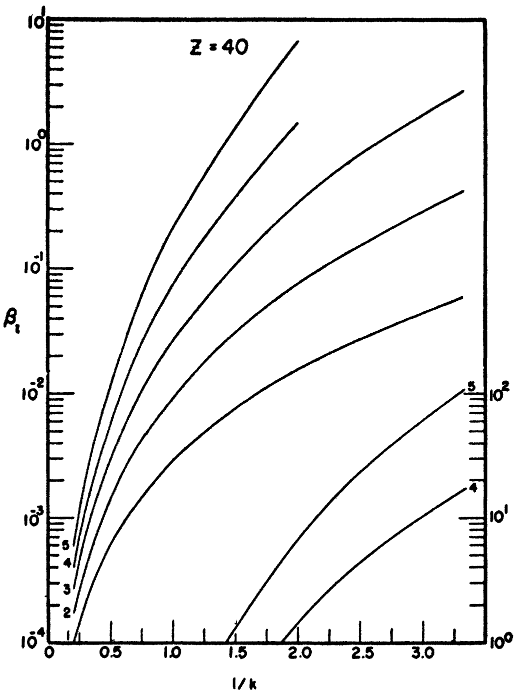

FIG. 7. Same as Fig. 1 but for magnetic conversion and $Z=4$

to unit energy range[17] and ψ_i, the initial state wave function, is normalized to unity in all configuration space. A sum over all final states, including magnetic substates, and an average over initial magnetic substates is implied by the sum sign. In (1) α is the Dirac matrix vector, α the fine structure constant, n is the number of electrons in the initial state, while ϕ and $\mathbf{A}$ are the scalar and vector potentials of the radiation field (see (3) below).

It is most convenient to use the following representa-

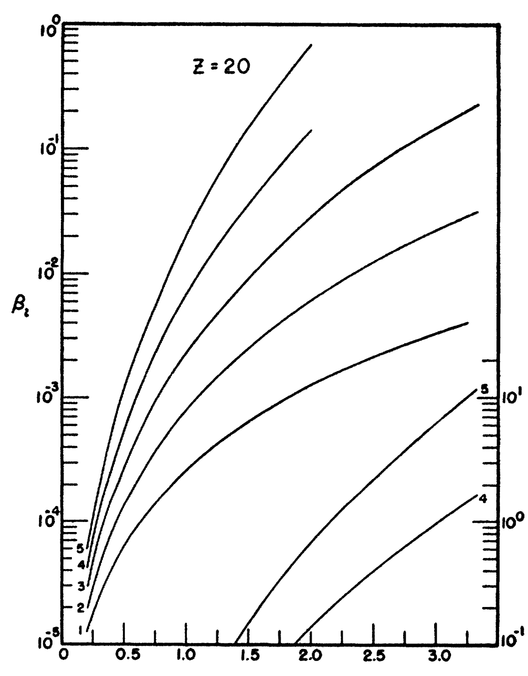

FIG. 6. Same as Fig. 1 but for magnetic multipole conversion.

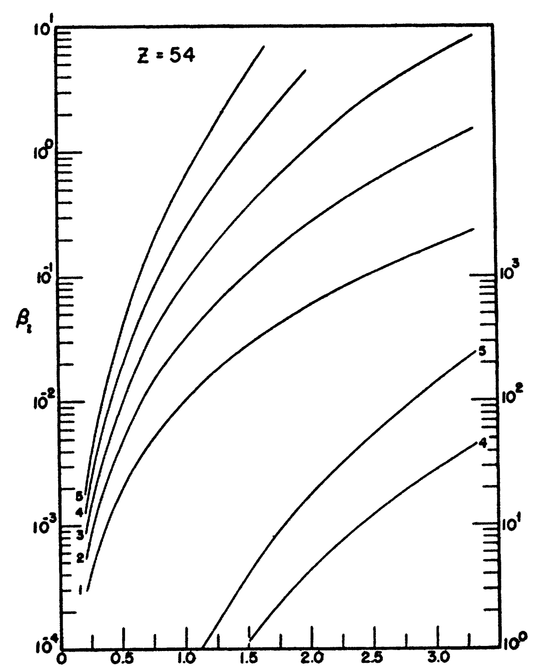

FIG. 8. Same as Fig. 1 but for magnetic conversion and $Z=54$.

[17] M. E. Rose, Phys. Rev. 51, 484 (1937).

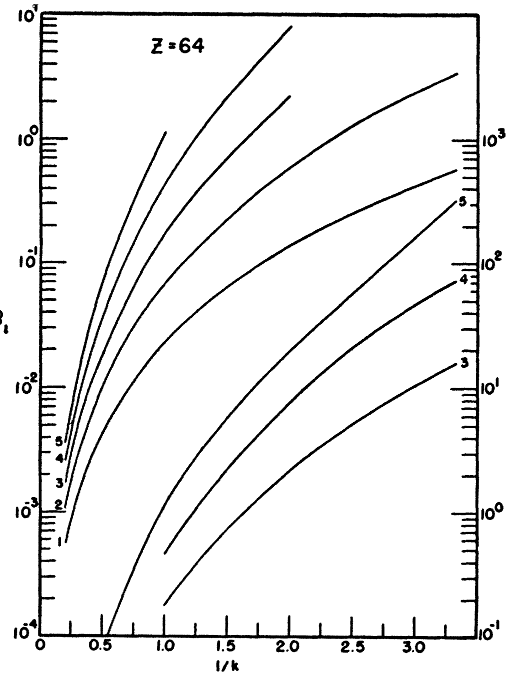

FIG. 9. Same as Fig. 1 but for magnetic conversion and $Z=64$.

tion of the multipole fields.[18] Designating the spherical hankel function by

$$\chi_l(x)=(\pi/2x)^{\frac{1}{2}}H_{l+\frac{1}{2}}^{(1)}(x), \tag{2}$$

where $H^{(1)}$ is the hankel function of the first kind, we

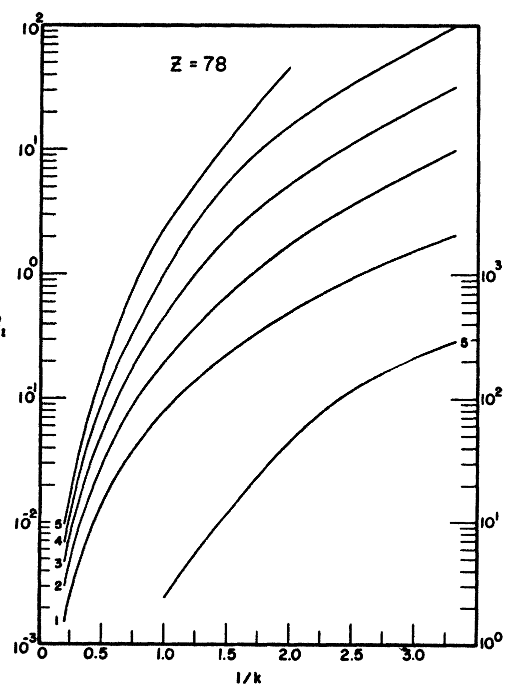

FIG. 10. Same as Fig. 1 but for magnetic conversion and $Z=78$.

[18] See, e.g., V. Berestetzky, J. Phys. U.S.S.R. **11**, 85 (1947); also W. Heitler, Proc. Cambridge Phil. Soc. **32**, 112 (1936).

have for the magnetic 2^l-pole

$$\begin{aligned}\mathbf{A}_l^m&=[2/\pi l(l+1)]^{\frac{1}{2}}\chi_l(kr)\mathbf{L}Y_l^m(\vartheta,\varphi),\\ \phi_l^m&=0,\end{aligned} \tag{3a}$$

where $\mathbf{L}$ is the (orbital) angular momentum operator

$$\mathbf{L}=-i\mathbf{r}\times\text{grad}, \tag{3b}$$

k is the wave number (numerically equal to the energy) of the radiation, and Y_l^m is a normalized spherical harmonic.[17] For electric 2^l-pole radiation

$$\begin{aligned}\mathbf{A}_l^m&=[2/\pi l(l+1)]^{\frac{1}{2}}\chi_{l-1}(kr)[r\,\text{grad}+l\mathbf{r}/r]Y_l^m(\vartheta,\varphi),\\ \phi_l^m&=i[2l/\pi(l+1)]^{\frac{1}{2}}\chi_l(kr)Y_l^m(\vartheta,\varphi).\end{aligned} \tag{3c}$$

With this normalization the number of quanta radiated per unit time is

$$N_q=1/\pi^2k. \tag{3d}$$

It will be noted that l and m correspond to quantum numbers for the angular momentum and z-component thereof for the radiation field.

The internal conversion coefficient is N_e/N_q, and is denoted by α_l and β_l for 2^l-pole electric and magnetic radiation, respectively.

The wave functions can be conveniently expressed in terms of the two-component spinors

$$\Omega(j-\tfrac{1}{2},j;\mu)=\begin{pmatrix}(j+\mu/2j)^{\frac{1}{2}} & Y_{j-\frac{1}{2}}^{\mu-\frac{1}{2}}\\ -(j-\mu/2j)^{\frac{1}{2}} & Y_{j-\frac{1}{2}}^{\mu+\frac{1}{2}}\end{pmatrix} \tag{4a}$$

and

$$\Omega(j+\tfrac{1}{2},j;\mu)=\begin{bmatrix}\left(\dfrac{j-\mu+1}{2j+2}\right)^{\frac{1}{2}} & Y_{j+\frac{1}{2}}^{\mu-\frac{1}{2}}\\ \left(\dfrac{j+\mu+1}{2j+2}\right)^{\frac{1}{2}} & Y_{j+\frac{1}{2}}^{\mu+\frac{1}{2}}\end{bmatrix}. \tag{4b}$$

Then the initial state wave function may be written in the form

$$\psi_i=\begin{pmatrix}if_{l_1j_1}(r)\sigma_r\Omega(1)\\ g_{l_1j_1}(r)\Omega(1)\end{pmatrix}. \tag{5a}$$

In Eq. (5a) and in the following we abbreviate $\Omega(l_n, j_n; m_n)$ by $\Omega(n)$. Here f and g are (real) radial functions, j_1 is the total angular momentum quantum number, m_1 corresponds to the z-component of j_1, and l_1 is either $j_1-\frac{1}{2}$ [Eq. (4a)], or $j_1+\frac{1}{2}$ [Eq. (4b)]. The operator σ_r is given in terms of Pauli spin matrices and can be written as

$$\sigma_r=\begin{pmatrix}\cos\vartheta & \sin\vartheta e^{-i\varphi}\\ \sin\vartheta e^{i\varphi} & -\cos\vartheta\end{pmatrix}, \tag{6}$$

so that it is hermitian and unitary.

In exactly the same way the final state wave function may be expressed in the form[19]

$$\psi_f=\begin{pmatrix}iF_{l_2j_2}(r)\sigma_r\Omega(2)\\ G_{l_2j_2}(r)\Omega(2)\end{pmatrix} \tag{5b}$$

[19] The (real) radial functions F and G are denoted by f and g, respectively, in reference 17.

with j_2 and m_2 interpreted as above; again $l_2=j_2\pm\frac{1}{2}$.

(a) Electric Multipole Conversion

Using

$$\boldsymbol{\alpha}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix}$$

we find that the following angular integrals appear:

$$T_1=(\Omega(2)\,|\,Y_l^m\,|\,\Omega(1)), \tag{7a}$$

$$T_2=(\sigma_r\Omega(2)\,|\,Y_l^m\,|\,\sigma_r\Omega(1)), \tag{7b}$$

$$T_3=(\sigma_r\Omega(2)\,|\,(r\,\mathrm{grad}+l\mathbf{r}/r)Y_l^m\,|\cdot\boldsymbol{\sigma}\Omega(1)), \tag{7c}$$

$$T_4=(\Omega(2)\,|\,\boldsymbol{\sigma}\cdot(r\,\mathrm{grad}+l\mathbf{r}/r)Y_l^m\,|\,\sigma_r\Omega(1)), \tag{7d}$$

where in T_3 and T_4 the grad operator acts on Y_l^m only. By the unitary property of σ_r we find $T_2=T_1$. Using the relations

$$r\sigma_r\boldsymbol{\sigma}\cdot\mathrm{grad}=\mathbf{r}\cdot\mathrm{grad}-\boldsymbol{\sigma}\cdot\mathbf{L},$$

$$\boldsymbol{\sigma}\cdot\mathbf{L}\Omega(i)=[j_i(j_i+1)-l_i(l_i+1)-\tfrac{3}{4}]\Omega(i),\quad i=1,2 \tag{8}$$

we find

$$T_3=T_1[l+j_1(j_1+1)-l_1(l_1+1)-j_2(j_2+1)+l_2(l_2+1)], \tag{9a}$$

$$T_4=T_1[l+j_2(j_2+1)-l_2(l_2+1)-j_1(j_1+1)+l_1(l_1+1)]. \tag{9b}$$

The evaluation of T_1 can be carried out in a straightforward manner. We have

$$T_1=Q(l;\,l_2j_2;\,l_1j_1)S_{j_2mm_1}{}^{(lj_1)}. \tag{10}$$

In Eq. (10) the $S_{l_2mm_1}{}^{(ll_1)}$ are transformation coefficients for vector addition corresponding to the vector addition of l and l_1 with z components m and m_1, respectively, to give the resultant l_2 with z component $m_2=m+m_1$.[20] The Q coefficients are independent of the magnetic quantum numbers. In fact,

$$Q(l;\,l_2j_2;\,l_1j_1)=[S_{j_2,0,\frac{1}{2}}{}^{(l_2\frac{1}{2})}S_{j_1,0,\frac{1}{2}}{}^{(l_1\frac{1}{2})}\times(Y_{l_2}^0\,|\,Y_l^0\,|\,Y_{l_1}^0)S_{j_1,1,-\frac{1}{2}}{}^{(l_1\frac{1}{2})}S_{j_2,1,-\frac{1}{2}}{}^{(l_2\frac{1}{2})}\times(Y_{l_2}^1\,|\,Y_l^0\,|\,Y_{l_1}^1)]\{S_{j_2,0,\frac{1}{2}}{}^{(lj_1)}\}^{-1}. \tag{11}$$

We make use of

$$(Y_{l_2}^{m_2}\,|\,Y_l^m\,|\,Y_{l_1}^{m_1})=\left[\frac{(2l+1)(2l_1+1)}{4\pi(2l_2+1)}\right]^{\frac{1}{2}}\times S_{l_2,0,0}{}^{(ll_1)}S_{l_2,m,m_1}{}^{(ll_1)} \tag{11a}$$

for the integral of three spherical harmonics. The average over m and m_1 and the sum over m_2 gives[20]

$$(2l+1)^{-1}(2j_1+1)^{-1}\sum_{mm_1m_2}|T_1|^2=(2j_2+1)(2l+1)^{-1}(2j_1+1)^{-1}[Q(l;\,l_2j_2;\,l_1j_1)]^2. \tag{12}$$

The internal conversion coefficient can be obtained in terms of radial integrals from (1), (3d), (9), and (12):

$$\alpha_l=\frac{N_e}{N_q}=\frac{2\pi\alpha kl}{(l+1)(2l+1)}U_l, \tag{13}$$

where

$$U_l=(2\pi/l^2)\sum_{l_2j_2}(2j_2+1)[Q(l;\,l_2j_2;\,l_1j_1)]^2\times|\,l(R_1+R_2-R_3+R_4)_{l_2j_2}+[j_2(j_2+1)-l_2(l_2+1)-j_1(j_1+1)+l_1(l_1+1)]\times(R_3+R_4)_{l_2j_2}|^2. \tag{13a}$$

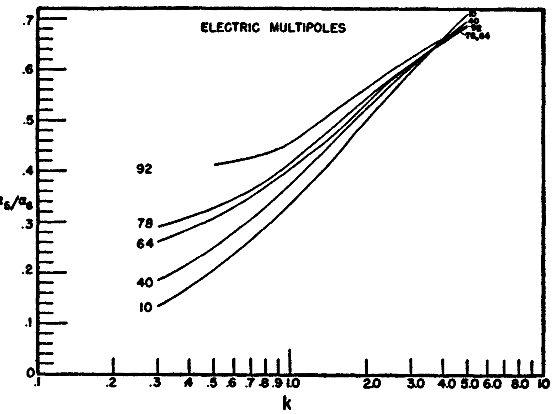

FIG. 11. The ratio of electric 2^5 to electric 2^6 conversion coefficients obtained by extrapolating α_l/α_{l+1}. The attached numbers refer to Z values.

For the K shell ($l_1=0$, $j=\frac{1}{2}$) we have

$$U_l=(l+1)\,|\,R_1+R_2+2R_4\,|^2{}_{l,\,l+\frac{1}{2}}+l\,|\,R_1+R_2-(2+1/l)R_3-R_4/l\,|^2{}_{l,\,l-\frac{1}{2}}, \tag{13b}$$

and $R_1\cdots R_4$ are the radial integrals

$$(R_1)_{l_2j_2}=\int_0^\infty F_{l_2j_2}\chi_l f_{l_1j_1}r^2dr, \tag{13c}$$

$$(R_2)_{l_2j_2}=\int_0^\infty G_{l_2j_2}\chi_l g_{l_1j_1}r^2dr, \tag{13d}$$

$$(R_3)_{l_2j_2}=\int_0^\infty F_{l_2j_2}\chi_{l-1}g_{l_1j_1}r^2dr, \tag{13e}$$

$$(R_4)_{l_2j_2}=\int_0^\infty G_{l_2j_2}\chi_{l-1}f_{l_1j_1}r^2dr. \tag{13f}$$

In (13b) the values of l_2j_2 for which the integrals must

[20] E. Wigner, *Gruppentheorie* (Friedrich Vieweg und Sohn, Braunschweig, 1931), Chapter XVII. The tabulation of these coefficients given in E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1935), pp. 76–77, suffices for all cases considered here. In the latter reference the coefficients are denoted by $(ll_1mm_1|ll_1l_2m_2)$.

be evaluated have been written as subscripts outside the absolute value signs. The selection rules giving the values of l_2 and j_2 over which the sum in (1) must be taken are

$$|l-l_1|\leqslant l_2\leqslant l+l_1,\quad |l-j_1|\leqslant j_2\leqslant l+j_1,\quad l+l_1+l_2 \text{ even}, \tag{14}$$

where the last is the parity selection rule.

(b) Magnetic Multipoles

The calculation for the magnetic multipole proceeds in similar fashion. The angular matrix elements which appear are

$$T_5=(\sigma_r\Omega(2)|\mathbf{L}Y_l^m|\cdot\boldsymbol{\sigma}\Omega(1)) \tag{15a}$$

and

$$T_6=(\Omega(2)|\mathbf{L}Y_l^m|\cdot\boldsymbol{\sigma}\sigma_r\Omega(1)), \tag{15b}$$

which can be transformed to a common form

$$T_5=-T_6=-T_1'[j_2(j_2+1)-l_2(l_2+1)+j_1(j_1+1)-l_1(l_1+1)+\tfrac{1}{2}],$$
$$T_1'=(\sigma_r\Omega(2)|Y_l^m|\Omega(1))=(\Omega(2j_2-l_2,j_2;m_2)|Y_l^m|\Omega(1)).$$

From the selection rules valid for magnetic multipoles

$$|l-l_1|-1\leqslant l_2\leqslant l+l_1+1,\quad |l-j_1|\leqslant j_2\leqslant l+j_1,\quad l+l_1+l_2 \text{ odd} \tag{16}$$

we recognize that the values of $2j_2-l_2$ which enter for the magnetic 2^l-pole are the same as the values of l_2 which enter for the electric 2^l-pole radiation so that (10) and (11) can be used in this case also.

Finally, the magnetic internal conversion coefficient is given by

$$\beta_l=\frac{2\pi\alpha kl}{(l+1)(2l+1)}U_l', \tag{17}$$

where

$$U_l'=(2\pi/l^2)\sum_{l_2j_2}(2j_2+1)[j_2(j_2+1)-l_2(l_2+1)+j_1(j_1+1)-l_1(l_1+1)+\tfrac{1}{2}]^2\times[Q(l;2j_2-l_2,j_2;l_1,j_1)]^2|R_3'+R_4'|^2{}_{l_2j_2} \tag{17a}$$

and

$$(R_3')_{l_2j_2}=\int_0^\infty F_{l_2j_2}\chi_l g_{l_1j_1}r^2dr, \tag{17b}$$

$$(R_4')_{l_2j_2}=\int_0^\infty G_{l_2j_2}\chi_l f_{l_1j_1}r^2dr. \tag{17c}$$

For the K-shell

$$U_l'=(l+1)|R_3'+R_4'|^2{}_{l+1,\,l+\frac{1}{2}}+(l+1)^2/l|R_3'+R_4'|^2{}_{l-1,\,l-\frac{1}{2}}. \tag{18}$$

(c) Radial Integration and Transformation of Results

The radial integrations are carried out exactly as in Hulme's calculation.[2] For the hankel function we use the series representation

$$H_{l+\frac{1}{2}}^{(1)}(x)=(-i)^{l+1}\left(\frac{\pi x}{2}\right)^{-\frac{1}{2}}e^{ix}\sum_{\nu=0}^{l}\frac{(l+\nu)!}{\nu!(l-\nu)!}\left(\frac{i}{2x}\right)^\nu, \tag{19}$$

and for the radial functions of the final state wave function the integral representations[17] are employed. The radial functions for the ground state are

$$f=-(1-\gamma)^{\frac{1}{2}}Dr^{\gamma-1}e^{-\alpha Zr},\quad g=(1+\gamma)^{\frac{1}{2}}Dr^{\gamma-1}e^{-\alpha Zr}, \tag{20}$$

where

$$\gamma=(1-\alpha^2Z^2)^{\frac{1}{2}} \tag{20a}$$

is the total energy (including rest energy) in the K-shell and

$$D=(2\alpha Z)^{\gamma+\frac{1}{2}}[2\Gamma(2\gamma+1)]^{-1} \tag{20b}$$

is a normalization factor.

We introduce the following notation for quantities occurring in the final state wave function:

$$\gamma'=[(j_2+\tfrac{1}{2})^2-\alpha^2Z^2]^{\frac{1}{2}}, \tag{21a}$$

$$N_{j_2}{}^2=\frac{e^{\pi\xi}(p/2)^{2\gamma'}}{4\pi p|\Gamma(\gamma'+i\xi)|^2}, \tag{21b}$$

$$\xi=\alpha ZW/p, \tag{21c}$$

where $p(>0)$ is the final state momentum and $W=(p^2+1)^{\frac{1}{2}}$ is the final state energy (including rest energy). Then we find for the electric multipole conversion (omitting common factors of modulus unity)

$$(R_1)_{l,\,l\pm\frac{1}{2}}=-i(W-1)^{\frac{1}{2}}(1-\gamma)^{\frac{1}{2}}DN_{l\pm\frac{1}{2}}\times[K_{l,\,l\pm\frac{1}{2}}-\exp(-2i\eta_\pm)K^*{}_{l,\,l\pm\frac{1}{2}}], \tag{22a}$$

$$(R_2)_{l,\,l\pm\frac{1}{2}}=(W+1)^{\frac{1}{2}}(1+\gamma)^{\frac{1}{2}}DN_{l\pm\frac{1}{2}}\times[K_{l,\,l\pm\frac{1}{2}}+\exp(-2i\eta_\pm)K^*{}_{l,\,l\pm\frac{1}{2}}], \tag{22b}$$

$$(R_3)_{l,\,l\pm\frac{1}{2}}=-(W-1)^{\frac{1}{2}}(1+\gamma)^{\frac{1}{2}}DN_{l\pm\frac{1}{2}}\times[K_{l-1,\,l\pm\frac{1}{2}}-\exp(-2i\eta_\pm)K^*{}_{l-1,\,l\pm\frac{1}{2}}], \tag{22c}$$

$$(R_4)_{l,\,l\pm\frac{1}{2}}=-i(W+1)^{\frac{1}{2}}(1-\gamma)^{\frac{1}{2}}DN_{l\pm\frac{1}{2}}\times[K_{l-1,\,l\pm\frac{1}{2}}+\exp(-2i\eta_\pm)K^*{}_{l-1,\,l\pm\frac{1}{2}}], \tag{22d}$$

and for the magnetic conversion

$$(R_3')_{l',\,l'\pm\frac{1}{2}}=-(W-1)^{\frac{1}{2}}(1+\gamma)^{\frac{1}{2}}DN_{l\mp\frac{1}{2}}\times[K_{l,\,l\mp\frac{1}{2}}-\exp(-2i\eta_\pm)K^*{}_{l,\,l\mp\frac{1}{2}}], \tag{22e}$$

$$(R_4')_{l',\,l'\pm\frac{1}{2}}=-i(W+1)^{\frac{1}{2}}(1-\gamma)^{\frac{1}{2}}DN_{l\mp\frac{1}{2}}\times[K_{l,\,l\mp\frac{1}{2}}+\exp(-2i\eta_\pm)K^*{}_{l,\,l\mp\frac{1}{2}}], \tag{22f}$$

where the $\pm$ in the second subscript in R_3', R_4' goes with $l'=l\mp 1$. Note that for both values of l', $2j-l'=l$. In (22) we have introduced

$$\exp(-2i\eta_\pm)=\frac{\gamma'+i\xi}{\pm(j+\frac{1}{2})+i\alpha Z/p} \quad (23)$$

and the value to be used for j is the same as the second subscript in K. Also K^* is obtained from K by changing the sign of p in the formulas given below:

$$K_{\lambda,j}=\sum_{\nu=0}^{\lambda}\frac{(\lambda+\nu)!}{(\lambda-\nu)!}E_{j\nu}, \quad (24a)$$

$$K_{\lambda,j}^*=\sum_{\nu=0}^{\lambda}\frac{(\lambda+\nu)!}{(\lambda-\nu)!}E_{j\nu}^*, \quad (24b)$$

and the $E_{j\nu}$, $E_{j\nu}^*$ are defined by

$$\begin{pmatrix}E_{j\nu}\\ E_{j\nu}^*\end{pmatrix}=\left(-\frac{p}{2k\zeta}\right)^\nu\frac{\Gamma(\gamma+\gamma'-\nu)}{\nu!\Gamma(2\gamma'+1)}\times\frac{2^{2\gamma'}}{k}\left(\frac{\zeta}{ip}\right)^{\gamma+\gamma'}|\Gamma(\gamma'+i\xi)|^2\times\begin{pmatrix}(\gamma'+i\xi)F(\gamma'+\gamma-\nu,\gamma'+1+i\xi;2\gamma'+1;2\zeta)\\ (\gamma'-i\xi)F(\gamma'+\gamma-\nu,\gamma'+i\xi;2\gamma'+1;2\zeta)\end{pmatrix}, \quad (25)$$

where

$$\zeta=ip/[\alpha Z+i(p-k)] \quad (25a)$$

and F is the hypergeometric function

$$F(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!}. \quad (25b)$$

In (25) we have used the relations

$$F(a,b;c;z)=(1-z)^{-a}F(a,c-b;c;z/(z-1)) \quad (25c)$$

and with

$$\zeta^*=-ip/[\alpha Z-i(k+p)]$$

we have

$$2\zeta^*/(2\zeta^*-1)=2\zeta. \quad (25d)$$

The result (25) applies only in the circle of convergence: $|2\zeta|<1$. Applying the conservation of energy

$$k+\gamma=W \quad (26)$$

it is seen that the region $|2\zeta|<1$ applies only in a narrow band of the $Z-k$ plane near the K-threshold. For all cases considered in the numerical work the analytic continuation of (25) is required; that is, we use

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}F(a,b;c;z)=\frac{\Gamma(a)\Gamma(a-b)}{\Gamma(a-c)}(-z)^{-a}F(a,1-c+a;1-b+a;1/z)+\frac{\Gamma(b)\Gamma(b-a)}{\Gamma(b-c)}(-z)^{-b}F(b,1-c+b;1-a+b;1/z). \quad (27)$$

Applied to E_{j0} and E_{j0}^* this gives

$$\begin{pmatrix}E_{j0}\\ E_{j0}^*\end{pmatrix}=\frac{2^{2\gamma'}}{k}\left(\frac{\zeta}{ip}\right)^{\gamma+\gamma'}\Bigg\{(-2\zeta)^{-\gamma-\gamma'}\frac{\Gamma(\gamma'+\gamma)\Gamma(-\gamma+i\xi)\Gamma(\gamma'-i\xi)}{\Gamma(\gamma'-\gamma+1)}\begin{pmatrix}(-\gamma+i\xi)F(\gamma'+\gamma,\gamma-\gamma';\gamma-i\xi;1/2\zeta)\\ (\gamma'-i\xi)F(\gamma'+\gamma,\gamma-\gamma';\gamma+1-i\xi;1/2\zeta)\end{pmatrix}$$
$$+(-2\zeta)^{-\gamma'-i\xi}\Gamma(\gamma'+i\xi)\Gamma(\gamma-1-i\xi)\begin{pmatrix}(\gamma'+i\xi)(-2\zeta)^{-1}F(\gamma'+1+i\xi,1-\gamma'+i\xi;2-\gamma+i\xi;1/2\zeta)\\ (\gamma-1-i\xi)F(\gamma'+i\xi,-\gamma'+i\xi;1-\gamma+i\xi;1/2\zeta)\end{pmatrix}\Bigg\}, \quad (28)$$

where we require $|2\zeta|>1$, $|\arg(-2\zeta)|<\pi$ and

$$|\arg(\zeta/ip)|<\pi/2.$$

The task of obtaining $E_{j\nu}$ and $E_{j\nu}^*$ for $\nu\neq 0$ is greatly simplified by the use of recurrence relations. With the aid of

$$(c-a)F(a-1,b;c;z)=b(1-z)F(a,b+1;c;z)+(c-b-a)F(a,b;c;z)$$

and

$$(c-a)F(a-1,b+1;c;z)=(b-a+1)(1-z)F(a,b+1;c;z)+(c-b-1)F(a,b;c;z)$$

we find the following recurrence relation which we write in matrix form:

$$\begin{pmatrix}E_{j,\nu+1}\\ E^*_{j,\nu+1}\end{pmatrix}=\frac{(-p/2k\zeta)}{(\nu+1)(\gamma+\gamma'+\nu-1)(\gamma'-\gamma+\nu+1)}\times\begin{pmatrix}\nu+1-\gamma+i\xi & \gamma'+i\xi\\ \gamma'-i\xi & \nu+1-\gamma-i\xi\end{pmatrix}\begin{pmatrix}(1-2\zeta)E_{j\nu}\\ E_{j\nu}^*\end{pmatrix}. \quad (29)$$

For all cases considered $|2\xi|>1$ so that (28) and (29) determine all the $E_{j\nu}$, $E_{j\nu}^*$ and from (13), (17), (18), (22), and (24) the internal conversion coefficients are obtained. Some further simplification may be made by introducing the transformations discussed by Gellman *et al.*[7]