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The Scattering of Light by Light*

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Cross sections for several processes involving electromagnetic fields in a nonlinear manner are derived from the electrodynamic scattering matrix and are expressed in terms of the fourth-order nonlinear vacuum polarization tensor. The differential cross section for the scattering of light by light is calculated as a function of energy and angle. Numerical values are given for scattering at zero and at ninety degrees in the center-of-mass system. Near 1.75 Mev the forward scattering cross section has its largest value of 4.1×10^{-31} cm^2/sterad, while the maximum right-angle scattering takes place near 0.7 Mev with a cross section of 2.8×10^{-31} cm^2/sterad, all for unpolarized radiation. Numerical results are also given for scattering at the above angles between circularly polarized states. The conclusions in this paper agree with all the results previously calculated for special cases.

IN an earlier paper[1] the nonlinear interactions between electromagnetic fields were expressed in terms of the polarization of the electron-positron vacuum. In this discussion the polarization tensor derived in I will be related to the cross sections for the occurrence of events which are consequences of the nonlinearity.

The processes to be considered are the scattering of light by light,[2] two-quantum pair creation,[3] the scattering of light in an external field,[4] and the creation of pairs in an external field.[5] An expression for the cross section for each of these is set up in Sec. II. The two cases involving only radiation fields are then treated in detail. A certain simplification occurs in the calculation of the appropriate element of the scattering matrix and of the polarization tensor, because the fact that all the 4-momentum vectors involved are null-vectors, together with the conservation laws, implies that the matrix element depends on the initial and final momenta only by way of two parameters; these may be taken as the incident photon energy and the scattering angle in the center-of-gravity system. The total cross section for two-quantum pair creation is easily obtained from the diagonal (forward scattering) element of the S-matrix, as has been described elsewhere;[5b] this relationship follows from the unitary character of the S-matrix.

To obtain the probability for the scattering of a photon by an external field, the electromagnetic potential in the S-matrix is replaced by the potential of a quantized wave (creation or annihilation operator) plus the external potential which causes the scattering. The matrix element of this operator between one-photon states of the appropriate momenta then is the probability amplitude of the scattering process. Here the evaluation of the polarization tensor is much more difficult because the momentum vectors are not null vectors. The forward scattering matrix element again is simply related to the total cross section for pair-creation in the external field.[5b]

Formulas for the differential cross sections for transitions between photon states of definite linear momentum and definite spin angular momentum (circular polarization) are given to summarize the results of the calculation of the scattering of light by light. This choice is convenient because the transitions fall into two groups: the more probable ones conserve spin angular momentum, while the less probable ones do not. In the limiting cases of low and high energy our results agree with the published ones.[2]

I. SCATTERING CROSS SECTION

The total probability W that a transition takes place from an initial state Ψ_i is given in terms of the scattering matrix S by

$$W=(\Psi_i,\,(S^+-1)(S-1)\Psi_i)$$
$$=-2\,\mathrm{Re}[(\Psi_i,\,(S-1)\Psi_i)];\quad (1)$$

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the second part of the identity is a consequence of the unitarity of the S-matrix. If not all transitions from Ψ_i are being considered, then the probability W' for the occurrence of a process that is of interest may be expressed in terms of a part S' of the scattering matrix by

$$W'=(\Psi_i, S'^{+}S'\Psi_i)=\sum_f(\Psi_i, S'^{+}\Psi_f)(\Psi_f, S'\Psi_i), \quad (2)$$

if S' is an operator whose matrix elements are only the relevant scattering amplitudes. S' need no longer be unitary, of course.

A. Two-Photon Scattering

In the calculation of the scattering cross section of photons by photons, the states Ψ_i and Ψ_f refer each to the presence of two light quanta. Hence, to the lowest order in the coupling constant, the operator S' of Eq. (2) is equal to the operator $S^{(4)}$ defined in Eq. (11) of reference I if the quantities $A_\mu(k^{(i)})$ are taken to be the quantized electromagnetic field variables and if the vacuum and single-particle parts are removed. Let the initial state be characterized by the presence of two quanta with momenta $\mathbf{P}_1=h\kappa\mathbf{p}$ and $\mathbf{P}_2=-h\kappa\mathbf{p}$, energies $c|\mathbf{P}_1|=c|\mathbf{P}_2|=hc\kappa\omega=mc^2\omega$, and polarization vectors $e_\mu^{\lambda_1}(-\mathbf{p})$ and $e_\mu^{\lambda_2}(-\mathbf{p})$, respectively, inclosed in a large but finite volume V. The expectation value indicated in Eq. (2) can now be easily evaluated with the help of the usual substitution

$$A_\mu(\mathbf{k}, k_0)\rightarrow V(2|\mathbf{k}|V)^{-\frac{1}{2}}[e_\mu^{\lambda}(\mathbf{k})a^{\lambda}(\mathbf{k})\delta(k_0-|\mathbf{k}|)+e_\mu^{\lambda*}(\mathbf{k})a^{\lambda*}(-\mathbf{k})\delta(k_0+|\mathbf{k}|)], \quad \left(\int d\mathbf{k}\rightarrow(1/V)\sum_{\mathbf{k}}\right), \quad (3)$$

where $a^{\lambda}(\mathbf{k})$ annihilates and $a^{\lambda*}(-\mathbf{k})$ creates a photon of momentum $\mathbf{k}$ and polarization $e_\mu^{\lambda}(\mathbf{k})$. The many terms that arise from different identifications of $\mathbf{P}_i$ with the $\mathbf{k}^{(i)}$ all give the same contributions because of the symmetry of the polarization tensor.

The matrix element of the operator part of $S^{(4)}$ to be evaluated is

$$Q=(P_1, \lambda_1; P_2, \lambda_2|(A_{\mu'}^{1'}A_{\nu'}^{2'}A_{\lambda'}^{3'}A_{\sigma'}^{4'})_2^{+}\times(A_\mu^1A_\nu^2A_\lambda^3A_\sigma^4)_2|P_1, \lambda_1; P_2, \lambda_2),$$

where $A_\mu(k^{(i)})$ is abbreviated by A_μ^i. The symbol $(\quad)_2$ denotes the two-particle part of the enclosed expression: the operator is to be decomposed into products of annihilation and creation operators; then the terms containing two factors of each are taken with the annihilation operators standing on the right. This operator, therefore, annihilates two particles and creates two others; in particular, it transforms a two-particle state into another two-particle state via the vacuum state. The latter fact suggests that Q be factored as follows:

$$\begin{aligned}Q&=\sum_{\text{perm}}(P_1, \lambda_1; P_2, \lambda_2|(A_{\mu'}^{1'}A_{\nu'}^{2'})^{+}(A_\mu^1A_\nu^2)|P_1, \lambda_1; P_2, \lambda_2)\times(0|(A_{\lambda'}^{3'}A_{\sigma'}^{4'})^{+}(A_\lambda^3A_\sigma^4)|0)\\&=\sum_{\text{perm}}(P_1; \lambda_1|A_{\mu'}^{1'+}A_\mu^1|P_1, \lambda_1)(P_2, \lambda_2|A_{\nu'}^{2'+}A_\nu^2|P_2\lambda_2)\times(0|A_{\lambda'}^{3'+}A_\lambda^3|0)(0|A_{\sigma'}^{4'+}A_\sigma^4|0).\end{aligned}$$

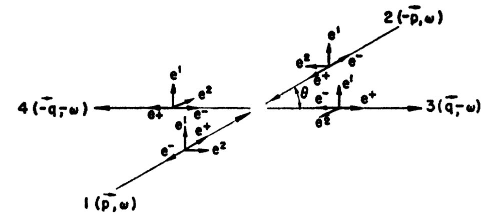

FIG. 1. Definition of the initial and final states in terms of the wave numbers $\mathbf{p}$ and $\mathbf{q}$ and the polarization vectors $\mathbf{e}^{\lambda}$. The directions of the vectors $\mathbf{e}^{\lambda}$ for circular polarization are those of the angular momenta associated with the rotation.

The sums are respectively over the 36 and 288 distinct permutations of the primed and unprimed indices separately. These all make the same contribution, however, because they are multiplied by the symmetrical function $G_{\mu\nu\lambda\sigma}(1234)$. Hence

$$\begin{aligned}Q\rightarrow288&[e_{\mu'}^{\lambda_1*}e_\mu^{\lambda_1}\delta(k^{(1)}-P_1)\delta(k^{(1')}-P_1)/2V|P_1|]\\&\times[e_{\nu'}^{\lambda_2*}e_\nu^{\lambda_2}\delta(k^{(2)}-P_2)\delta(k^{2'}-P_2)/2V|P_2|]\\&\times[\delta_{\lambda\lambda'}\delta(k^{(3)}-k^{(3')})\delta(k_0^{(3)}+|\mathbf{k}^{(3)}|)/2(2\pi)^3|\mathbf{k}^{(3)}|]\\&\times[\delta_{\sigma\sigma'}\delta(k^{(4)}-k^{(4')})\delta(k_0^{(4)}+|\mathbf{k}^{(4)}|)/2(2\pi)^3|\mathbf{k}^{(4)}|].\end{aligned}$$

The probability W' then becomes

$$W_s'=\left[\frac{2}{V^2}\int d^4x\right]\frac{\alpha^4}{128\pi^2}\frac{1}{\omega^2\kappa^2}\int d\Omega(\mathbf{k})\left|e_\mu^{\lambda_1}e_\nu^{\lambda_2}G_{\mu\nu\lambda\sigma}\left(\mathbf{p}, \omega; -\mathbf{p}, \omega; \frac{\mathbf{k}}{\kappa}, -\omega; -\frac{\mathbf{k}}{\kappa}, -\omega\right)\right|^2. \quad (4)$$

Over-all momentum and energy conservation have reduced the number of independent arguments of G, whose space and time components have been written separately for greater clarity; the scale factor κ has been dropped from the arguments because G is dimensionless and, therefore, homogeneous of degree zero in κ. The factor $[(2/V^2)\int d^4x]$ is interpreted as $(2c/V)T$, the incident photon flux multiplied by the time during which transitions have been taking place.[6] The total cross section is, therefore, given by the right side of Eq. (4) with this factor omitted.

The differential cross section for scattering into the state containing one photon with momentum in the solid angle $d\Omega$ around the direction of $\mathbf{P}_3=h\kappa\mathbf{q}$ and polarization $e_\mu^{\lambda_3}$, while the other photon has momentum $-\mathbf{P}_3$ and polarization $e_\mu^{\lambda_4}$, is clearly

$$\sigma_s(\theta, \phi; \omega)=\frac{\alpha^4}{4\pi^2\kappa^2}\frac{1}{16\omega^2}|e_\mu^{\lambda_1}e_\nu^{\lambda_2}e_\lambda^{\lambda_3*}e_\sigma^{\lambda_4*}G_{\mu\nu\lambda\sigma}(\mathbf{p}, \omega; -\mathbf{p}, \omega; -\mathbf{q}, -\omega; \mathbf{q}, -\omega)|^2; \quad (5)$$

energy conservation requires that the scattered quanta still have the energy $hc\kappa\omega=hc\kappa|\mathbf{q}|$, as is indicated by the arguments of the polarization tensor in Eqs. (4) and (5).

We now introduce some conventions with regard to the unit vectors specifying states of polarization. The

[6] B. A. Lippmann and J. Schwinger, Phys. Rev. 79, 469 (1950).

symbols $e_\mu^1(P)$ and $e_\mu^2(P)$ will denote spacelike unit vectors lying perpendicular to and in the scattering plane, respectively. The set $(\mathbf{e}^1, \mathbf{e}^2, \mathbf{P})$ forms a right-handed system. The unit vectors for right and left handed circular polarization are given by $e^+ = 2^{-\frac{1}{2}}(e^1 + ie^2)$ and $e^- = 2^{-\frac{1}{2}}(e^1 - ie^2)$, respectively. For simplicity in writing, the following abbreviation will now be introduced:

$$M_{\lambda_1\lambda_2\lambda_3\lambda_4}(\theta, \omega) = \tfrac{1}{4} e_\mu^{\lambda_1} e_\nu^{\lambda_2} e_\lambda^{\lambda_3 *} e_\sigma^{\lambda_4 *} G_{\mu\nu\lambda\sigma}(\mathbf{p}, \omega; -\mathbf{p}, \omega; -\mathbf{q}, -\omega; \mathbf{q}, -\omega), \quad (6)$$

where

$$\theta = \cos^{-1}(\mathbf{p}\cdot\mathbf{q}/\omega^2)$$

is the angle of scattering in the scattering plane. Figure 1 illustrates the symbols that have been introduced.

An examination of I, Eqs. (21) to (24) indicates the possibility of five distinct scattering configurations in the case of linearly and circularly polarized light. In the former, these correspond to M_{1111}, M_{2222}, M_{1122}, M_{1221} and M_{1212}, with weight factors 1, 1, 2, 2, and 2; in the latter they are M_{++++}, M_{++--}, M_{+-+-}, M_{+--+}, and M_{+++-} with weights 2, 2, 2, 2, and 8. The differential scattering cross section for unpolarized radiation is then given by two alternative formulas

$$\bar{\sigma}_s(\theta, \omega) = \frac{\alpha^4}{4\pi^2\kappa^2}\frac{1}{\omega^2}\langle |M|^2\rangle_{\text{Av}},$$

$$\begin{aligned}\langle |M|^2\rangle_{\text{Av}} &= \tfrac{1}{4}[|M_{1111}|^2 + |M_{2222}|^2 + 2|M_{1122}|^2 + 2|M_{1221}|^2 + 2|M_{1212}|^2] \\ &= \tfrac{1}{2}[|M_{++++}|^2 + |M_{++--}|^2 + |M_{+-+-}|^2 + |M_{+--+}|^2 + 4|M_{+++-}|^2].\end{aligned} \quad (7)$$

B. Two-Photon Pair Creation

In the treatment of this process, the states Ψ_i and Ψ_f, Eq. (2) refers to the presence of two photons and of an electron-positron pair, respectively. It is easy to verify[5b] that to the lowest order in the coupling constant the matrix S' appropriate to pair production is equal to S, so that Eq. (1) may be used. The terms of order α^2 in this equation then give

$$W_p = -2\,\text{Re}[(\psi_i, S^{(4)}\psi_i)] = -\left[\frac{2}{V^2}\int d^4x\right]\frac{2\alpha^2}{\kappa^2\omega^2}\,\text{Im}[M_{\lambda_1\lambda_2\lambda_1\lambda_2}(0,\omega)] \quad (8)$$

for the total probability that a pair be produced by two photons of momenta $\mathbf{p}_1 = \hbar\kappa\mathbf{p}$, $\mathbf{p}_2 = -\hbar\kappa\mathbf{p}$, energies $c|\mathbf{p}_1| = c|\mathbf{p}_2| = \hbar c\kappa\omega$, and polarizations $e_\mu^{\lambda_1}$, $e_\mu^{\lambda_2}$. The total cross section is therefore

$$\sigma(\omega) = -2(\alpha^2/\kappa^2\omega^2)\,\text{Im}[M_{\lambda_1\lambda_2\lambda_1\lambda_2}(0,\omega)]. \quad (9)$$

C. Scattering of Photons by an External Static Field

Here again, S' of Eq. (2) is equal to $S^{(4)}$, but now the quantities $A_\mu(k^{(i)})$ must be taken as the sum of a quantized wave amplitude $A_\mu(k^{(i)})$ and a fourier component of the external field $A_\mu^e(k^{(i)})$. Only the one-particle part is to be used. The quantized $A_\mu(k^{(i)})$ are again expressed as creation and annihilation operators, Eq. (3), while the three-dimensional fourier transform $\bar{A}_\mu^e$ can be introduced by setting

$$A_\mu^e(k) = \kappa^{-3}\bar{A}_\mu^e(\mathbf{k}/\kappa)\delta(k_0/\kappa). \quad (10)$$

In terms of this quantity, then

$$W_s'^e = \left[\frac{\int dx_0}{V}\right]\frac{(2\pi)^4\alpha^4}{4\kappa^2}\int d\Omega(\mathbf{k}_2) \times\left|\int d\mathbf{k}_3 d\mathbf{k}_4 e_\mu^{\lambda_1}(\mathbf{p})\bar{A}_\lambda^e(\mathbf{k}_3)\bar{A}_\sigma^e(\mathbf{k}_4) \times G_{\mu\nu\lambda\sigma}(\mathbf{p},\omega;\mathbf{k}_2,-\omega;\mathbf{k}_3,0;\mathbf{k}_4,0) \times\delta(\mathbf{p}-\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\right|^2. \quad (11)$$

Dropping the factor $[\int dx_0/V]$ yields the total cross section, while the differential cross section for scattering into the solid angle $d\Omega$ around the direction $-\mathbf{k}_2 = \mathbf{q}$ and to the polarization $e_\mu^{\lambda_2}(\mathbf{q})$ is

$$\sigma_s^e(\theta,\omega) = 4\pi^4\frac{\alpha^4}{\kappa^2}\left|\int d\mathbf{R}\,e_\mu^{\lambda_1}(\mathbf{p})e_\nu^{\lambda_2 *}(\mathbf{q}) \times\bar{A}_x^e\left(-\frac{\mathbf{P}}{2}+\mathbf{R}\right)\bar{A}_\sigma^e\left(-\frac{\mathbf{P}}{2}-\mathbf{R}\right) \times G_{\mu\nu\lambda\sigma}\left(\mathbf{p},\omega;-\mathbf{q},-\omega;-\frac{\mathbf{P}}{2}+\mathbf{R},0;-\frac{\mathbf{P}}{2}-\mathbf{R},0\right)\right|^2, \quad (12)$$

where $\mathbf{P} = \mathbf{p}-\mathbf{q}$ and $\theta = \cos^{-1}(\mathbf{p}\cdot\mathbf{q}/\omega^2)$ are the momentum transfer and scattering-angle, respectively. The basic polarization vectors are again taken in the scattering plane and perpendicular to it, for convenience.

D. Pair-Creation by Photons in an External Static Field

By an argument corresponding to the one in B, the total cross section for the production of electron-positron pairs by a quantum of energy $\hbar c\kappa\omega$ and polarization e_μ^λ is

$$\sigma_p^e(\omega) = -\frac{4\pi^3\alpha^2}{\kappa^2\omega}\,\text{Im}\left[\int d\mathbf{R}\,e_\mu^\lambda e_\nu^{\lambda *}\bar{A}_\lambda^e(\mathbf{R})\bar{A}_\sigma^e(-\mathbf{R}) \times G_{\mu\nu\lambda\sigma}(\mathbf{p},\omega;-\mathbf{p},-\omega;\mathbf{R},0;-\mathbf{R},0)\right]. \quad (13)$$

II. CONSTRUCTION OF THE POLARIZATION TENSOR

In this section we shall evaluate the five distinct invariants that determine the polarization tensor of quantized fields by calculating the eleven quantities $A_1^n(1234)$ (I, Eqs. (30), (31), (33), (34), and (35)) and applying I, Eqs. (36)–(41). The denominator $D(1234)$ (I, Eq. (27)) simplifies greatly when the values of the four momenta from Eq. (5) are introduced. Thus, in terms of the parameters

$$\begin{aligned}
\alpha &= -(k^{(1)}k^{(2)})/2\kappa^2 = -(k^{(3)}k^{(4)})/2\kappa^2 = \omega^2,\\
\beta &= -(k^{(1)}k^{(3)})/2\kappa^2 = -(k^{(2)}k^{(4)})/2\kappa^2 = -\omega^2\sin^2(\theta/2),\\
\gamma &= -(k^{(1)}k^{(4)})/2\kappa^2 = -(k^{(2)}k^{(3)})/2\kappa^2 = -\omega^2\cos^2(\theta/2),\\
&\alpha+\beta+\gamma=0,
\end{aligned} \tag{14}$$

we have

$$\begin{aligned}
&D(1234)=D(3412)=D(2143)=D(4321)=D(\alpha,\gamma),\\
&D(2341)=D(4123)=D(1432)=D(3214)=D(\gamma,\alpha),\\
&D(1243)=D(4312)=D(2134)=D(3421)=D(\alpha,\beta),\\
&D(2431)=D(3124)=D(4213)=D(1342)=D(\beta,\alpha),\\
&D(1324)=D(2413)=D(3142)=D(4231)=D(\beta,\gamma),\\
&D(3241)=D(4132)=D(2314)=D(1423)=D(\gamma,\beta);\\
&D(u,v)=[1-i\epsilon-4(uy_2y_4+vy_1y_3)]^{-2}.
\end{aligned} \tag{15}$$

To carry out the integration over the hyperplane defined in I, Eq. (25), it is convenient to introduce three new variables by relations like these:

$$\begin{aligned}
y_1&=(1-x)(1-y), & y_3&=(1-z)y,\\
y_2&=x(1-y), & y_4&=zy.
\end{aligned} \tag{16}$$

Then integrals of the form

$$\int_{\Sigma y_i=1,\, y_i>0} d\tau P(y)D(u,v)\rightarrow\int_0^1 dy\cdot y(1-y)\int_0^1 dx \times\int_0^1 dz P'(x,y,z)D'(u,v), \tag{17}$$

$$D'(u,v)=\{1-i\epsilon-4y(1-y)\times[uxz+v(1-x)(1-z)]\}^{-2},$$

where P and P' are polynomials, must be evaluated. One can deduce useful algebraic relationships among many of these by equating two expressions that contain different functions P' obtained by different permutations of the substitutions Eq. (16) applied to the same function $P(y_i)$ in the integrand.

Straightforward integration, together with the judicious use of the relationships just mentioned, yields lengthy expressions for the $A_1^n(1234)$, and so for the $A^n(1234)$, as functions of α, β, and γ. They are listed in Appendix A. It is useful to give special designations to the three transcendental functions that appear in the A^n in addition to rational functions of α, β, and γ. They are

$$\begin{aligned}
B(u)&=\frac{1}{2}\int_0^1 dy\log[1-i\epsilon-4uy(1-y)]\\
&=\left(\frac{u-1}{u}\right)^{\frac{1}{2}}\sinh^{-1}(-u)^{\frac{1}{2}}-1;\quad u<0\\
&=\left(\frac{1-u}{u}\right)^{\frac{1}{2}}\sin^{-1}u^{\frac{1}{2}}-1;\quad 0<u<1\\
&=\left(\frac{u-1}{u}\right)^{\frac{1}{2}}\cosh^{-1}u^{\frac{1}{2}}-1-\frac{\pi i}{2}\left(\frac{u-1}{u}\right)^{\frac{1}{2}};\quad 1<u,
\end{aligned} \tag{18}$$

$$\begin{aligned}
T(u)&=\int_0^1\frac{dy}{4y(1-y)}\log[1-i\epsilon-4uy(1-y)]\\
&=[\sinh^{-1}(-u)^{\frac{1}{2}}]^2;\quad u<0\\
&=-[\sin^{-1}u^{\frac{1}{2}}]^2;\quad 0<u<1\\
&=[\cosh^{-1}u^{\frac{1}{2}}]^2-\tfrac{1}{4}\pi^2-i\pi\cosh^{-1}u^{\frac{1}{2}};\quad 1<u
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
I(u,v)=I(v,u)&=\int_0^1\frac{dy}{4y(1-y)-(u+v)/uv}\\
&\times\{\log[1-i\epsilon-4uy(1-y)]\\
&\quad+\log[1-i\epsilon-4vy(1-y)]\}
\end{aligned} \tag{20}$$

with the real part

$$\begin{aligned}
\mathrm{Re}[I(u,v)]=\frac{1}{2a}\mathrm{Re}\Bigg[&\phi\left(\frac{a+1}{a+b(u)}\right)+\phi\left(\frac{a+1}{a-b(u)}\right)\\
&-\phi\left(\frac{a-1}{a+b(u)}\right)-\phi\left(\frac{a-1}{a-b(u)}\right)\\
&+\phi\left(\frac{a+1}{a+b(v)}\right)+\phi\left(\frac{a+1}{a-b(v)}\right)\\
&-\phi\left(\frac{a-1}{a+b(v)}\right)-\phi\left(\frac{a-1}{a-b(v)}\right)\Bigg],
\end{aligned} \tag{20'}$$

where

$$\begin{aligned}
a&=[1-(u+v)/uv]^{\frac{1}{2}},\\
b(u)&=[(u-1)/u]^{\frac{1}{2}},\quad u<0,\quad u>1\\
&=i[(1-u)/u]^{\frac{1}{2}},\quad 0<u<1
\end{aligned} \tag{21}$$

and[7]

$$\phi(z)=\int_0^z\log(1-t)dt/t; \tag{22}$$

[7] The Spence function $\phi(z)$ has been encountered in electrodynamic problems by G. Racah, Nuovo cimento **11**, No. 7 (1939). It is discussed at length by K. Mitchell, Phil. Mag. **40**, 351 (1949), who also gives tables of numerical values.

TABLE I. Expression of the matrix element $M(\pi/2, \omega)$ in terms of the basic invariants A^n for transitions between circularly polarized states.

	$A^{2143}(1234)$	$A^{2143}(1324)$	$A^{2341}(1234)$	$A^{2341}(1324)$	$A^{2111}(1234)$	$A^{2111}(1432)$	$A^{2121}(1234)$	$A^{2121}(1432)$	$A^{2121}(1423)$	$A^{2311}(1234)$
$M_{++++}/\omega^4\kappa^4$	1	$\frac{1}{2}$	2	$-\frac{1}{2}$	0	0	-2	-2	1	-4
$M_{++--}/\omega^4\kappa^4$	1	0	2	1	0	0	-2	0	0	0
$M_{+-+-}/\omega^4\kappa^4$ $=M_{+--+}/\omega^4\kappa^4$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0	0	0	-1	$\frac{1}{2}$	0
$M_{+++-}/\omega^4\kappa^4$	0	0	0	0	$-\frac{1}{2}$	2	$\frac{1}{2}$	-1	-1	1

u and v are two distinct members of the triplet (α, β, γ). It should be noted that $(u+v)/uv \leqslant 0$ then follows from the values of α, β, and γ in Eq. (14). Hence care must be taken in the integration only if the logarithms have branch points, only, therefore, if u or v is greater than one. In this case, the imaginary part of $I(u, v)$ is found to be

$$\begin{aligned}\mathrm{Im}[I(u, v)] &= -(\pi/2a)\log[v(a+b(u))^2], \quad u \geqslant 1 \\ &= -(\pi/2a)\log[u(a+b(v))^2], \quad v \geqslant 1.\end{aligned} \tag{20''}$$

Some numerical values of these three functions and their behavior in the limits of large and small arguments are given in Appendix B and Table IV.

It is clear from the discussion of the scattering cross section that the imaginary parts of the functions B, T, and I are connected with the contribution of real intermediate pairs to the cross section. The consistency of this interpretation follows from the fact that the imaginary parts are zero unless $1 < u$, $v = \alpha = \omega^2$: the total energy must exceed the rest energy of a pair.

With these results we can obtain the cross section for two-photon scattering and two-photon pair creation by inserting the values of the photon momenta and their polarizations. In the remainder of the paper this is done for several special cases.

IV. LOW ENERGY APPROXIMATION

The result obtained for the polarization tensor in I, Eq. (48) is applicable here. The evaluation of M from Eq. (6) gives the following values for circularly polarized radiation:

$$\begin{aligned}M_{++++}(\theta, \omega) &= (1/15)\omega^4(3+\cos^2\theta), \\ M_{++--}(\theta, \omega) &= -(22/45)\omega^4, \\ M_{+-+-}(\theta, \omega) &= -(11/90)\omega^4(1+\cos\theta)^2, \\ M_{+--+}(\theta, \omega) &= -(11/90)\omega^4(1-\cos\theta)^2,\end{aligned} \tag{23}$$

and finally, for unpolarized radiation,

$$\bar{\sigma}_s(\theta, \omega) = (\alpha^4/4\pi^2\kappa^2)[139/(90)^2]\omega^6(3+\cos^2\theta)^2. \tag{24}$$

These results, of course, are in agreement with those derived from formulas (10, 9)[8] of Euler.[2a]

V. RIGHT-ANGLE SCATTERING

For this scattering angle $\alpha = -2\beta = -2\gamma = \omega^2$. Because β and γ are the same, those permutations of the $A(1234)$ which differ only by the interchange of the variables $k^{(3)}$ and $k^{(4)}$ are numerically equal. Hence the tensor $G_{\mu\nu\lambda\sigma}$ simplifies somewhat to the following form:

$$\begin{aligned}G_{\mu\nu\lambda\sigma}(1234) = \{&A^{2143}(1234)g^{(1)}(1234)+A^{2143}(1324) \\ &\times[g^{(1)}(1324)+g^{(1)}(2341)]+A^{2341}(1234)[g^{(2)}(1234) \\ &+g^{(2)}(1243)]+A^{2341}(1324)g^{(2)}(1324) \\ &-(1/2\omega^2)A^{2111}(1234)[g^{(3)}(1234)+g^{(3)}(3412) \\ &+g^{(3)}(2143)+g^{(3)}(4321)]+(1/\omega^2)A^{2111}(1432) \\ &\times[g^{(3)}(1432)+g^{(3)}(4123)+g^{(3)}(3214)+g^{(3)}(2341) \\ &+g^{(3)}(1342)+g^{(3)}(4213)+g^{(3)}(3124)+g^{(3)}(2431)] \\ &-(1/2\omega^2)A^{2121}(1234)[g^{(4)}(1234)+g^{(4)}(3412) \\ &+g^{(4)}(3421)+g^{(4)}(2134)]+(1/\omega^2)A^{2121}(1432) \\ &\times[g^{(4)}(1432)+g^{(4)}(3214)+g^{(4)}(3124)+g^{(4)}(2431)] \\ &+(1/\omega^2)A^{2121}(1423)[g^{(4)}(1423)+g^{(4)}(3241) \\ &+g^{(4)}(2413)+g^{(4)}(1324)]+(1/\omega^2)A^{2311}(1234) \\ &+[g^{(5)}(1234)+g^{(5)}(4312)+g^{(5)}(1243) \\ &+g^{(5)}(4321)]\}_{\mu\nu\lambda\sigma}.\end{aligned} \tag{25}$$

The quantity $M_{\lambda_1\lambda_2\lambda_3\lambda_4}$ is a linear combination of the functions A^n. The coefficients depend, of course, on the particular values of λ_i that are being considered. For transitions between states of circular polarization they are listed in Table I. Once the A^n are obtained, they

TABLE II. Expression of the matrix element $M(\pi/2, \omega)$ in terms of the transcendental functions B, T, and I.

	Constant	$(B(\omega^2)-B(-\omega^2/2))$	$T(\omega^2)$	$T(-\omega^2/2)$	$I(\omega^2, -\omega^2/2)$	$I(-\omega^2/2, -\omega^2/2)$
M_{++++}	-1	0	0	0	$-2/\omega^4$	$2/\omega^4$
M_{++--}	1	0	0	$2-4/\omega^2$	$(4/\omega^2)-2/\omega^4$	$-1+2/\omega^2$
M_{+-+-} $=M_{+--+}$	1	-6	$10+4/\omega^2$	$10+4/\omega^2$	$-10-(10/\omega^2)-2/\omega^4$	$(2/\omega^2)+2/\omega^4$
M_{+++-}	-1	0	$3/\omega^2$	$6/\omega^2$	$-(4/\omega^2)-2/\omega^4$	$(1/\omega^2)+2/\omega^4$

[8] This differential cross section gives the number of scattered particles rather than the number of scattering events, so that it is twice as large as the cross section discussed in this paper.

may be combined to give the matrix elements M directly in terms of the three transcendental functions B, T, and I. The coefficients in these expressions are given in Table II. The matrix elements for transitions between states of plane polarized photons can be derived easily from Table II. Since they are more complicated and have no special interest, they are not given here. In Fig. 2 the values of $|M(\pi/2, \omega)|^2/\omega^2$ are plotted as functions of ω; they represent the complete dependence of the cross sections on energy, since they are equal to the cross section when multiplied by the scale factor $\alpha^4/(4\pi^2\kappa^2)$. Although it is the square of the absolute value of M that determines the cross section, separate curves have been drawn for the squares of the real and imaginary parts of M. These represent the contributions to the scattering of virtual and real intermediate pairs, respectively. The latter vanish, of course, at energies below the pair threshold. In Fig. 3 the energy dependence of the cross section for unpolarized radiation is shown. It may be noticed in Fig. 2 that at intermediate energies the cross section for transitions which involve a change in the spin-angular momentum of the system, namely, $^+_-\}\rightarrow\{^+_-$ and $^+_+\}\rightarrow\{^+_-$, are considerably smaller than the other two, and that of these two the one which involves a change in the magnitude of the spin, $^+_+\}\rightarrow\{^+_-$, is smaller than the other, which involves only a change in the direction of the spin. At very high energy, the quantities M approach constant values:

$$M_{++++}(\pi/2, \omega) \sim -1,$$

$$M_{++--}(\pi/2, \omega) \sim 1+\pi^2/4 \cong 3.47,$$

$$M_{+--+}(\pi/2, \omega) \sim 1-3\log 2+(5/2)\log^2 2 + \pi i(3-5\log 2) \cong 0.12-0.47\pi i, \tag{26}$$

$$M_{+++-}(\pi/2, \omega) \sim -1.$$

At high energy, therefore, the cross section decreases as ω^{-2}, while it increases as ω^6 for small values of the energy (see preceding section; the same result is obtained by expanding the function $M(\omega)$ in a McLaurin series).

VI. FORWARD SCATTERING

Forward scattering ($\theta=0$) is characterized by $\alpha=-\gamma=\omega^2$, $\beta=0$. To obtain the values of the function $A^n(1234)$, the expansions of $B(u)$, $T(u)$, and $I(u, v)$ given in Appendix B are very useful. The $A^n(1234)$ simplify greatly because the Spence functions disappear on going to the limit $\beta=0$. Another simplification results from the fact that the four collinear space-parts of the momentum vectors are orthogonal to all the space-like polarization vectors. Most terms in the tensors $g_{\mu\nu\lambda\sigma}^{(i)}(1234)$ therefore do not contribute to the

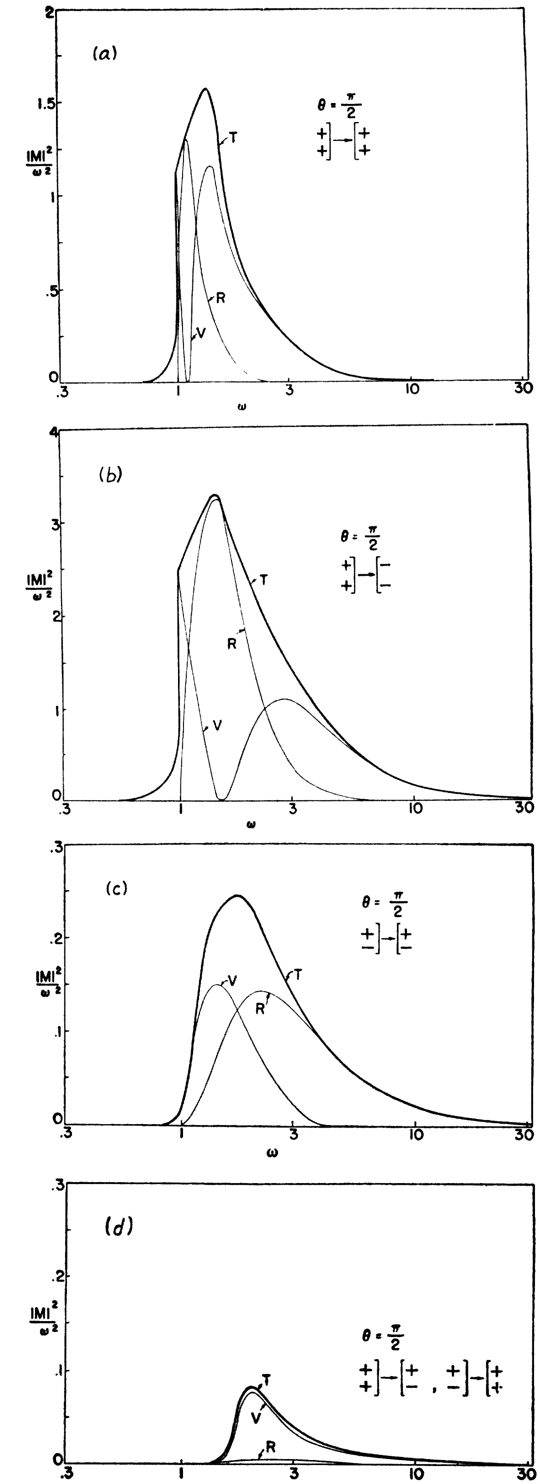

FIG. 2. Differential cross sections for scattering at right angles (cg system) between the indicated states of circular polarization. The unit is 1.07×10^{-31} cm²/sterad; the unit of energy is mc^2. R, V, and T represent the contributions of real intermediate pairs, of virtual intermediate pairs, and their sum, respectively. Note the different scales.

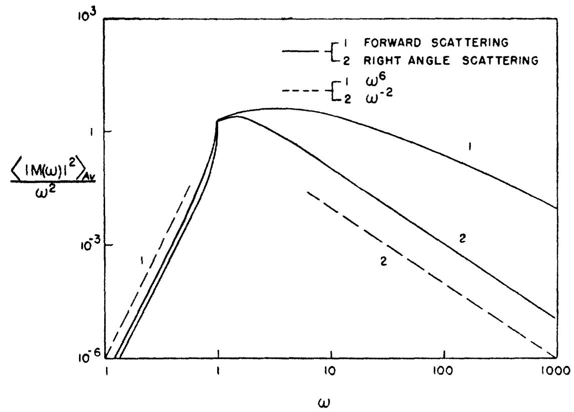

FIG. 3. Differential cross section for scattering unpolarized light (cg system). The unit is 1.07×10^{-31} cm²/sterad; the unit of energy is mc^2.

matrix element. In fact, $M_{\lambda_1\lambda_2\lambda_3\lambda_4}$ becomes

$$\begin{aligned}M_{\lambda_1\lambda_2\lambda_3\lambda_4}(0,\omega) &= (e_\mu^{\lambda_1}(\mathbf{p})e_\mu^{\lambda_2}(-\mathbf{p}))(e_\nu^{\lambda_3*}(\mathbf{p})e_\nu^{\lambda_4*}(-\mathbf{p}))F(\alpha)\\ &+(e_\mu^{\lambda_1}(\mathbf{p})e_\mu^{\lambda_4*}(-\mathbf{p}))(e_\nu^{\lambda_2}(-\mathbf{p})e_\nu^{\lambda_3*}(\mathbf{p}))F(-\alpha)\\ &+(e_\mu^{\lambda_1}(\mathbf{p})e_\mu^{\lambda_3*}(\mathbf{p}))(e_\nu^{\lambda_2}(-\mathbf{p})e_\nu^{\lambda_4*}(-\mathbf{p}))H(\alpha),\end{aligned} \quad (27)$$

where

$$\begin{aligned}F(\alpha) &= \alpha^2\kappa^4[A^{2143}(1234)+A^{2341}(1234)-2A^{2121}(1234)]_{\beta=0,\,\gamma=-\alpha}\\ &=\tfrac{1}{2}\{-1+(6-1/\alpha)[B(\alpha)-B(-\alpha)]\\ &\quad-(2+1/2\alpha^2)T(\alpha)+(2-1/2\alpha^2)T(-\alpha)\},\end{aligned} \quad (28)$$

and

$$\begin{aligned}H(\alpha)=H(-\alpha) &= \alpha^2\kappa^4[A^{2341}(1324)+A^{2341}(1243)+4A^{2111}(1324)-2A^{2121}(1342)\\ &\quad-2A^{2121}(1324)]_{\beta=0,\,\gamma=-\alpha}\\ &=\tfrac{1}{2}\{3-(2+1/\alpha)B(\alpha)-(2-1/\alpha)B(-\alpha)\\ &\quad+(2+(1/\alpha)-1/2\alpha^2)T(\alpha)\\ &\quad+(2-(1/\alpha)-1/2\alpha^2)T(-\alpha)\}.\end{aligned} \quad (29)$$

From these relationships it is easy to obtain the matrix elements for transitions between states of circular polarization. The results are summarized in Table III.

In Fig. 4 the energy dependence of $|M(0,\omega)|^2/\omega^2$ is shown; again the contributions of real and virtual pairs are indicated separately. The forward scattering of unpolarized radiation is illustrated in Fig. 3. The complete absence of a contribution from $M_{+--+}(0,\omega)$ can be attributed to the conservation of angular momentum: this transition involves a change in spin angular momentum and, therefore, in the component of the total angular momentum along the direction of propagation. Figure 5 is a composition of the two curves in Fig. 3; it shows a plausible angular dependence of the cross section for unpolarized light at a number of energies. Although the cross section changes from being almost independent of angle at intermediate energies to having a strong maximum in the forward direction, this peak does not contribute significantly to the total cross section because the solid angle over which it extends becomes small too rapidly.[2b]

At very high energy the quantities M approach the following values:

$$\begin{aligned}&M_{++++}(0,\omega)\sim\tfrac{1}{2}(\log 4\omega^2-1)^2+(5/2)-\pi i(\log 4\omega^2-2)\\ &M_{++--}(0,\omega)\sim-1\\ &M_{+-+-}(0,\omega)\sim\tfrac{1}{2}(\log 4\omega^2-1)^2+(5/2)-\tfrac{1}{2}\pi^2-\pi i.\end{aligned} \quad (30)$$

The cross section, therefore, decreases as $(\alpha^4/\pi^2\kappa^2)\times(\log\omega)^4/\omega^2$, in agreement with the result of Achieser.[2b]

The total cross section for two-quantum pair creation can be deduced by application of Eqs. (9) and (18) and (19). It is

$$\begin{aligned}\sigma_p &= -2(\alpha^2/\kappa^2\omega^2)\,\mathrm{Im}[H(\alpha)+\eta(F(\alpha)+F(-\alpha))]\\ &=\pi(\alpha^2/\kappa^2\omega^2)\{2(1+(1/\omega^2)-1/\omega^4)\cosh^{-1}\omega\\ &\quad-(1+1/\omega^2)(1-1/\omega^2)^{\frac{1}{2}}+(1-2\eta)[(1/2\omega^2)(1-1/\omega^2)^{\frac{1}{2}}\\ &\quad+(1/4\omega^2)\cosh^{-1}\omega]\},\end{aligned} \quad (31)$$

where $\eta=1$ if the incident photons of energy $mc^2\omega$ have parallel plane polarizations, and $\eta=0$ if they have perpendicular polarizations. This result is identical with that of Breit and Wheeler.[3,9]

Finally, we may discuss the experimental significance of the results. Let us consider two photon beams of equal energy and of intensity n photons/sec colliding with each other; scattered photons are collected from a length L of their common path. Then the number N of collisions occurring per second is

$$N=(n^2/A)L(\sigma/c)\sim10^{-40}n^2(L/A)\ \mathrm{sec}^{-1}, \quad (32)$$

where A is the area of the beam in cm², and $\sigma\sim3\times10^{-30}$ cm² was taken as the cross section for scattering into a fair solid angle at an energy of about 1 Mev (see Eq. (7) and Fig. 5). For presently attainable values of the experimental parameters, therefore, it would seem that N is too small to be detected in the presence of the probable background radiation.

We wish to express our appreciation to Dr. Hartland S. Snyder for helpful discussion and criticism. We are grateful to Elizabeth Karplus, Dale Meyer, Rika Sarfaty, and William F. Donoghue for aid with the numerical work.

APPENDIX A

The five invariants appearing in I, Eq. (47) have the general structure

[9] In a recent interesting paper (to be published), Toll and Wheeler essentially reverse the procedure by which Eq. (31) was derived. They start with this result and calculate the entire functions $H(\alpha)$ and $(F(\alpha)+F(-\alpha))$ from their imaginary part by assuming that these functions are analytic over one-half of the α-plane. The present calculation indicates that this assumption is justified.

TABLE III. Expression of the matrix element $M(0, \omega)$ in terms of the transcendental functions B and T.

	Constant	$B(\omega^2)$	$B(-\omega^2)$	$T(\omega^2)$	$T(-\omega^2)$
$M_{++++}(0, \omega)=F(\alpha)+H(\alpha)$	1	$2-1/\omega^2$	$-4+1/\omega^2$	$1/\omega^2-1/2\omega^4$	$2-1/\omega^2-1/2\omega^4$
$M_{++--}(0, \omega)=F(\alpha)+F(-\alpha)$	-1	$-2/\omega^2$	$1/\omega^2$	$-1/2\omega^4$	$-1/2\omega^4$
$M_{+-+-}(0, \omega)=F(-\alpha)+H(\alpha)$	1	$-4-1/\omega^2$	$2+1/\omega^2$	$2+1/\omega^2-1/2\omega^4$	$-1/\omega^2-1/2\omega^4$
$M_{+--+}(0, \omega)=0$	0	0	0	0	0

TABLE IV. Numerical values of the transcendental functions B, T, and I for some values of the arguments.

u	$B(u)$		$T(u)$		$I(u, -u/2)$		$I(u, u)$
-25	1.3583		5.3474				
-12.5	1.0530		3.9024				5.2977
-4	0.6141		2.0841				
-3	0.5207		1.7394				
-2	0.4038		1.3138				0.9286
-1.5	0.3319		1.0644				
-1	0.2465		0.7768				0.3454
-0.5	0.1405		0.4336				0.1176
-0.25	0.0760		0.2315				
0	0.0000		0.0000		0.0000		0.0000
0.25	-0.0933		-0.2741				
0.5	-0.2146		-0.6169				
1	-1.0000		-2.4674		-0.9095		
1.5	-0.6199	$-0.2887\pi i$	-2.0343	$-0.6581\pi i$			
2	-0.3768	$-0.3536\pi i$	-1.6906	$-0.8814\pi i$	-0.2571	$-0.5376\pi i$	
3	-0.0634	$-0.4083\pi i$	-1.1502	$-1.1472\pi i$			
4	0.1405	$-0.4330\pi i$	-0.7330	$-1.3170\pi i$	0.7675	$-0.9223\pi i$	
12.5	0.8564	$-0.4796\pi i$	1.2783	$-1.9354\pi i$			
25	1.2462	$-0.4899\pi i$	2.7878	$-2.2950\pi i$	6.5442	$-1.9178\pi i$	

$$\kappa^4 A(1,2,3,4)=r+b_{+\alpha\beta}[B(\alpha)+B(\beta)]+b_{+\beta\gamma}[B(\beta)+B(\gamma)] +b_{+\gamma\alpha}[B(\gamma)+B(\alpha)]+b_{-\alpha\beta}[B(\alpha)-B(\beta)]+b_{-\beta\gamma}[B(\beta)-B(\gamma)] +b_{-\gamma\alpha}[B(\gamma)-B(\alpha)]+t_{\alpha\beta}[T(\alpha)+T(\beta)]+t_{\beta\gamma}[T(\beta)+T(\gamma)] +t_{\gamma\alpha}[T(\gamma)+T(\alpha)]+i_{\alpha\beta}I(\alpha,\beta)+i_{\beta\gamma}I(\beta,\gamma)+i_{\gamma\alpha}I(\gamma,\alpha). \quad \text{(A1)}$$

The r's, b's, t's, and i's are rational functions of α, β, and γ. A number of these vanish for certain values of the indices. Thus

$$b_-^{2341}=b_-^{2111}=b_-^{2121}=b_-^{2311}=0, \quad b_+^{2143}=0. \quad \text{(A2)}$$

In addition,

$$b_{-\beta\gamma}^{2143}=0. \quad \text{(A3)}$$

Many others differ only by a permutation of arguments. Thus

$$\begin{aligned} f_{\gamma\alpha}^{2143}(\alpha,\beta,\gamma)&=f_{\alpha\beta}^{2143}(\alpha,\gamma,\beta),\\ f_{\beta\gamma}^{2341}(\alpha,\beta,\gamma)&=f_{\alpha\beta}^{2341}(\gamma,\beta,\alpha),\\ f_{\gamma\alpha}^{2111}(\alpha,\beta,\gamma)&=f_{\alpha\beta}^{2111}(\alpha,\gamma,\beta),\\ f_{\gamma\alpha}^{2311}(\alpha,\beta,\gamma)&=-f_{\beta\gamma}^{2311}(\beta,\alpha,\gamma), \end{aligned} \quad \text{(A4)}$$

where f stands for b_+, t, and i. For b_- we have the identity

$$b_{-\gamma\alpha}^{2143}(\alpha,\beta,\gamma)=-b_{-\alpha\beta}^{2143}(\alpha,\gamma,\beta). \quad \text{(A5)}$$

We now list the "irreducible" rational functions:

$$\begin{aligned} r^{2143}&=(3\beta^2\gamma^2)^{-1}(3\alpha^2-10\beta\gamma)\\ r^{2341}&=(3\alpha\beta^2\gamma)^{-1}(3\alpha\gamma-5\beta^2)\\ r^{2111}&=(3\beta\gamma)^{-1}\\ r^{2121}&=-(3\beta^2\gamma)^{-1}(3\alpha+2\beta)\\ r^{2311}&=(3\alpha\beta\gamma)^{-1}(\alpha-\beta) \end{aligned} \quad \text{(A6)}$$

$$\begin{aligned} b_{+\alpha\beta}^{2341}&=-(3\alpha\beta\gamma^2)^{-1}(4\alpha^2+21\alpha\beta-12\gamma^2)\\ b_{+\gamma\alpha}^{2341}&=(3\alpha^2\gamma^2)^{-1}(4\beta^2+6\alpha\gamma)\\ b_{+\alpha\beta}^{2111}&=(3\alpha\beta^2\gamma^2)^{-1}(\alpha^3+\beta^3-\gamma^3)\\ b_{+\beta\gamma}^{2111}&=-(3\alpha\beta^2\gamma^2)^{-1}(3\alpha^3+\beta^3+\gamma^3)\\ b_{+\alpha\gamma}^{2121}&=-(3\alpha\gamma^2)^{-1}(6\alpha+2\beta)\\ b_{+\alpha\beta}^{2121}&=(3\alpha\beta^3\gamma)^{-1}(12\alpha^2\gamma+6\alpha\beta^2-2\beta\gamma^2)\\ b_{+\beta\gamma}^{2121}&=-(3\alpha\beta^3\gamma^2)^{-1}(12\alpha^2\gamma^2-6\alpha\beta^3+2\beta^4)\\ b_{+\alpha\beta}^{2311}&=(3\alpha^2\beta^2)^{-1}(2\alpha^2-2\beta^2)\\ b_{+\beta\gamma}^{2311}&=-(3\alpha\beta^2\gamma^2)^{-1}(2\alpha^3-2\alpha^2\beta+3\beta^2\gamma) \end{aligned} \quad \text{(A7)}$$

$$b_{-\alpha\beta}^{2143}=(3\alpha\gamma^3)^{-1}(12\beta^2+2\beta\gamma-6\gamma^2) \quad \text{(A8)}$$

$$\begin{aligned} t_{\alpha\beta}^{2143}&=(3\gamma^4)^{-1}(12\beta^2+4\alpha\gamma)+(\alpha\gamma^3)^{-1}2\beta\\ t_{\beta\gamma}^{2143}&=(3\alpha^2)^{-1}4\\ t_{\alpha\beta}^{2341}&=(3\gamma^3)^{-1}(3\gamma-4\beta)+(2\alpha^2\beta\gamma^2)^{-1}(\beta^2+\gamma^2-\alpha^2)\\ t_{\gamma\alpha}^{2341}&=-(3\beta^4)^{-1}(4\beta^2+12\alpha\gamma)\\ &\qquad +(2\alpha^2\beta^3\gamma^2)^{-1}(\beta^4+\alpha^2\beta^2+\beta^2\gamma^2-4\alpha^2\gamma^2)\\ t_{\alpha\beta}^{2111}&=(3\gamma^3)^{-1}(2\alpha-3\gamma)-(4\beta^2\gamma^2)^{-1}\alpha\\ t_{\beta\gamma}^{2111}&=(3\alpha^2)^{-1}2-(4\beta^2\gamma^2)^{-1}\alpha\\ t_{\alpha\beta}^{2121}&=-(3\gamma^3)^{-1}(\alpha-3\beta)-(2\alpha\beta\gamma)^{-1}\\ t_{\beta\gamma}^{2121}&=(3\alpha^2)^{-1}2+(2\alpha\beta\gamma^2)^{-1}(\alpha-\beta)\\ t_{\gamma\alpha}^{2121}&=(3\beta^4)^{-1}(4\alpha^2-8\alpha\beta-3\beta^2)-(2\alpha\gamma^2\beta^3)^{-1}(\alpha^3-\beta^3+3\alpha\gamma^2)\\ t_{\beta\gamma}^{2311}&=(3\alpha^3)^{-1}(2\alpha-2\beta)-(4\alpha^2\beta^2\gamma^2)^{-1}(\alpha-\beta)(\alpha^2+\alpha\beta+\beta^2)\\ t_{\alpha\beta}^{2311}&=(3\gamma^3)^{-1}(2\alpha-2\beta)-(4\alpha^2\beta^2\gamma^2)^{-1}(\alpha-\beta)(\alpha^2+\alpha\beta+\beta^2) \end{aligned} \quad \text{(A9)}$$

$$\begin{aligned} i_{\alpha\beta}^{2143}&=-(3\gamma^4)^{-1}(12\beta^2+4\alpha\gamma)-(3\alpha\beta\gamma^3)^{-1}(12\beta^2+2\alpha\gamma)\\ &\qquad +(6\alpha\beta^3\gamma^2)^{-1}(3\alpha^2-8\beta\gamma)\\ i_{\beta\gamma}^{2143}&=-(3\alpha^2)^{-1}4-(3\alpha\beta\gamma)^{-1}8+(6\beta^3\gamma^3)^{-1}(3\alpha^2-8\beta\gamma)\\ i_{\alpha\beta}^{2341}&=(3\gamma^3)^{-1}(4\beta-3\gamma)+(6\alpha^2\beta^2\gamma^2)^{-1}(6\alpha^2\gamma+7\alpha\beta^2-3\beta^3)\\ &\qquad +(6\alpha^2\beta^3\gamma)^{-1}(2\beta^2+3\alpha\gamma)\\ i_{\gamma\alpha}^{2341}&=(3\beta^4)^{-1}(4\beta^2+12\alpha\gamma)-(6\alpha^2\beta^3\gamma^2)^{-1}(3\alpha^2\beta^2+3\beta^2\gamma^2\\ &\qquad -24\alpha^2\gamma^2-4\alpha\beta^2\gamma)+(6\alpha^2\beta^2\gamma^2)^{-1}(2\beta^2+3\alpha\gamma)\\ i_{\alpha\beta}^{2111}&=-(3\gamma^3)^{-1}(2\alpha-3\gamma)-(6\alpha\beta\gamma^2)^{-1}(5\alpha-6\gamma)-(6\alpha\beta^2\gamma)^{-1}\\ i_{\beta\gamma}^{2111}&=-(3\alpha^2)^{-1}2-(3\alpha\beta\gamma)^{-1}-(6\beta^2\gamma^2)^{-1}\\ i_{\alpha\beta}^{2121}&=+(3\gamma^3)^{-1}(\alpha+3\beta)-(6\alpha\beta^2\gamma^2)^{-1}(3\alpha^2-\alpha\beta+6\beta^2)\\ &\qquad -(6\alpha\beta^3\gamma)(3\alpha+\beta)\\ i_{\beta\gamma}^{2121}&=-(3\alpha^2)^{-1}2+(6\alpha\beta^2\gamma^2)^{-1}(\alpha^2+4\alpha\beta+2\gamma^2)\\ &\qquad -(6\beta^3\gamma^2)^{-1}(3\alpha+\beta)\\ i_{\gamma\alpha}^{2121}&=-(3\beta^4)^{-1}(4\alpha^2-8\alpha\gamma-3\beta^2)+(6\alpha\beta^3\gamma^2)^{-1}(24\alpha^3+38\alpha^2\beta\\ &\qquad +11\alpha\beta^2-6\beta^3)-(6\alpha\beta^2\gamma^2)^{-1}(3\alpha+\beta)\\ i_{\alpha\beta}^{2311}&=-(3\gamma^3)^{-1}(2\alpha-2\beta)-(6\alpha\beta\gamma^2)^{-1}(5\alpha-5\beta)\\ &\qquad -(6\alpha^2\beta^2\gamma)^{-1}(\alpha-\beta)\\ i_{\beta\gamma}^{2311}&=-(3\alpha^3)^{-1}(2\alpha-2\beta)+(6\alpha^2\beta\gamma^2)^{-1}(2\alpha^2-5\beta^2)\\ &\qquad -(6\alpha\beta^2\gamma^2)(\alpha-\beta). \end{aligned} \quad \text{(A10)}$$

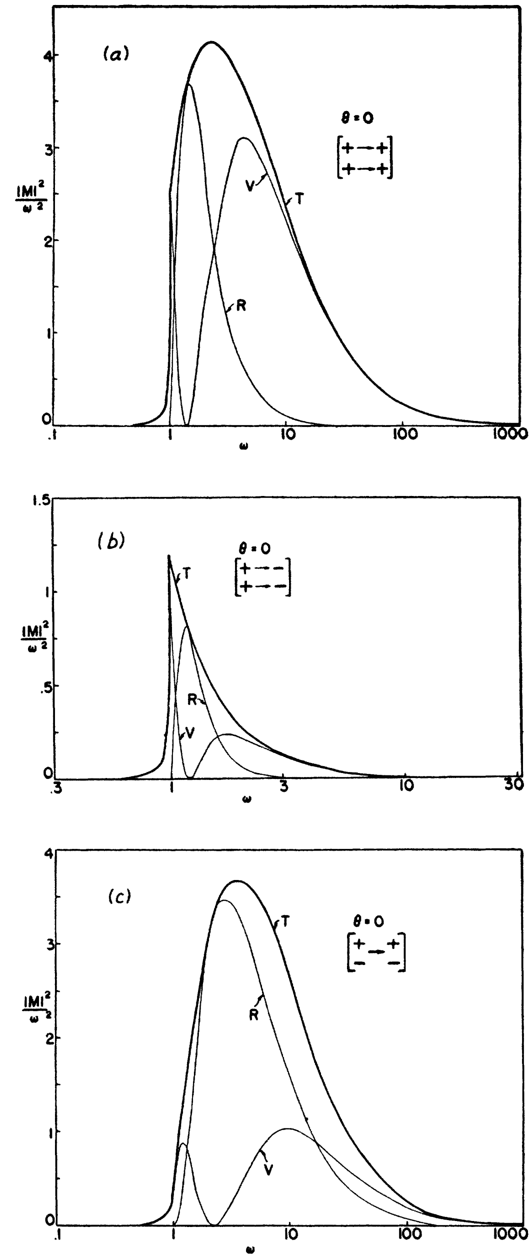

FIG. 4. Differential cross section for forward scattering (cg system) between the indicated states of circular polarization. The unit is 1.07×10^{-31} cm^2/sterad; the unit of energy is mc^2. R, V, and T represent the contributions of real intermediate pairs, of virtual intermediate pairs, and their sum, respectively. Note the different scales.

APPENDIX B

Definitions

$$B(u)=\tfrac{1}{2}\int_0^1 dy\,\log[1-i\epsilon-4uy(1-y)]. \tag{B1}$$

$$T(u)=\int_0^1 dy[4y(1-y)]^{-1}\log[1-i\epsilon-4uy(1-y)]. \tag{B2}$$

$$I(u,v)=\int_0^1 dy[4y(1-y)-(uv)^{-1}(u+v)]^{-1}\times\log[1-i\epsilon-4uy(1-y)]. \tag{B3}$$

Approximate Expressions

$$\begin{aligned}&B(u)\sim-(1/3)u-(2/15)u^2-(8/105)u^3;\quad |u|\ll1.\\&B(u)\sim(1/2)\log|4u|-1-(1/2)i\pi\theta(u)\quad |u|\gg1.\end{aligned} \tag{B4}$$

$$\begin{aligned}&T(u)\sim-u-(1/3)u^2-(8/45)u^3\quad |u|\leqslant1.\\&T(u)\sim(1/4)\log^2|4u|-((1/4)\pi^2+(1/2)i\pi\log|4u|)\theta(u)\quad |u|\gg1.\\&\theta(|u|)=1,\quad \theta(-|u|)=0.\end{aligned} \tag{B5}$$

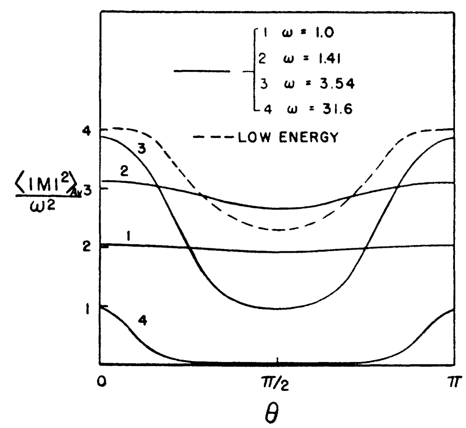

FIG. 5. Angular dependence of the cross section for scattering unpolarized quanta at selected energies (cg system). These curves were interpolated from the values of the cross section at 0°, 90°, and 180°. The shape at low energy is not to scale.

Denoting $(u+v)/uv$ by x we get

$$\begin{aligned}I(u,v)\sim&[T(u)+T(v)][1+(1/2)x+(3/8)x^2+(5/16)x^3]\\&+B(u)x[u+(1/4)xf_1(u)+(1/24)x^2f_2(u)]\\&+B(v)x[v+(1/4)xf_1(v)+(1/24)x^2f_2(v)]\\&-(1/4)(u+v)x[1+x+(1/24)x^2(45+4uv)]\end{aligned} \tag{B6a}$$

for

$$|x|\ll1,$$

where

$$f_1(u)=3u+2u^2,$$
$$f_2(u)=15u+10u^2+8u^3.$$

$$I(u,v)\sim-(2/x)[B(v)-(4/3)-(4/15)v^2(u+v)^{-2}],\quad |u|\ll1, \tag{B6b}$$

$$\begin{aligned}I(u,v)\sim(1/4)\{&\log^2|4u|+\log^2|4v|-\log^2[|u|\,|v|^{-1}]\\&-\pi^2-\pi i\theta(u)\log|v|\},\\&|u|\gg1,\\&v\ll-1.\end{aligned} \tag{B6c}$$