The Magneto-Resistance Effect in Oriented Single Crystals of Germanium

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This paper describes an extensive study of the magneto-resistance effect in germanium as a function of crystal orientation. Experimental measurements establish the constants involved in the dependence of the effect on orientation of magnetic field and electric current relative to the crystal axes. The measurements are internally consistent with existing phenomenological theory based on cubic crystal symmetry, in which terms involving the magnetic field to higher than the second order are neglected. It is shown that such deviations as do occur arise from higher terms in the field, since an extension of the phenomenological theory to the fourth order predicts their symmetry. Relations are established between the experimentally observed phenomenological constants and those constants appearing in existing magneto-resistance electronic theories. It is concluded that no electronic theory yet worked out is entirely consistent with experiment. The present electronic theories are special cases of a very general theory recently proposed by Shockley, and it is possible that agreement can be obtained as soon as the computational difficulties of the latter theory are overcome.

I. INTRODUCTION

THE effect of magnetic fields on the electrical resistance of semiconductors has been obscure until quite recently. Although simplified theories pertaining to parallel and perpendicular electric and magnetic fields in isotropic media[1] had been formulated, no equations applicable to experimental measurements on allotropic semiconducting materials were available. Adequate experimental data were likewise lacking. Those most often quoted in the literature were obtained by Kapitza[2] on multicrystalline germanium and silicon samples of unknown impurity content.

The phenomenological theory of the magneto-resistance effect in semiconductors has recently been clarified by Seitz,[3] who succeeded in deriving general equations in a closed form relating to semiconducting crystals possessing cubic symmetry. Important contributions on the experimental side have been made by Estermann and Foner,[4] who measured the effect in germanium samples of known impurity content as a function of the temperature and the magnitude and relative directions of the electric and magnetic fields. Their results, however, have limited application to Seitz's theory, since their germanium samples were composed of many small crystals having random orientation.

The present paper reports magneto-resistance measurements on single crystal samples of germanium as a function of the orientation of the electric and magnetic fields with respect to the crystal axes. Other variables studied were strength of magnetic field, temperature, and type of conduction (either by holes or electrons). The numerical results are compared with Seitz's theoretical equations, from which the fundamental magneto-resistance constants of germanium are obtained. Certain discrepancies between theory and experiment lend weight to a more sophisticated model recently treated by Shockley.[5]

II. THEORETICAL DISCUSSION

The theoretical treatment of the effect of magnetic fields on the electrical resistance of semiconductors is simplified over that of metals, in that the concentration of free electrons is usually so low that the electron gas obeys classical statistics. The first study of this problem was made by Ganz,[1] who assumed perfectly free electrons, an isotropic medium, and an energy-independent mean free path. His calculations were limited to the case in which the applied electric and magnetic fields are normal. He carrried the solutions to terms in the current which depend on the first power of the electric field strength and on arbitrary powers of the magnetic field strength, and obtained a solution in closed form for the transverse Hall effect and for the conductivity as a function of the magnetic field.

These calculations have been extended by Harding,[1] Wilson,[1] and Davis[6] to include the case of an allotropic medium. In addition, Davis generalized the treatment to include electric and magnetic fields oriented arbitrarily relative to one another, electrons that are not perfectly free, and a mean free path dependent on the energy in an arbitrary manner. His solution is so general that it cannot be expressed in a closed form but is in terms of series in powers of the magnetic field, the coefficients being integrals which must be evaluated in particular cases.

Seitz[3] has recently established the phenomenological relation between the current density vector $\mathbf{i}$ and the electric and magnetic field $\mathbf{E}$ and $\mathbf{H}$ that must hold in any material having cubic symmetry. His equation, which is perfectly general and subject only to the

[1] R. Ganz, Ann. Physik **20**, 293 (1906). A. Sommerfeld and N. H. Frank, Revs. Modern Phys. **3**, 1 (1931). J. W. Harding, Proc. Roy. Soc. (London) **140**, 205 (1933). A. H. Wilson, *The Theory of Metals* (Cambridge University Press, London, 1936).
[2] P. Kapitza, Proc. Roy. Soc. (London) **123**, 291 (1929).
[3] F. Seitz, Phys. Rev. **79**, 372 (1950)
[4] I. Estermann and A. Foner, Phys. Rev. **79**, 365 (1950).
[5] W. Shockley, Phys. Rev. **78**, 173 (1950); Phys. Rev. **79**, 191 (1950).
[6] Leverett Davis, Jr., Phys. Rev. **56**, 93 (1939).

limitation that terms involving the magnetic field to higher than the second power are neglected, is

$$\mathbf{i}=\sigma_0\mathbf{E}+\alpha(\mathbf{E}\times\mathbf{H})+\beta\mathbf{E}H^2+\gamma\mathbf{H}(\mathbf{E}\cdot\mathbf{H})+\delta T\mathbf{E}, \quad (1)$$

where T is a diagonal tensor with elements H_1^2, H_2^2, and H_3^2 (the coordinate axes 1, 2, and 3 being taken along the crystal axes), σ_0 is the electronic conductivity under zero field conditions, $-\alpha/(\sigma_0)^2$ is the Hall coefficient,[7] and β, γ, δ are the small-field magneto-resistance constants deducible from electronic theory. Any electronic theory must lead to the form of Eq. (1).

All of the magneto-resistance theories mentioned above explicitly determine the current as a function of the magnetic and electric fields applied to the specimen. In magneto-resistance experiments, on the other hand, the change in voltage across the sample due to $\mathbf{H}$ is taken as a measure of the resistance increment due to $\mathbf{H}$. In other words, $\mathbf{i}$ is held constant and $\mathbf{E}$ is permitted to adjust itself accordingly. The experiment is therefore only indirectly related to the constants α, β, γ, and δ. Experiment is more closely related to the constants a, b, c, and d of a reciprocal equation

$$\mathbf{E}=\rho_0[\mathbf{i}+a(\mathbf{i}\times\mathbf{H})+b\mathbf{i}H^2+c\mathbf{H}(\mathbf{i}\cdot\mathbf{H})+dT\mathbf{i}] \quad (2)$$

which is also a consequence of cubic symmetry. Straightforward matrix algebra, with neglect of powers of H higher than the second, shows that

$$\left.\begin{aligned} a&=-\alpha\rho_0, \quad b=-(\beta+\rho_0\alpha^2)\rho_0,\\ d&=-\delta\rho_0, \quad c=-(\gamma-\rho_0\alpha^2)\rho_0,\end{aligned}\right\} \quad (3)$$

where $\rho_0=1/\sigma_0$.

In terms of this new set of constants, the resistance increment due to an increase in magnetic field from 0 to H is[8]

$$\frac{\Delta\rho}{\rho H^2}=\frac{(\mathbf{E}-\mathbf{E}_{H=0})\cdot\mathbf{i}}{(\mathbf{E}_{H=0}\cdot\mathbf{i})}=b+c\frac{(\mathbf{i}\cdot\mathbf{H})^2}{i^2H^2}+d\frac{i_1^2H_1^2+i_2^2H_2^2+i_3^2H_3^2}{i^2H^2}. \quad (4)$$

More simply, if ι_1, ι_2, ι_3 and η_1, η_2, η_3 are the direction cosines of $\mathbf{i}$ and $\mathbf{H}$, respectively,

$$\Delta\rho/(\rho_0H^2)=b+c(\Sigma\iota\eta)^2+d\Sigma\iota^2\eta^2. \quad (5)$$

Thus, by measuring the magneto-resistance effect in certain selected directions one can evaluate the constants b, c, and d after which Eq. (5) permits one to calculate the effect for arbitrary directions of $\mathbf{i}$ and $\mathbf{H}$. Likewise, after experimentally determining b, c, and d, one can use Eq. (3) to calculate the constants β, γ, and δ for comparison with electronic theory.

For the simple case of isotropic materials having a mean free path independent of energy, Seitz has shown from electronic theory that $\beta=-\gamma=-9\pi\mu^3ne/16c^2$, where μ is the mobility of the carriers, n is their density, e is the electronic charge, and c is the velocity of light. Substituting these values in Eqs. (3) and (4) it is deduced that

$$\Delta\rho/(\rho H^2)=0, \quad (6)$$

for $\mathbf{i}$ parallel to $\mathbf{H}$, and

$$\Delta\rho/(\rho H^2)=81\times10^4\pi\mu^2(1-\tfrac{1}{4}\pi)/16c^2=3.8\times10^{-17}\mu^2, \quad (7)$$

for $\mathbf{i}$ perpendicular to $\mathbf{H}$. Equations (6) and (7) are in quantitative agreement with the earlier relations[1] for the simple isotropic case at low magnetic field intensities. Harding[1] studied the case of transverse magnetic fields in isotropic semiconductors beyond the quadratic range and predicted that

$$\lim_{H\to\infty}\frac{\Delta\rho}{\rho}=0.13. \quad (8)$$

Seitz[3] has applied an electronic theory of Davis[6] to find the form of α, β, γ, and δ on the hypothesis that the energy-momentum surfaces of the electrons are spheres, whereas the relaxation time τ is an allotropic function of the momentum vector $\mathbf{k}$ such that

$$\tau=\tau_0(k)+\tau_1(k)Y(\theta,\varphi),$$

where $Y(\theta,\varphi)$ is the polar form of the cubically invariant quartic $x^4+y^4+z^4-3(x^2y^2+y^2z^2+z^2x^2)$.* τ_0 and τ_1 depend on the length of $\mathbf{k}$ only and have a constant ratio. His final expressions are

$$\left.\begin{aligned}\beta&=[1+(\tau_1/\tau_0)^2(532/231)]I,\\ \gamma&=-[1+(\tau_1/\tau_0)^2(212/231)]I,\\ \delta&=-(\tau_1/\tau_0)^2(240/231)I,\end{aligned}\right\} \quad (9)$$

where I is a certain integral depending on τ_0 and on the distribution function of $\mathbf{k}$. In order to compare the experimental results described below with this theory we note that

$$b+c+\tfrac{1}{2}d=-(\beta+\gamma+\tfrac{1}{2}\delta)\rho_0=-(\tau_1/\tau_0)^2(200/231)I\rho_0, \quad (10)$$

and

$$b+c+d=-(\beta+\gamma+\delta)\rho_0=-(\tau_1/\tau_0)^2(80/231)I\rho_0. \quad (11)$$

Physically, Eq. (10) is the value of $\Delta\rho/(\rho H^2)$ when $\mathbf{H}$ and $\mathbf{i}$ are parallel and directed along the 110 crystal axis, while Eq. (11) is the value of $\Delta\rho/(\rho H^2)$ when $\mathbf{H}$ and $\mathbf{i}$ are parallel and directed along the 100 crystal axis. The above electronic theory predicts that the ratio of these two magneto-resistance effects should be 2.5.

III. PREPARATION OF SAMPLES

The germanium used in this study was large single crystal ingots prepared by the pulling technique of Teal and Little.[9] As drawn, the conductivity was n-type and

[7] If i is in amp/cm², E in volt/cm, and H in gauss, then $-\alpha/\sigma_0^2$ is in volt cm/amp gauss.

[8] This equation was first called to our attention by J. Bardeen who derived it from symmetry considerations.

* Where $x^2+y^2+z^2=1$.

[9] G. K. Teal and J. B. Little, Phys. Rev. **78**, 647 (1950).

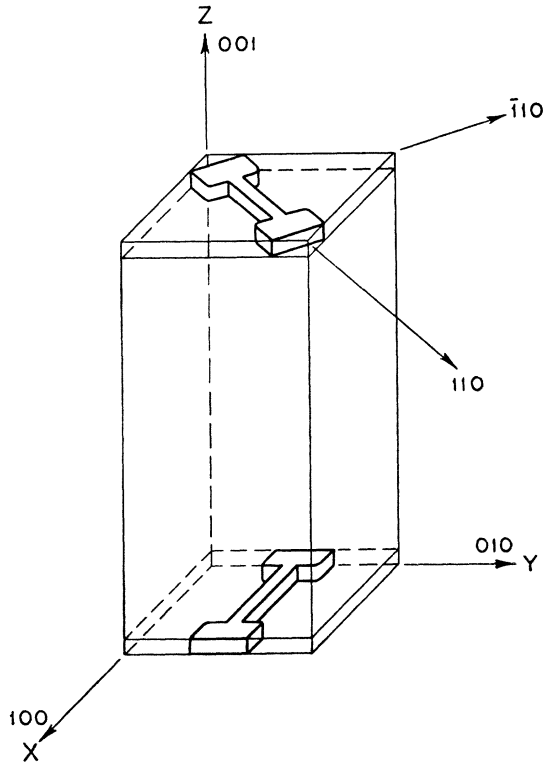

FIG. 1. Orientation of rod samples as cut from a single crystal of germanium.

the purity high as evidenced by a resistivity of 11.5 ohm cm. The crystal structure of germanium is cubic and of the diamond lattice type. The ingots were ground on a carborundum belt grinder into elongated square prisms having their edges parallel to the x, y, and z crystal axes with the help of x-rays as described by Bond.[10] Figure 1 is a sketch of the oriented crystal after grinding. The crystals were then sawed in half along the xy plane and one section of each crystal converted to p-type by heat treating at 900°C for a period of six hours in a helium atmosphere followed by an air quench.[11] The resistivity of the converted sections was 0.38 ohm cm.

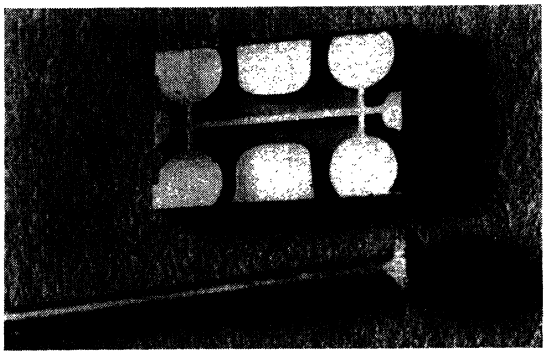

FIG. 2. Photograph of oriented germanium rod for measuring magneto-restistance effect. A common pin is shown for size comparison.

[10] W. L. Bond, Phys. Rev. **78**, 646 (1950).

Samples suitable for making magneto-resistance measurements were prepared by sawing and lapping thin plates as shown in Fig. 1. After etching to clean up the surfaces the plates were cemented to glass substrates and rod samples with intimately attached current, and voltage electrode areas were cut by means of a magnetostrictive driven die as described by Bond.[10] Further steps in the preparation included rhodium electroplating of the electrode surfaces to obtain good ohmic contact and cleaning up of the germanium surfaces by etching. Figure 2 is a photograph of a completed rod sample. The two end electrodes are for current and the four side electrodes for potentiometric measurements. All of the samples were cut with the same die, which resulted in identical dimensions. The length of the rod between potential electrodes is 0.500 cm, the width 0.025 cm, and the thickness 0.025 cm. Several samples from both the n- and p-type sections were made from a number of ingots. They were cut in each of the two different crystal orientations shown in Fig. 1, that is, with the length along either the 100 or the 110 crystal axis and with the thickness always along the 001 axis.

The magneto-resistance measurements were made by passing a fixed current through the sample and measuring the voltage between the potential probes with and without a magnetic field. The voltage drops were measured with a Leeds and Northrup type K2 potentiometer. Since the effect was found to be independent of current magnitude, this was altered for the different samples to always give a convenient maximum of about 1 volt and was usually of the order of 1 ma. Magnetic fields up to 20,000 gauss were furnished by a large electromagnet, and the field strengths were measured by a synchronously driven rotating coil and Ballantine voltmeter. Accurate calibration was obtained by comparison with a permanent magnet which had been calibrated at the U. S. Bureau of Standards. The samples were enclosed in a Dewar flask during the measurements. This eliminated temperature drifts at room temperature, served as a liquid nitrogen and liquid hydrogen container at low temperatures, and eliminated all photoeffects. Temperatures were measured with a Mueller bridge and a platinum resistance thermometer.

IV. EXPERIMENTAL RESULTS

A. Magneto-Resistance Effect *versus* Field Strength

Measurements were first made of $(\rho-\rho_0)/\rho_0=\Delta\rho/\rho$ as a function of $\mathbf{H}$ for given directions of $\mathbf{i}$ and $\mathbf{H}$ with respect to the crystal axes. Since current is always along the length of the rods, the direction of $\mathbf{i}$ is fixed for any given sample and may be either along the 100 or the 110 axis. Runs were made with $\mathbf{H}$ parallel to $\mathbf{i}$ and

[11] H. C. Theuerer and J. H. Scaff, Trans. Am. Inst. Mining Engrs. **189**, 59 (1951).

perpendicular to **i**. The results for p-type rods at both 300°C and 77°C are given in Fig. 3. The general features of these curves include: (1) $\Delta\rho/\rho$ proportional to H^2 at low field strengths with a break in the curve between 5000 and 10,000 gauss and proportional to the first power of H or lower at high fields, (2) at a given field $\Delta\rho/\rho$ increases with decrease in temperature, and (3) the magneto-resistance is much greater for **H** perpendicular to **i** than for **H** parallel to **i**. These features are in agreement with the simplified electronic theory as given by Eqs. (6) and (7). The experiments do not agree with this theory in that they predict a hole mobility three times that measured by other means;[12] also, $\Delta\rho/\rho$ values greater than 0.13 have been achieved at high field strengths without reaching saturation.

The direction of **i** could be reversed without altering the magnitude of $\Delta\rho/\rho$, as could that of **H**. In the sample having **i** along the 100 axis, Fig. 3(A), identical results were obtained with **H** along either the 010 or 001 axis. This is to be expected, since these two axes have complete crystal symmetry. In the sample with **i** along the 110 axis, Fig. 3(B), slightly different results were obtained with **H** along the $1\bar{1}0$ and the 001 unsymmetrical axes.

A corresponding set of measurements on the n-type samples are shown in Fig. 4. The results for **H** perpendicular to **i** are quite similar to those for p-type, and the calculated electron mobility checks that determined by the drift method within 25 percent. In the n-type samples, however, the magneto-resistance effect with **H** parallel to **i** is completely out of line with the simple theory. Instead of being small in comparison with the transverse effect, it is twice as large. Through the cooperation of the Magnetics Laboratory at the Massachusetts Institute of Technology we were able to extend the room temperature measurements on the 100 sample to 100,000 gauss as shown in Fig. 4(a). Notice that saturation has not yet been attained at this high field although $\Delta\rho/\rho$ transverse has reached 1.1 and $\Delta\rho/\rho$ longitudinal has reached 2.2. At 77°K $\Delta\rho/\rho$ longitudinal reached 2.0 for a field of only 20,000 gauss.[12] In the n-type 110 sample, Fig. 4(b), the effect is markedly higher with **H** in the perpendicular 110 direction than in the perpendicular 001 direction. Saturation is fairly complete at $\Delta\rho/\rho$ equal to 0.5 with $H=20{,}000$ gauss, $T=77$°K, and **H** parallel to **i**.

In order to calculate certain fundamental constants to be described in Sec. V it is necessary to determine $\Delta\rho/\rho H^2$ as H approaches zero for the various directions of **i** and **H** shown in Fig. 3 and Fig. 4. These data are given in Fig. 5 and Fig. 6, which are replots of the previous data excepting that $\Delta\rho/\rho H^2$ is plotted along the vertical scale. Notice that at room temperature the curves are horizontal up to 3000 gauss indicating an exact square law relationship between $\Delta\rho/\rho$ and H. At low temperatures the curves do not flatten even at low fields indicating a power of H slightly less than 2.

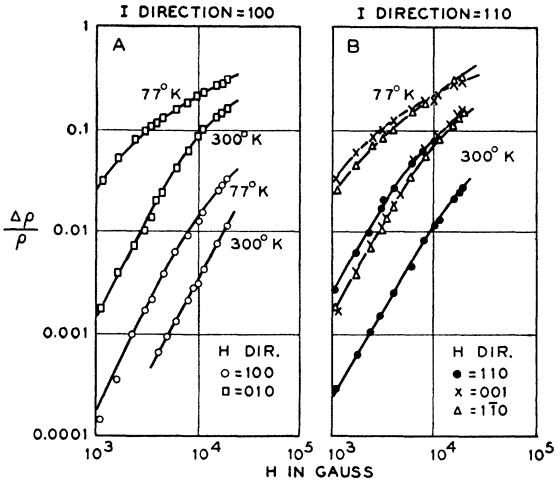

FIG. 3. $\Delta\rho/\rho$ *vs* H in p-type germanium for fixed directions of **I** and **H**; the values of T are as shown.

B. Magneto-Resistance Effect *versus* Temperature

The magneto-resistance effect in an n-type sample with **i** along the 100 direction was measured as a function of temperature for a fixed magnetic field of 4000 gauss in both the 100 longitudinal direction and the 010 transverse direction. These results are shown in Fig. 7, where $\Delta\rho/\rho H^2$ is plotted vertically and the temperature in degrees Kelvin horizontally, both scales being logarithmic. Theory indicates, as shown by Eq. (7) that $\Delta\rho/\rho H^2$ should vary as μ^2. Since μ varies as $T^{-\frac{3}{2}}$ in the lattice scattering range, $\Delta\rho/\rho G^2$ should vary as T^{-3}. This is verified by the data for temperatures between 150 and 300 degrees Kelvin. At lower temperatures the curves droop as is to be expected because

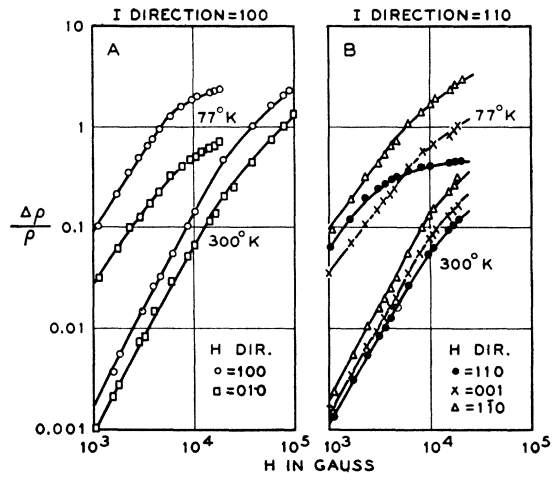

FIG. 4. $\Delta\rho/\rho$ *vs* H in n-type germanium for fixed directions of **I** and **H** and fixed values of T as shown.

[12] Pearson, Haynes, and Shockley, Phys. Rev. **78**, 295 (1950). Hole mobility in germanium at room temperature as determined by the drift method is 1700 cm²/volt sec and electron mobility 3600 cm²/volt sec.

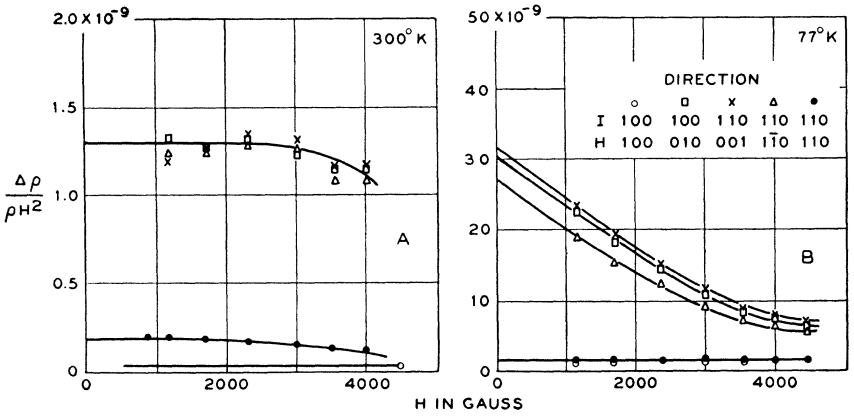

FIG. 5. $\Delta\rho/\rho H^2$ *vs* H in p-type germanium. The fixed directions of **I** and **H** and the values of T are as shown.

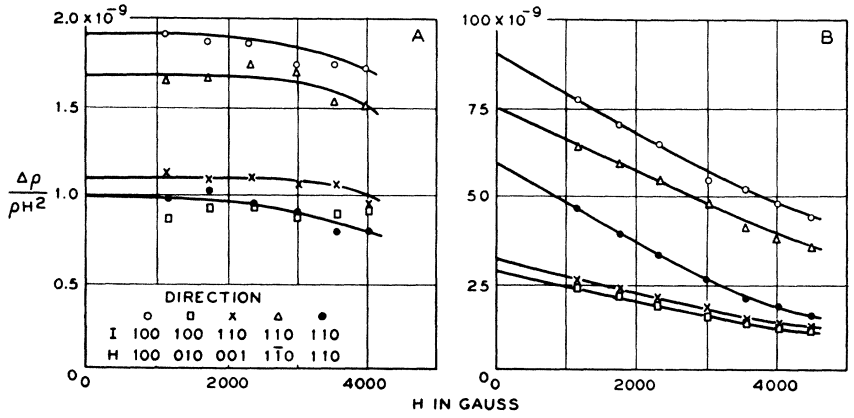

FIG. 6. $\Delta\rho/\rho H^2$ *vs* H in n-type germanium. The fixed directions of **I** and **H** and the values of T are as shown.

of impurity scattering. Results similar to these were obtained for other directions of **i** and **H** and with p-type germanium.

C. Magneto-Resistance Effect *versus* Angle θ

Measurements were next made of $\Delta\rho/\rho$ for fixed values of H and T and a fixed direction of **i** but with varying directions of **H** with respect to the crystal axes. For this purpose the magnetic field was set at 4000 gauss and the electromagnet mounted so that its lines of force were in a fixed horizontal direction. The germanium sample was placed in the magnetic field and attached to a vertically mounted brass rod which could be rotated through 360 degrees about its vertical axis. The germanium sample could be mounted with its length, width, or thickness dimension parallel with the axis of rotation. Measurements were made at either 300 or 77 degrees Kelvin and the current was of course along the lengthwise dimension.

Figure 8 gives the results of such measurements on p-type samples. Figures 8(A) and 8(B) are for a sample cut with its length along the 100 axis, the former giving room temperature data and the latter liquid nitrogen temperature data. Figures 8(C) and 8(D) are for a sample cut with its length along the 110 axis. Figure 8(C) was obtained at room temperature and Fig. 8(D) at liquid nitrogen temperature. In all of these figures $\Delta\rho/\rho$ is plotted vertically and the angle of rotation θ is plotted horizontally. The curves with solid circles apply to the case where the axis of rotation is the sample length, with open circles the axis of rotation is the sample width and with crosses the sample thickness. At θ equal to 0, 90, 180, 270, and 360, the crystal axis parallel to **H** is indicated.

As shown in Fig. 8(A), which applies to a p-type sample at room temperature cut with its length along the 100 axis, $\Delta\rho/\rho$ is constant for all values of θ when the sample length is the axis of rotation. In this experimental arrangement, although **H** rotates through 360 degrees around the 100 crystal axis, it is always perpendicular to **i**. Because of the cubic crystal structure of germanium the experimental result obtained here is to be expected providing H is within the square law range. If the experiment were repeated at 77 degrees Kelvin with the same field strength of 4000 gauss the magneto-resistance effect would be increased by a factor of 6 and the square law range would be exceeded as shown by Fig. 3(A). That such is the case is shown by the upper curve of Fig. 8(B). With **H** parallel to the 001 axis $\Delta\rho/\rho$ has increased from 0.019 to 0.12. Although 90-degree symmetry has been retained, we now have maxima with **H** midway between principal axes.

When the sample was mounted so that its width was along the axis of rotation, the curves designated by

crosses in Fig. 8(A) and Fig. 8(B) were obtained. At θ equal to 0, 180, and 360 degrees **H** is perpendicular to **i** and at θ equal to 90 and 270 **H** is parallel to **i**. As expected from crystal symmetry, rotation with either the width or thickness along the axis of rotation gave identical results.

Figure 8(C) and Fig. 8(D) give the results of similar measurements on a p-type germanium sample cut with its length along the 110 axis. In this case the crystal symmetry is destroyed and $\Delta\rho/\rho$ is no longer constant for rotation about the length axis, the value being greater with **H** along the 001 direction than with **H** along the 110 direction. Although the longitudinal magneto-resistance effect, with **H** and **i** both in the 110 direction, is less than the transverse effect, it does not approach zero as in the 100 cut sample.

Similar rotation measurements on n-type germanium samples are shown in Fig. 9. The more significant features of these curves are: (1) the longitudinal effect is much larger than the transverse effect, (2) in the 110 cut sample the transverse effect with **H** in the 110 direction is larger than in the 001 direction which is opposite to the effect in p-type germanium, and (3) at liquid nitrogen temperatures the same 90-degree symmetry is found as in p-type.

V. CALCULATION OF ANGULAR DEPENDENCE CONSTANTS

The experimentally determined values of $\Delta\rho/\rho H^2$ for selected directions of **i** and **H** as H approaches zero are given in Table I for the n- and p-type germanium samples at 300°K and 77°K. The right-hand column in Table I gives the theoretical values of $\Delta\rho/\rho H^2$ for the selected directions in terms of the phenomenological angular dependence constants b, c, and d obtained by substituting the direction cosines of **i** and **H** in Eq. (5). By taking appropriate combinations of the $\Delta\rho/\rho H^2$ values given in the lower three lines of Table I, the values of b, c, and d shown in Table II are obtained. Using the relations given in Eq. (3) the corresponding phenomenological theory constants β, γ, and δ have been calculated as shown in Table II. The related constants $\alpha=-a\sigma_0=-R_H\sigma_0^2$ are also given in Table II together with the experimentally determined values of zero field conductance (σ_0) and Hall coefficient (R_H in volt cm/amp gauss).

Having obtained the phenomenological constants b, c, and d, it is now possible to calculate the small field magneto-resistance effect in germanium for **i** and **H** oriented in any arbitrary manner. The fact that the experimentally determined curves shown in Fig. 8 and Fig. 9 fit Eq. (5) for most conditions indicates that the phenomenological relationships are in good agreement with experiment.

The particular electronic theory developed by Seitz can be tested by means of Eqs. (9), (10), and (11). The last two predict that $\Delta\rho/\rho H^2$ with **i** and **H** parallel and along 110 should be 2.5 times $\Delta\rho/\rho H^2$ for **i** and **H** along 100. The experimentally determined ratios as found from Table I are 5.2 and 1.4 for p-type germanium at 300°K and 77°K, respectively. For n-type the corresponding ratios are 0.51 and 0.66.

A much more serious discrepancy between the Seitz theory and the experimental results is apparent in the first of Eqs. (9). The integral I can be evaluated for a mean free path independent of energy and is found to be positive.[3] But the experimentally determined β are negative in all cases. Therefore, the first of Eqs. (9) can hold only for imaginary τ_1/τ_0. We are forced to conclude that a theory based on anisotropies in the relaxation time alone does not agree with the observed results.

VI. MAGNETO-RESISTANCE EFFECT OF HIGHER ORDER IN H

The phenomenological theory described above, which extends only to terms of the second order in H, predicts a constant value $\Delta\rho/\rho H^2=b$ when **i** is in the 100 direction and **H** is rotated about the current vector in a plane normal to it. Figures 8(A) and 9(A), which apply to p- and n-type materials respectively at 300°K and 4000 gauss, verify this theory. In Figs. 8(B) and 9(B) the temperature was lowered to 77°K which increased the effect several-fold and induced small modulations in the observed values of $\Delta\rho/\rho$. These have maxima at 45° inclinations (that is with **H** along 011, $0\bar{1}1$, $0\bar{1}\bar{1}$, $0\bar{1}\bar{1}$) and minima at 90° inclinations (010, 001, $0\bar{1}0$, $00\bar{1}$).

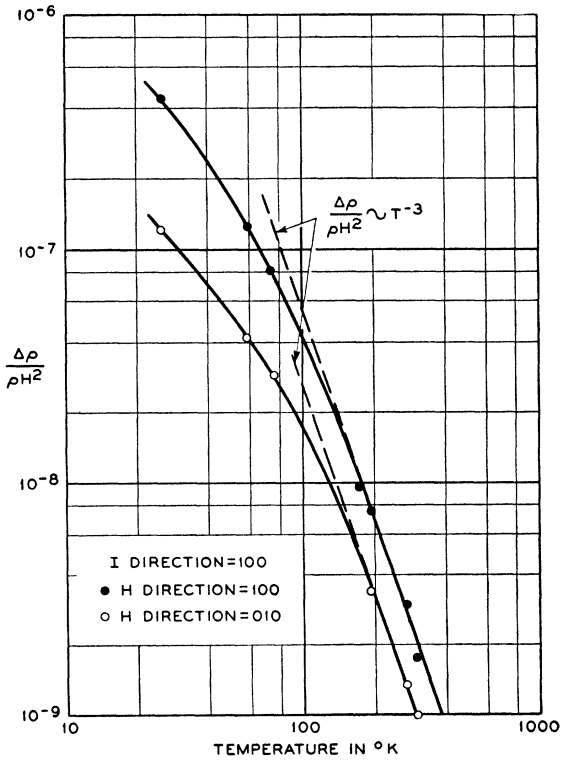

FIG. 7. $\Delta\rho/\rho H^2$ *vs* temperature. $H=4000$ gauss. Directions of **I** and **H** are as shown.

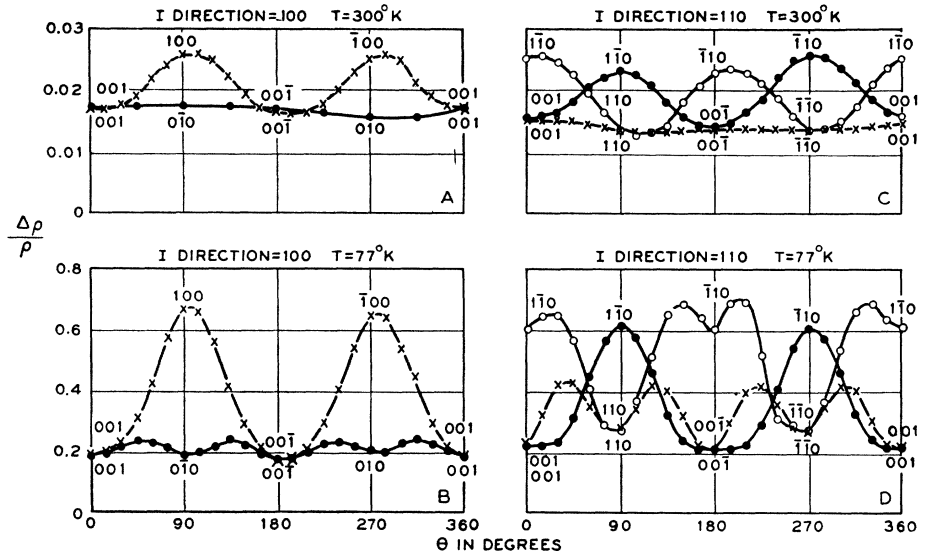

FIG. 8. Variations of $\Delta\rho/\rho$ in p-type germanium as **H** is rotated through an angle θ from 0 to 360 degrees. The magnitude of H is 4000 gauss, and certain cardinal directions of **H** are as indicated. The fixed directions of **I** and the values of T are as shown.

As indicated in Sec. IV above, this new effect arises from exceeding the square law range in H. This may be shown quantitatively by developing the phenomenological expression for $\Delta\rho/\rho$ one stage further, that is to fourth powers in H.

In the same notation as used in Sec. II this extension, obtained from group theory, is

$$\Delta\rho/\rho=H^2[b+c(\sum\iota\eta)^2+d\sum\iota^2\eta^2]+H^4[b'+c'(\sum\iota\eta)^2+d'\sum\iota^2\eta^2+e\sum\eta^4+f\sum\eta_1^2\eta_2^2+g\sum\iota^2\eta^4+h\sum\iota_1\iota_2\eta_1^2\eta_2\eta_3], \quad (12)$$

where $\sum$ signifies the sum over all cyclic permutations of the type indicated. It is seen that the fourth approximation involves the seven additional constants b', c', d', e, f, g, h. This expression is greatly simplified for **i** along the 100 direction and **H** in any direction perpendicular to it. For this condition

$$\left.\begin{aligned}&\iota_1=1, \quad \iota_2=0, \quad \iota_3=0,\\ &\eta_1=0, \quad \eta_2=\cos\theta, \quad \eta_3=\sin\theta,\end{aligned}\right\} \quad (13)$$

and it is easily shown that

$$\Delta\rho/\rho=(b+b'H^2+eH^2)H^2+\tfrac{1}{4}H^4(f-e)\sin^2 2\theta, \quad (14)$$

where θ is the angle between the magnetic vector and the 001 crystal direction. The second term on the right

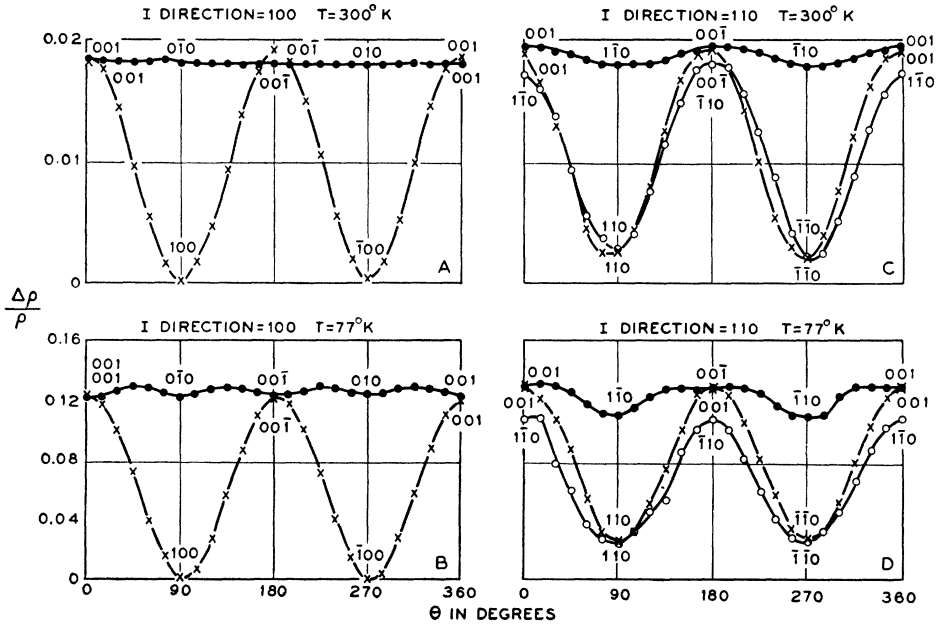

FIG. 9. Variations of $\Delta\rho/\rho$ in n-type germanium as **H** is rotated through an angle θ from 0 to 360 degrees. The magnitude of H is 4000 gauss, and certain cardinal directions of **H** are as indicated. The fixed directions of **I** and the values of T are as shown.

contributes the observed modulations shown in Figs. 8(B) and 9(B), since it has maxima at the 45° inclinations and minima equal to zero at the 90° inclinations.

The explanation of the modulation superposed on the sine curve predicted by second-order theory for the case of **i** parallel to 110 and **H** rotating within the (100, 010) plane is similar. (The experimental result is shown in Fig. 9(D).) In this case

$$\left.\begin{aligned} &\iota_1=1/\sqrt{2}, \quad \iota_2=1/\sqrt{2}, \quad \iota_3=0, \\ &\eta_1=\cos\theta, \quad \eta_2=\sin\theta, \quad \eta_3=0, \end{aligned}\right\} \quad (15)$$

so that the contribution from fourth-order terms to the angular dependence is $\frac{1}{4}H^4(f-2e-g)\sin^2 2\theta$, where θ is the angle between the magnetic vector and the 100 crystal direction.

In either case the amplitude of the modulation should vary as the fourth power of the magnetic field. Figure 10 shows the experimentally observed power law in the case of n-type germanium at 77°K for the conditions applying in Eq. (14). The discrepancy between the theoretical value of 4.0 and the experimental value of 3.4 could either be due to higher power terms or to the fact that an expansion in ascending powers of H is no longer appropriate at the fields employed in these measurements. Such an expansion is particularly inappropriate when saturation effects are approached as in the case of parallel **i** and **H**, Fig. 4. Measurements in fields up to 170 kilogauss and 77°K carried out at the Bell Telephone Laboratories[13] showed full saturation of $\Delta\rho/\rho$ for all parallel directions of field and current, while saturation was approaching in some perpendicular directions.

TABLE I. Measured values of magneto-resistance in germanium as H approaches zero.

I Direction	H Direction	$\Delta\rho/\rho H^2$ p-type 300°K	p-type 77°K	n-type 300°K	n-type 77°K	Theory Eq. (5)
100	100	0.04×10^{-9}	1.4×10^{-9}	1.92×10^{-9}	88×10^{-9}	$b+c+d$
100	010	1.3	30.4	0.90	28.5	b
110	001	1.3	32.2	1.03	31.5	b
110	$1\bar{1}0$	1.3	27	1.68	75	$b+d/2$
110	110	0.21	2	0.99	58	$b+c+d/2$

TABLE II. The magneto-resistance constants of germanium at low magnetic field strengths.

	p-type 300°K	p-type 77°K	n-type 300°K	n-type 77°K
σ_0	2.6	4.4	0.087	0.59
R_H	8.4×10^{-6}	4.4×10^{-5}	-4.4×10^{-4}	-4.8×10^{-4}
a	2.18×10^{-5}	1.94×10^{-4}	-3.83×10^{-5}	-2.84×10^{-4}
α	-5.7×10^{-5}	-8.5×10^{-4}	3.33×10^{-6}	1.68×10^{-4}
b	1.3×10^{-9}	32.2×10^{-9}	1.03×10^{-9}	31.5×10^{-9}
c	-1.09×10^{-9}	-25.0×10^{-9}	-0.69×10^{-9}	-17.0×10^{-9}
d	0	-10.4×10^{-9}	1.3×10^{-9}	87.0×10^{-9}
β	-4.62×10^{-9}	-307×10^{-9}	-0.205×10^{-9}	-65×10^{-9}
γ	4.07×10^{-9}	274×10^{-9}	0.189×10^{-9}	57×10^{-9}
δ	0	45.7×10^{-9}	-0.113×10^{-9}	-51.4×10^{-9}

VII. CONCLUSIONS

It has been shown that the magneto-resistance in single crystals of germanium conforms with the phenomenological relation. However, the constants characteristic of this relation have experimental values that differ from those calculated on the basis of the two electronic theories so far proposed.

Recently magneto-resistance theory has been formulated in great generality by W. Shockley.[5] His results are not restricted to small magnetic fields, but reduce to the existing theories for small **H**. He suggests that the nonspherical shape of the energy surface may be responsible for magneto-resistance. He shows that for germanium the energy surface for electrons at the bottom of the conduction band may consists of either two or three nonspherical sheets. The parameters specifying these should enter the expression for magneto-resistance and thus be determined from experiment. However, the calculations involved are of considerable algebraic complexity and have so far not been carried out.

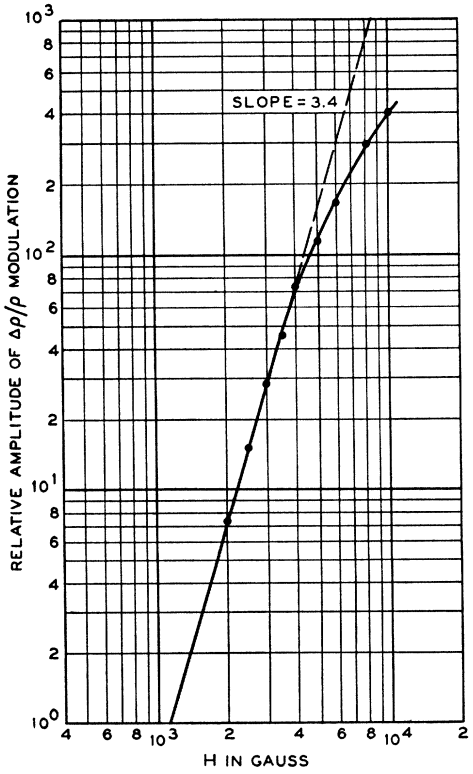

FIG. 10. Amplitude of higher order angular modulation with **i** in the 100 direction and **H** rotating in a plane normal to **i**.

[13] One of the authors has extended these measurements to 170 kilogauss using pulse techniques. H. Suhl, Phys. Rev. **78**, 646 (1950).

VIII. ACKNOWLEDGMENTS

The research described above was carried out under the direction of W. Shockley, to whom the authors are indebted for valuable discussion of the general problem of magneto-resistance. They are also obliged to J. Bardeen, who deduced the second-order dependence of $\Delta\rho/\rho$ as given by Eq. (2) and was frequently consulted as the work progressed, and to C. Herring for help with the group theory involved in the fourth-order expression. Thanks are also due G. K. Teal for supplying the excellent single crystal germanium ingots and to P. W. Foy for assistance with the experimental work.

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The Scattering of Light by Light*

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Cross sections for several processes involving electromagnetic fields in a nonlinear manner are derived from the electrodynamic scattering matrix and are expressed in terms of the fourth-order nonlinear vacuum polarization tensor. The differential cross section for the scattering of light by light is calculated as a function of energy and angle. Numerical values are given for scattering at zero and at ninety degrees in the center-of-mass system. Near 1.75 Mev the forward scattering cross section has its largest value of 4.1×10^{-31} cm²/sterad, while the maximum right-angle scattering takes place near 0.7 Mev with a cross section of 2.8×10^{-31} cm²/sterad, all for unpolarized radiation. Numerical results are also given for scattering at the above angles between circularly polarized states. The conclusions in this paper agree with all the results previously calculated for special cases.

IN an earlier paper[1] the nonlinear interactions between electromagnetic fields were expressed in terms of the polarization of the electron-positron vacuum. In this discussion the polarization tensor derived in I will be related to the cross sections for the occurrence of events which are consequences of the nonlinearity.

The processes to be considered are the scattering of light by light,[2] two-quantum pair creation,[3] the scattering of light in an external field,[4] and the creation of pairs in an external field.[5] An expression for the cross section for each of these is set up in Sec. II. The two cases involving only radiation fields are then treated in detail. A certain simplification occurs in the calculation of the appropriate element of the scattering matrix and of the polarization tensor, because the fact that all the 4-momentum vectors involved are null-vectors, together with the conservation laws, implies that the matrix element depends on the initial and final momenta only by way of two parameters; these may be taken as the incident photon energy and the scattering angle in the center-of-gravity system. The total cross section for two-quantum pair creation is easily obtained from the diagonal (forward scattering) element of the S-matrix, as has been described elsewhere;[5b] this relationship follows from the unitary character of the S-matrix.

To obtain the probability for the scattering of a photon by an external field, the electromagnetic potential in the S-matrix is replaced by the potential of a quantized wave (creation or annihilation operator) plus the external potential which causes the scattering. The matrix element of this operator between one-photon states of the appropriate momenta then is the probability amplitude of the scattering process. Here the evaluation of the polarization tensor is much more difficult because the momentum vectors are not null vectors. The forward scattering matrix element again is simply related to the total cross section for pair-creation in the external field.[5b]

Formulas for the differential cross sections for transitions between photon states of definite linear momentum and definite spin angular momentum (circular polarization) are given to summarize the results of the calculation of the scattering of light by light. This choice is convenient because the transitions fall into two groups: the more probable ones conserve spin angular momentum, while the less probable ones do not. In the limiting cases of low and high energy our results agree with the published ones.[2]

I. SCATTERING CROSS SECTION

The total probability W that a transition takes place from an initial state Ψ_i is given in terms of the scattering matrix S by

$$W=(\Psi_i,\,(S^+-1)(S-1)\Psi_i)$$
$$=-2\,\mathrm{Re}[(\Psi_i,\,(S-1)\Psi_i)]; \quad (1)$$

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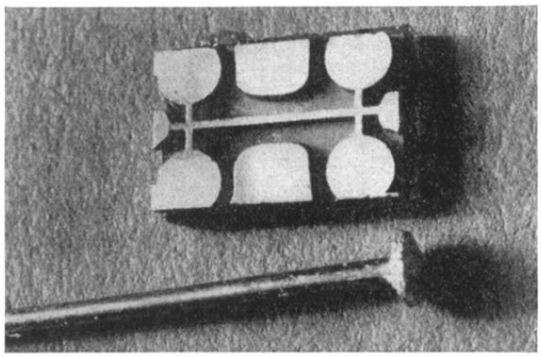

FIG. 2. Photograph of oriented germanium rod for measuring magneto-restistance effect. A common pin is shown for size comparison.