Heisenberg Operators in Quantum Electrodynamics. II

F. J. DYSON
Department of Mathematical Physics, University of Birmingham, Birmingham, England
(Received March 29, 1951)

The equations of motion of quantum electrodynamics are set up in the interaction representation, using a formalism due to Gupta. A new representation, called the intermediate representation, is defined by constructing explicitly a unitary operator $S(t)$, which transforms the state-vector of the interaction representation into the state-vector of the new representation. The intermediate representation is intermediate in behavior between the interaction and Heisenberg representations. In it the low frequency changes in the state of a system are represented by changes in the state-vector, as in the interaction representation, while the high frequency fluctuations are represented by the time-variation of the field operators, as in the Heisenberg representation.

The program of this and a forthcoming paper is to prove that the intermediate representation provides a complete and divergence-free formulation of quantum electrodynamics, with a divergence-free Schrödinger equation which describes accurately the behavior of any physical system. In this paper the mathematical technique is developed which will enable the divergences to be eliminated from the Schrödinger equation. For simplicity, the technique is explained by applying it first to the analysis of the electromagnetic potential operators. The greater part of the paper is occupied with a proof that any fourier component of an electromagnetic potential operator in the intermediate representation is divergence-free after renormalizations have been consistently carried out. The cancellation of the divergences by appropriate compensating terms arising from $S(t)$ is an extremely intricate process, the success of which could not be foreseen without carrying through the calculations in detail.

In a final section, it is explained how Heisenberg operators are to be regarded as a special limiting case of intermediate representation operators. It follows from the preceding analysis, that averages over finite space-time regions of Heisenberg field operators in quantum electrodynamics are divergence-free after renormalization.

I. INTRODUCTION

ONE aim of this paper is to carry through in detail the renormalization program for Heisenberg operators in quantum electrodynamics, as promised in an earlier paper.[1] Since that paper was written, the scope of the whole investigation has been widened by considering instead of Heisenberg operators a more general class of operators, intermediate representation operators, of which the Heisenberg operators are a special limiting case. The physical meaning and purpose of the intermediate representation have been described in a separate publication,[2] in a qualitative way. The second aim of the present paper is to give an exact mathematical definition of the intermediate representation, and to show how a renormalization program can be consistently carried out in it. The greater part of the paper will be occupied with the proof that averages of certain intermediate representation operators over finite space-time regions are divergence-free after renormalization. At the end will be a discussion of the way in which Heisenberg operators can be obtained as limiting forms of intermediate representation operators, and this will complete the proof of the finiteness of Heisenberg operators. The second main step in the program described in RM, the proof that the Schrödinger equation in the intermediate representation is divergence-free, is left for later publication.

II. THE GUPTA FORMALISM

The starting point of the analysis is a formulation of quantum electrodynamics due to Gupta,[3] which differs from the usual formulation in the treatment of charge-renormalization. The Gupta formulation describes charge-renormalization from the beginning as a renormalization of the unit in which the electromagnetic field is measured, and not as a renormalization of the coupling constant e. In conforming correctly to the physical interpretation and logical structure of the theory, the point of view of Gupta has substantial advantages; it is therefore not surprising to find that the use of the Gupta formalism is also demanded in order to make the renormalization of intermediate representation operators mathematically manageable.

The idea of Gupta is to distinguish at the outset between the unrenormalized fields $\mathbf{F}_{\mu\nu}^*$, which appear in the field equations of electrodynamics in the Heisenberg representation, and the renormalized fields $\mathbf{F}_{\mu\nu}$, which are physically observable. Here, as always, heavy type is used for Heisenberg operators. The $\mathbf{F}_{\mu\nu}$ are defined by

$$\mathbf{F}_{\mu\nu}=\theta\mathbf{F}_{\mu\nu}^*, \quad \theta=(e/e_1), \tag{1}$$

where e is the unrenormalized electronic charge and e_1 the renormalized charge. The definitions of e and e_1 have been given elsewhere.[4]

The complete lagrangian density in quantum electrodynamics may be written, with the usual notations, as

$$L^*=-\tfrac{1}{4}\mathbf{F}_{\mu\nu}^*\mathbf{F}_{\mu\nu}^*-\tfrac{1}{2}(\partial\mathbf{A}_\mu^*/\partial x_\mu)^2+L_D +ie\mathbf{A}_\mu^*\bar{\psi}\gamma_\mu\psi+\delta mc^2\bar{\psi}\psi. \tag{2}$$

Here L_D is the lagrangian for a free Dirac field with the observed electronic rest-mass. The second term in

[1] F. J. Dyson, Phys. Rev. **82**, 428 (1951), referred to hereafter as HO.I (Heisenberg Operators. I.).
[2] F. J. Dyson, Proc. Roy. Soc. (London) A**207**, 395 (1951), referred to hereafter as RM (Renormalization Method).
[3] S. N. Gupta, Proc. Phys. Soc. (London) A**64**, 426 (1951). The author wishes to thank Dr. Gupta for letting him see this paper before publication; he is also greatly indebted to Mr. Abdus Salam, who first informed him that Gupta's formalism existed and would be effective in overcoming some serious difficulties which had arisen in the analysis.
[4] F. J. Dyson, Phys. Rev. **75**, 486 and 1736 (1949). These papers will be referred to as I and II.

L^* is zero according to the supplementary condition; it is inserted in order to obtain independent dynamical equations for the potentials $\mathbf{A}_\mu^*$, without changing the physical content of the theory.[5] Gupta proposes to use instead of L^* the lagrangian

$$L=L_0+L_1, \tag{3}$$

$$L_0=-\tfrac{1}{4}\mathbf{F}_{\mu\nu}\mathbf{F}_{\mu\nu}-\tfrac{1}{2}(\partial\mathbf{A}_\mu/\partial x_\mu)^2+L_D, \tag{4}$$

$$L_1=ie_1\mathbf{A}_\mu\bar{\psi}\gamma_\mu\psi+\delta mc^2\bar{\psi}\psi+\tfrac{1}{4}f\mathbf{F}_{\mu\nu}\mathbf{F}_{\mu\nu}, \tag{5}$$

$$f=1-\theta^{-2}. \tag{6}$$

Note that L is not exactly equal to L^* expressed in terms of renormalized fields and potentials. The inserted term, the second term in L_0, differs by a numerical factor from the second term in L^*. The difference in the inserted term, while not affecting the physical consequences of the formalism, causes a substantial difference in the mathematical details of calculations carried out with the two lagrangians.

In the usual treatment of electrodynamics,[6] the interaction representation is set up as follows. The interaction representation potentials A_μ, ψ are operators satisfying field equations derived from the free-field lagrangian L_0. The operators $\mathbf{A}_\mu^*$ and ψ are related to A_μ and ψ by a contact transformation

$$\mathbf{A}_\mu^*=(S^*)^{-1}A_\mu S^*, \quad \psi=(S^*)^{-1}\psi S^*, \tag{7}$$

because both sets of operators satisfy the same canonical commutation relations. An equation of motion for S^* is derived, by means of which S^* can be expressed explicitly in terms of interaction representation operators.

In the Gupta formalism the interaction representation is introduced by finding a unitary operator S directly relating the renormalized potentials $\mathbf{A}_\mu$ to the free-field potentials A_μ, thus:[7]

$$\mathbf{A}_\mu=S^{-1}A_\mu S, \quad \psi=S^{-1}\psi S. \tag{8}$$

The equation of motion for S is found to be

$$ih(\partial S/\partial t)=H_1S, \tag{9}$$

$$H_1=H_1(t)=\int H^I(x)d_3x, \tag{10}$$

$$H^I(x)=H_i(x)+H_s(x)+H_p(x), \tag{11}$$

$$H_i(x)=-ie_1A_\mu(x)\bar{\psi}(x)\gamma_\mu\psi(x), \tag{12}$$

$$H_s(x)=-\delta mc^2\bar{\psi}(x)\psi(x), \tag{13}$$

$$H_p(x)=-\tfrac{1}{4}fF_{\mu\nu}(x)F_{\mu\nu}(x)-\tfrac{1}{2}\theta^2f^2F_{\mu4}(x)F_{\mu4}(x). \tag{14}$$

Equation (9) can be formally integrated, and then Eq. (8) gives the series expansions of renormalized Heisenberg operators directly in terms of interaction representation operators. Thus, one obtains

$$\mathbf{A}_\mu(x)=\sum_{n=0}^{\infty}\left(\frac{i}{hc}\right)^n\int_{-\infty}^{t}dx_1\int_{-\infty}^{t_1}dx_2\cdots\int_{-\infty}^{t_{n-1}}dx_n \times[H^I(x_n),[\cdots[H^I(x_1),A_\mu(x)]\cdots]]. \tag{15}$$

The extra term (14) in the interaction hamiltonian compensates exactly all charge-renormalization effects produced by the radiation interaction (12). Accordingly, the operator (15) will be found to be divergence-free as it stands, without any further renormalization of the unit of potential. Just for this reason, the Gupta formalism is the appropriate one to use for a proof of finiteness of Heisenberg operators.

III. THE INTERMEDIATE REPRESENTATION

Let $g(a)$ be a function of the real variable a, defined for $a\geq0$, with the following properties:

$$g(a)=\int_0^\infty G(\Gamma)e^{-c\Gamma a}d\Gamma, \tag{16}$$

where $G(\Gamma)$ is a function of the real variable Γ, the integral $\int_0^\infty|G(\Gamma)|d\Gamma$ being finite, so that Eq. (16) is uniformly convergent for all $a\geq0$:

$$g(0)=\int_0^\infty G(\Gamma)d\Gamma=1, \tag{17}$$

$$g'(0)=-c\int_0^\infty\Gamma G(\Gamma)d\Gamma=0, \tag{18}$$

$$g(a)\to0 \quad \text{as} \quad a\to\infty. \tag{19}$$

The convergence (19) is supposed sufficiently rapid, so that all integrations in which $g(a)$ appears as a factor will be convergent at infinity. The function $G(\Gamma)$ is allowed to have singularities at most of a simple δ-function character at a set of discrete values of Γ.

Let the interaction $H^I(x)=H^I(x,e_1)$ given by Eq. (11) be written explicitly as a function of the true electronic charge e_1. According to Eq. (104) of II, $(\delta m/m)$ is a power series in e_1, with coefficients which are divergent integrals formally independent of e_1. Also,

$$f=[(e^2-e_1^2)/e^2], \quad \theta^2f^2=[(e^2-e_1^2)^2/e^2e_1^2] \tag{20}$$

may be written as power series of the same kind, in virtue of Eq. (105) of II. Let x' be any space-time point with time-coordinate t' not later than t, the time-coordinate of x. Then the operator H_g^0, a function of the two points x' and x, is defined by

$$H_g^0(x,x')=H^I(x',e_1g(t-t')) =H_{gi}+H_{gs}+H_{gp}, \tag{21}$$

[5] Following E. Fermi, Revs. Modern Phys. 4, 87 (1932). See also G. Wentzel, *Einführung in die Quantentheorie der Wellenfelder* (F. Deuticke, Wien, 1943), p. 112.

[6] J. Schwinger, Phys. Rev. 74, 1439 (1948).

[7] It may appear at first sight paradoxical that the operators $\mathbf{A}_\mu^*$ and $\mathbf{A}_\mu$ should both satisfy the same canonical commutation relations, since they differ essentially by a factor not equal to unity. The commutation relations between the $\mathbf{A}_\mu$ and their derivatives $(\partial\mathbf{A}_\mu/\partial t)$ are not the same as the corresponding commutators of $\mathbf{A}_\mu^*$ with $(\partial\mathbf{A}_\mu^*/\partial t)$. The solution of the paradox is that the momenta conjugate to $\mathbf{A}_\mu^*$ and $\mathbf{A}_\mu$ in the two formalisms are different functions of the time-derivatives.

$$H_{gi}=-ie_1g(t-t')A_\mu\bar{\psi}\gamma_\mu\psi(x'), \tag{22}$$

$$H_{gs}=-\delta_g mc^2\bar{\psi}\psi(x'), \tag{23}$$

$$H_{gp}=-\tfrac{1}{4}f_gF_{\mu\nu}F_{\mu\nu}(x')-\tfrac{1}{2}\theta_g{}^2f_g{}^2F_{\mu4}F_{\mu4}(x'). \tag{24}$$

Writing explicitly the expansions of the coefficients,

$$\delta_g=\sum_{d=2}^{\infty}A_d[e_1g(t-t')]^d, \tag{25}$$

$$f_g=\sum_{d=2}^{\infty}B_d[e_1g(t-t')]^d, \tag{26}$$

$$\theta_g{}^2=[1-f_g]^{-1}. \tag{27}$$

Up to this point, $H_g{}^0(x, x')$ is identical, apart from notations, with the $H_g(t, t')$ described in RM. However, as was already remarked in RM, the correct definition of $H_g(x, x')$ contains additional terms

$$H_g(x, x')=H_g{}^0(x, x')+H_{ga}(x, x')+H_{gb}(x, x'), \tag{28}$$

where H_{ga} is of the form of a transient photon self-energy appearing only while $g(t-t')$ is varying. The precise definitions of H_{ga} and H_{gb} are

$$H_{ga}=-\frac{1}{2c^2}\left[\frac{d}{dt}\left(\frac{g'}{g}(t-t')f_g\right)-\left(\frac{g'}{g}(t-t')\right)^2f_g\right]A_jA_j(x'), \tag{29}$$

$$H_{gb}=-\frac{i}{c}\left[\frac{g'}{g}(t-t')f_g\right]A_4(x')\left(\frac{\partial}{\partial x_j'}\right)A_j(x'). \tag{30}$$

In Eqs. (29) and (30) the index j is summed over the values 1, 2, 3 only.

The intermediate representation is set up by constructing the operator $U(t, t')$, which satisfies the initial condition,

$$U(t, -\infty)=I, \tag{31}$$

and the differential equation for $t'\leq t$,

$$ih(\partial U(t, t')/\partial t')=\left[\int H_g(x, x')d_3x'\right]U(t, t'). \tag{32}$$

Let $\Psi(t)$ be the state-vector of the interaction representation. Then the state-vector of the intermediate representation is $\Phi(t)$, where

$$\Psi(t)=S(t)\Phi(t) \tag{33}$$

and

$$S(t)=U(t, t). \tag{34}$$

By Eqs. (31) and (32), $S(t)$ has the formal expansion

$$S(t)=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-i}{hc}\right)^n\int_{-\infty}^{t}dx_1\cdots\int_{-\infty}^{t}dx_n\times P(H_g(x, x_1), H_g(x, x_2), \cdots, H_g(x, x_n)), \tag{35}$$

the P representing a chronological product as defined in I, and the fourfold integrals dx_r extending over the whole of space-time previous to the time t. Note that the operator factors in $H_g(x, x_m)$ refer to the point x_m and not to the point x, so that the chronological ordering is with respect to x_m.

The meaning of $S(t)$ has been explained in some detail in RM. A formal proof will now be given of the statement made there that

$$S(t)=\exp[iH_0t/\hbar]S\exp[-iH_0t/\hbar], \tag{36}$$

where H_0 is the hamiltonian of the noninteracting fields and S is a time-independent operator. Imagine each operator $H_g(x, x_m)$ to be expanded as a sum of products Π_m of particle emission and absorption operators multiplied by appropriate exponential factors representing plane waves in the variable x_m. Then $S(t)$ is a sum of terms each having a structure similar to

$$A\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\cdots\int_{-\infty}^{t_{n-1}}dt_n[(g(t-t_1))^{\alpha_1}\cdots(g(t-t_n))^{\alpha_n}]\times[\Pi_1\exp(-it_1\Delta E_1/\hbar)]\cdots[\Pi_n\exp(-it_n\Delta E_n/\hbar)]=AJ[\Pi_1\Pi_2\cdots\Pi_n]\exp[-it(\Delta E_1+\cdots+\Delta E_n)/\hbar], \tag{37}$$

where J is an absolutely convergent integral over the variables $(t-t_1)$, $\cdots$, $(t-t_n)$, and is independent of t. Since ΔE_m is just the energy difference between initial and final states in a transition effected by the operator Π_m, Eq. (37) may be written

$$\exp[iH_0t/\hbar]\{AJ\Pi_1\cdots\Pi_n\}\exp[-iH_0t/\hbar]. \tag{38}$$

Since $S(t)$ is a sum of terms (38), it is of the form (36) as required. It is important that this argument fails completely in the limiting case $g(t-t')=1$, when $S(t)$ becomes the transformation function leading from the interaction to the Heisenberg representation. In that case the integrals J diverge whenever real energy-conserving processes are occurring, and $S(t)$ depends on t in a much more complicated way than is indicated by Eq. (36).

The electromagnetic potentials in the intermediate representation are given by

$$A_{\mu g}(x)=S^{-1}(t)A_\mu(x)S(t), \tag{39}$$

and other intermediate representation operators are defined similarly. As in HO.I, averages of such operators over finite space-time regions are the primary object of study. These averages are linear superpositions of fourier-transformed operators such as

$$A_{\mu g}(p)=(2\pi)^{-4}\int A_{\mu g}(x)\exp(-ip\cdot x)dx. \tag{40}$$

The main program of the present paper is to prove that $A_{\mu g}(p)$ is free of divergences. This will be accomplished in Secs. IV–X. A similar analysis can be applied to the matter-field operator $\psi_g(p)$ or to the current-operator

$j_{\mu g}(p)$, leading to results which are discussed in Sec. XI. It is found that $\psi_g(p)$ is a divergence-free expression multiplied by a constant divergent renormalization factor. In general, $j_{\mu g}(p)$ does not have a similar property; only in the limit, as the intermediate representation tends to the Heisenberg representation, does $j_{\mu g}(p)$ become divergence-free after renormalization.

IV. GRAPHICAL REPRESENTATION OF OPERATORS

Let $A_{\mu g}(p)$ be expanded into its normal constituents according to the rules given in Sec. II of HO.I. As in Sec. IV of HO.I, let M be the coefficient in the expansion multiplying a particular normal product

$$N(\bar{\psi}_\alpha(p_1)\cdots\psi_\delta(p_i)\cdots A_\lambda(p_j)\cdots A_\nu(p_l)). \quad (41)$$

The objective of Secs. IV–V is to obtain an expression for M in terms of integrals in momentum-space, making use of the results summarized in Sec. VIII of HO.I.

By the definitions (35), (39), and (40), one has

$$A_{\mu g}(p)=\frac{1}{(2\pi)^4}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{\hbar c}\right)^n\int_{-\infty}^{\infty}e^{-ip\cdot x}dx \times\int_{-\infty}^{\infty}dx_1\cdots\int_{-\infty}^{\infty}dx_n S_g, \quad (42)$$

$$S_g=\sum\theta(x-x_1)\theta(x_1-x_2)\cdots\theta(x_{n-1}-x_n) \times[H_g(x,x_n),[\cdots[H_g(x,x_1),A_\mu(x)]\cdots]], \quad (43)$$

the summation in Eq. (43) being over the $(n!)$ permutations of the points $x_1, \cdots, x_n$. Each of the factors $H_g(x, x_i)$ in Eq. (43) is a sum of a great variety of terms, according to Eqs. (21)–(30). By a "term" here is meant either Eq. (22), or a single term of a particular order in e_1 chosen from one of the infinite series occurring in Eqs. (23), (24), (29), and (30). With this meaning of the word "term," let S_g be split up into parts

$$S_g=\sum_Z S_Z, \quad (44)$$

where

$$S_Z=\sum\theta(x-x_1)\cdots\theta(x_{n-1}-x_n) \times[Z_n(x,x_n),[\cdots[Z_1(x,x_1),A_\mu(x)]\cdots]]. \quad (45)$$

For each i, $Z_i(x, x_i)$ is a single term derived from $H_g(x, x_i)$. The summation in Eq. (44) is over the possible ways of choosing one Z_i for every i. The summation in Eq. (45) is over the $(n!)$ permutations of the indices $1, \cdots, n$, the permutation being applied simultaneously to the indices of the Z_i and of the x_i.

Each Z_i consists of an operator Y_i referring to the point x_i, multiplied by a coefficient which contains e_1 raised to a certain power m_i. If Z_i is taken from Eqs. (22), (23), or (24), then one has

$$Z_i=Q_i[e_1 g(t-t_i)]^{m_i}Y_i(x_i), \quad (46)$$

where Q_i is a numerical coefficient. If Z_i is taken from Eq. (30), then one has

$$Z_i=Q_i e_1^{m_i}(g'(t-t_i))(g(t-t_i))^{m_i-1}Y_i(x_i), \quad (47)$$

and if from Eq. (29),

$$Z_i=Q_i e_1^{m_i}[g''(t-t_i)g(t-t_i)+(m_i-2)(g'(t-t_i))^2] \times(g(t-t_i))^{m_i-2}Y_i(x_i). \quad (48)$$

Let

$$N=\sum m_i \quad (49)$$

be the degree of S_Z in e_1. Then, one obtains

$$S_Z=Q_Z e_1^N\Pi_g S_Y, \quad (50)$$

$$S_Y=\sum\theta(x-x_1)\cdots\theta(x_{n-1}-x_n) \times[Y_n(x_n),[\cdots[Y_1(x_1),A_\mu(x)]\cdots]], \quad (51)$$

where Q_Z is the product of the Q_i, and Π_g is the product of functions g, g', g'' occurring in Eqs. (46), (47), and (48).

Let the factors in Π_g be expanded by means of the integral representation (16). Then, one obtains

$$\Pi_g=\int_0^\infty G(\Gamma_{10})d\Gamma_{10}\cdots\int_0^\infty G(\Gamma_{N0})d\Gamma_{N0}E_\Gamma F_\Gamma. \quad (52)$$

Here E_Γ is a product of factors Γ_{j0} and Γ_{j0}^2 arising from the differentiations in Eqs. (47) and (48). If no terms of the form (47) or (48) occur in S_Z, then one has $E_\Gamma=1$. The last factor in Eq. (52) is

$$F_\Gamma=\exp[\Lambda_1\cdot(x-x_1)+\cdots+\Lambda_n\cdot(x-x_n)], \quad (53)$$

arising from the exponential factors in Eq. (16). As in Sec. IV of HO.I, each Γ_{j0} is regarded as the fourth component of a vector

$$\Gamma_j=(0,0,0,\Gamma_{j0}), \quad (54)$$

and each Λ_i is a sum of m_i vectors Γ_j arising from the m_i factors g, g', g'' in Z_i.

Summarizing the results of the analysis so far,

$$A_{\mu g}(p)=\frac{1}{(2\pi)^4}\sum_{n=0}^{\infty}\left(\frac{i}{\hbar c}\right)^n\sum_Z Q_Z e_1^N J_Z, \quad (55)$$

$$J_Z=\int_0^\infty G(\Gamma_{10})d\Gamma_{10}\cdots\int_0^\infty G(\Gamma_{N0})d\Gamma_{N0}E_\Gamma I_n(\Lambda,p), \quad (56)$$

$$I_n(\Lambda,p)=\frac{1}{n!}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dx_1\cdots\int_{-\infty}^{\infty}dx_n e^{-ip\cdot x}F_\Gamma S_Y. \quad (57)$$

This $I_n(\Lambda, p)$ is of precisely the same form as the $I_n(\Gamma, p)$ defined in Sec. IV of HO.I, bearing in mind the remarks at the end of Sec. III of HO.I. There is only the important difference that the exponential damping factors (53) now appear naturally instead of being arbitrarily inserted. The results of HO.I are now directly applicable to the evaluation of $A_{\mu g}(p)$.

The normal constituents of the multiple commutator S_Y will be enumerated by means of Feynman graphs G

with $(n+1)$ vertices x, x_1, $\cdots$, x_n. These graphs are slightly different from those which were used in HO.I and in earlier papers for two reasons. First, only a single operator A_μ acts at the point x, and therefore only a single photon line is incident at the vertex x of G. Second, the operators Y_i may be of six distinct forms, namely,

$$A_\mu\bar{\psi}\gamma_\mu\psi,\quad \bar{\psi}\psi,\quad F_{\mu\nu}F_{\mu\nu},\quad F_{\mu 4}F_{\mu 4},\quad A_iA_i,\quad A_4\,\mathrm{div}A_i. \tag{58}$$

The first two of these forms for Y_i occurred also in earlier calculations. The corresponding vertices x_i in G were, respectively, vertices with two electron lines and one photon line incident, and vertices with only two electron lines incident. Now, whenever Y_i has one of the last four forms (58), the corresponding vertex x_i is of a new type, having two photon lines and no electron line incident.

The coefficient M is a sum of contributions M_Z from the various terms in the sum (55). Each M_Z is a sum of contributions $M'(G)$ from the various graphs which represent possible factor pairings of S_Y and which have the correct external lines to give rise to normal constituents with the operator factor (41). Each $M'(G)$ is further subdivided into contributions $M(G)$ with a particular association of each factor in (41) to one external line of G, so that each external line represents a particle carrying a certain definite momentum p_r.

In order to evaluate $M(G)$ by the methods of HO.I, it is not necessary to enumerate all the doubled graphs G_T derived from G. It is necessary only to choose arbitrarily one doubled graph, which will remain fixed throughout the subsequent discussion; the results are independent of which doubled graph is chosen. Therefore, the notation G_T may now be dropped, and G will denote either the original Feynman graph or the chosen doubled graph derived from it. The doubled lines in G form a "tree," which is connected and simply connected. Regarding the vertex x as the "root" of the tree, a unique "up" direction is fixed in every doubled line, namely, the direction pointing away from x. Every line of G has, independently of the doubling, an inherent direction which is supposed to be marked in it by an arrow. For each doubled line λ_j, there is defined the index η_j which takes the value ± 1 according as the arrow in λ_j points down or up. Further, for each λ_j there is defined the vector Δ_j, which is the sum of the Λ_i corresponding to all the vertices x_i on that part of the tree which is "above" λ_j. This definition of Δ_j corresponds to Eq. (40) of HO.I.

V. MOMENTUM-SPACE INTEGRALS

By Eq. (56), $M(G)$ may be written in the form,

$$M(G)=\int_0^\infty G(\Gamma_{10})d\Gamma_{10}\cdots \times\int_0^\infty G(\Gamma_{N0})d\Gamma_{N0}E_\Gamma M(G,\Lambda). \tag{59}$$

The rules for expressing $M(G,\Lambda)$ as a momentum-space integral will now be formulated. Consider the corresponding coefficient $M^F(G)$, which is obtained from $M(G,\Lambda)$ by replacing the factors $F_\Gamma S_Y$ in Eq. (57) by a chronological product

$$P(Y_1(x_1),\cdots,Y_n(x_n),A_\mu(x)) \tag{60}$$

simply. This $M^F(G)$ is a coefficient of the kind which arises in the evaluation of the S-matrix; it can immediately be expressed as a momentum-space integral by following the rules given in II. Certain complications, which arise from the presence of time-derivatives in the third and fourth of the operators (58), are here temporarily ignored and will be discussed in the following section. The form of the integral representation of $M^F(G)$ is

$$M^F(G)=A\int_{-\infty}^{\infty}dk_{n+1}\cdots\int_{-\infty}^{\infty}dk_b\Phi(k_{n+1},\cdots,k_b), \tag{61}$$

where A is a numerical coefficient and the integration is over the $(b-n)$ independent 4-vectors $k_{n+1},\cdots,k_b$, one corresponding to each undoubled internal line of G. The function Φ is a product of factors contributed by the various vertices and lines of G individually. An undoubled internal electron line contributes a factor

$$(k_{a\mu}\gamma_\mu-\mathrm{im})^{-1}. \tag{62}$$

An undoubled internal photon line contributes a factor

$$(k_a^2)^{-1}. \tag{63}$$

A doubled electron line λ_j contributes a factor

$$(l_{j\mu}\gamma_\mu-\mathrm{im})^{-1}, \tag{64}$$

where l_j is a certain linear combination of the vectors k_a and the external momenta p_r appearing in (41); in the notations of Sec. V of HO.I, we have

$$l_j=-\eta_j(q_j+u_j). \tag{65}$$

A doubled photon line λ_j contributes a factor

$$(l_j^2)^{-1}. \tag{66}$$

Each external line contributes a certain constant spinor or polarization vector. Each vertex at which the first type of interaction (58) is operating contributes a factor γ_μ. All the factors so far enumerated occur exactly as described in the S-matrix analysis II. Finally, each vertex x_i at which the third or the sixth type of interaction (58) is operating contributes a factor

$$(q^2\delta_{\alpha\beta}-q_\alpha q_\beta)\quad\text{or}\quad(\delta_{\alpha 4}q_\beta-\delta_{\beta 4}q_\alpha), \tag{67}$$

respectively, where q is the momentum vector belonging to both of the photon lines incident at x_i. In the following section it will be shown that the correct treatment of any graphs involving the fourth interaction (58) is to omit them entirely in the calculation of $M^F(G)$. Therefore, Φ is a product of factors (62), (63),

(64), (66), (67), apart from numerical coefficients. The integration in Eq. (61) is to be taken in the usual Feynman sense, i.e., to each of the factors in the denominator of Φ is added a small negative imaginary term $-i\epsilon$, which is made to tend to zero after the integration is performed.

The integral representation for $M(G, \Lambda)$ is obtained from Eq. (61) by making the following three changes. First, the factors (63) and (66) are changed to

$$(k_a^2+\lambda^2)^{-1}, \quad (l_j^2+\lambda^2)^{-1}, \tag{68}$$

where λ is a fictitious photon mass. Second, each vector l_j in (64) and (66) is changed to

$$R_j=l_j+i\eta_j\Delta_j; \quad j=1, \cdots, n. \tag{69}$$

Third, if x_i is a vertex at which the third or the sixth type of interaction (58) is operating, then the two photon lines incident at x_i both carry the same momentum q, and the vertex gives rise to a factor (67) in $M^F(G)$. The factor (67) is now to be written

$$(q'\cdot q''\delta_{\alpha\beta}-q_\beta' q_\alpha'') \quad \text{or} \quad (\delta_{\alpha 4}q_\beta''-\delta_{\beta 4}q_\alpha'), \tag{70}$$

where q' is the momentum vector carried by one of the two lines and q'' is carried by the other. Actually, it is true that $q'=q''=q$; but it is necessary to distinguish formally between q' and q'' in order to specify the third change to be made in Eq. (61) as follows. If q' belongs to a doubled line λ_j so that $q'=l_j$, then q' is to be replaced in Eq. (70) by R_j; if q' belongs to an undoubled line or to an external line, then q' is left unchanged; and similarly for q''.

It is convenient to write

$$R_a=k_a; \quad a=n+1, \cdots, b. \tag{71}$$

Then the integral $M(G, \Lambda)$ becomes

$$M(G, \Lambda)=A\int_{-\infty}^{\infty} dk_{n+1}\cdots\int_{-\infty}^{\infty} dk_b\Phi_\Lambda(k_{n+1}, \cdots, k_b), \tag{72}$$

where Φ_Λ is a product containing a factor

$$(R_{s\mu}\gamma_\mu-\mathrm{im})^{-1} \tag{73}$$

for every internal electron line of G, doubled or undoubled, and a factor

$$(R_s^2+\lambda^2)^{-1} \tag{74}$$

for every internal photon line. Also, Φ_Λ contains factors

$$(R_r\cdot R_s\delta_{\alpha\beta}-R_{r\beta}R_{s\alpha}), \tag{75}$$

$$(\delta_{\alpha 4}R_{s\beta}-\delta_{\beta 4}R_{r\alpha}), \tag{76}$$

arising from vertices at which two internal photon lines are incident. The integration in Eq. (72) is defined in the following way. First, the integral is to be evaluated as a Feynman integral, like Eq. (61), with the quantities $(i\Delta_j)$ treated as real numbers and the masses m and λ treated as positive and very large. The result is an analytic function E of m, λ, and the Λ_j. Then, while m and λ are kept large and positive, the Λ_j are continuously varied from imaginary to positive real values; next, while the Λ_j are kept real and positive, m and λ are decreased through real values continuously from their fictitious large values to the physical value of m and the value $\lambda=0$. During these successive variations the function E remains analytic, and the analytic continuation of E obtained in this way defines $M(G, \Lambda)$ for the values of m and Λ_j which are physically of interest.

To summarize the results of this section, the coefficient M is a sum of terms $M(G)$ given by Eq. (59), where $M(G, \Lambda)$ is an integral of the form (72). To evaluate $M(G, \Lambda)$ the analytic continuation method of HO.I is in general required, but it should be remarked that in practice the process of analytic continuation is usually trivial. The evaluation of Eq. (72), treated as a Feynman integral, can be carried out by using the well-known methods of Feynman,[8] leading to an explicit analytic formula. The analytic continuation then consists merely in substituting the physical values of m and the Γ_i into this analytic formula.

VI. DISCUSSION OF DERIVATIVE INTERACTIONS

The Gupta formalism introduces into the interaction lagrangian the term

$$T_1=\tfrac{1}{4}fF_{\mu\nu}(x)F_{\mu\nu}(x), \tag{77}$$

containing time-derivatives of the potentials. This term appears in the interaction hamiltonian together with a supplementary term T_2, given by Eq. (14). T_2 is the familiar "normal-dependent" term which always appears in the interaction hamiltonian in theories with interactions involving derivatives.[9] It is well-known[10] that when the S-matrix is calculated in such theories, the effects of T_2 are always exactly cancelled by certain singular expressions arising from the higher order effects of T_1. In fact, the correct procedure in calculating the S-matrix is to ignore T_2 entirely, at the same time using the simple rules of Feynman to represent the effects of T_1 as integrals in momentum-space, since these rules omit just the singular expressions arising from T_1. Gupta[3] has pointed out that such a procedure can easily be justified in the case of his formalism.

It is generally true that an exactly similar procedure is correct for the calculation of Heisenberg operators or of intermediate representation operators by the methods of HO.I, in all theories with derivative interactions. This will now be demonstrated for the special case of the Gupta formalism.

---

[8] R. P. Feynman, Phys. Rev. **76**, 769 (1949), Appendix.
[9] S. Tomonaga and S. Kanesawa, Prog. Theor. Phys. **3**, 1 and 101 (1948).
[10] P. T. Matthews, Phys. Rev. **76**, 684 (1949).

By Eq. (27), the interaction (24) may be written

$$H_{gp}=H_{gm}+H_{ge}, \tag{78}$$

$$H_{gm}=-\tfrac{1}{4}f_g F_{ij}(x')F_{ij}(x'), \tag{79}$$

$$H_{ge}=-\tfrac{1}{2}f_{4g}F_{\mu 4}(x')F_{\mu 4}(x'), \tag{80}$$

where

$$f_{4g}=[f_g/(1-f_g)]. \tag{81}$$

The term H_{gm} involves only the space-derivatives of the potentials, and gives rise to no singular expressions. Therefore, only H_{ge} need be further discussed. Consider the evaluation of any coefficient $M(G, \Lambda)$ derived from a graph G containing a vertex x_1 at which H_{ge} is operating. Let λ_j and λ_k be the two photon lines incident at x_1. If the ends of λ_j and λ_k remote from x_1 are vertices at which the interactions H_{gi}, H_{gm}, H_{ga}, or H_{gb} are operating, then the operator $H_{ge}(x_1)$ introduces into $M(G, \Lambda)$ only the first time-derivatives of D-functions. Just as in the calculation of the S-matrix, these first derivatives give rise to no singular expressions. Singular expressions will be obtained only when a line λ_j joins two vertices x_1, x_2 at both of which H_{ge} is operating.

In order to determine the nature of the singular expressions, it is necessary to return temporarily to the analysis in Sec. IV of HO.I, where $M(G, \Lambda)$ is expressed as a sum of contributions $M(G_T, \Lambda)$ from all the doubled graphs G_T derived from G. Suppose first that G_T is a doubled graph in which λ_j is an undoubled line. Then λ_j contributes to $M(G_T, \Lambda)$ a factor

$$D(x_1-x_2) \quad \text{or} \quad D^1(x_1-x_2). \tag{82}$$

The derivatives at x_1 and x_2 operate directly on the factor (82), introducing a second time-derivative of (82); but, since no θ-function or ϵ-function is associated with (82), the second time-derivative can be represented by a momentum-space integral of the usual simple form, and no singular term is left over. The singular expressions are obtained only when λ_j is a doubled line.

Suppose now that λ_j is doubled, and write down explicitly the factors in $M(G_T, \Lambda)$ arising from the vertices x_1, x_2 and the line λ_j. These factors are, according to (44) of HO.I,

$$\begin{aligned}F(\lambda_j)=&F_{\mu 4}(x_1)F_{\nu 4}(x_2)\theta(x_1-x_2)\exp(\Delta_j\cdot(x_1-x_2))\\&\times[\{\delta_{\mu\alpha}(\partial/\partial x_{14})-\delta_{4\alpha}(\partial/\partial x_{1\mu})\}\{\delta_{\nu\beta}(\partial/\partial x_{24})\\&-\delta_{4\beta}(\partial/\partial x_{2\nu})\}\delta_{\alpha\beta}D(x_1-x_2)].\end{aligned} \tag{83}$$

The part of $F(\lambda_j)$ containing second time-derivatives is

$$\begin{aligned}&-F_{\mu 4}(x_1)F_{\mu 4}(x_2)\theta(x_1-x_2)\exp(\Delta_j\cdot(x_1-x_2))\\&\qquad\times[(\partial^2/\partial x_{14}{}^2)D(x_1-x_2)].\end{aligned} \tag{84}$$

The usual momentum space integral formula for $M(G_T, \Lambda)$ is obtained if one replaces (84) in $F(\lambda_j)$ by

$$\begin{aligned}&-F_{\mu 4}(x_1)F_{\mu 4}(x_2)\exp(\Delta_j\cdot(x_1-x_2))\\&\qquad\times(\partial^2/\partial x_{14}{}^2)(\theta(x_1-x_2)D(x_1-x_2))].\end{aligned} \tag{85}$$

Therefore, the additional singular contribution to $M(G_T, \Lambda)$ is obtained if one replaces $F(\lambda_j)$ by the difference between (84) and (85), which after an integration by parts is

$$\begin{aligned}F_s(\lambda_j)=&F_{\mu 4}(x_1)F_{\mu 4}(x_2)\exp(\Delta_j\cdot(x_1-x_2))\\&\times[(\partial/\partial x_{14})\theta(x_1-x_2)][(\partial/\partial x_{14})D(x_1-x_2)]\end{aligned} \tag{86}$$

$$=F_{\mu 4}(x_1)F_{\mu 4}(x_2)\delta(x_1-x_2). \tag{87}$$

The exponential factor has disappeared from Eq. (87); the δ denotes an ordinary 4-dimensional δ-function. In virtue of Eq. (87), the singular contribution to $M(G_T, \Lambda)$ from the line λ_j is equal to a multiple of $M(G_T', \Lambda)$, where G_T' is the doubled graph obtained from G_T by removing λ_j and replacing x_1 and x_2 by a single vertex at which H_{ge} is operating. The G_T' derived in this way is a correctly constructed doubled graph when λ_j is doubled in G_T, whereas if λ_j were not doubled, G_T' would not be simply connected and would not be admissable.

It is clear from the preceding analysis that the singular contribution to $M(G, \Lambda)$, obtained when two operators H_{ge} occur at adjacent vertices of G, is always a multiple of $M(G', \Lambda)$, where G' is the graph obtained from G by collapsing the two vertices into one. It can be proved by a more detailed calculation that the effect of these singular contributions from $M(G, \Lambda)$ is accurately reproduced, if $M(G', \Lambda)$ is calculated with the coefficient f_{4g}, appearing in H_{ge} according to Eq. (80), changed to the new value

$$f_{4g}-f_{4g}{}^2. \tag{88}$$

However, it may happen that G' itself still contains two adjacent vertices at which H_{ge} operates, which will give rise to further singular contributions. To avoid this complication, it is best to consider from the beginning the possibility that G contains a string of r vertices $x_1, \cdots, x_r$, joined by $(r-1)$ photon lines, the interaction H_{ge} operating at each vertex of the string. Such a string of vertices, if G_T is any doubled graph in which all the $(r-1)$ photon lines are doubled, will make a singular contribution to $M(G_T, \Lambda)$ with the factor

$$F_{\mu 4}(x_1)F_{\mu 4}(x_r)\delta(x_1-x_2)\cdots\delta(x_{r-1}-x_r) \tag{89}$$

analogous to Eq. (87). Arguing in this way, one finds that all singular contributions to $M(G, \Lambda)$ will be correctly allowed for if the nonsingular terms are written down in the usual way, with the coefficient f_{4g} in H_{ge} changed to

$$f_{4g}-f_{4g}{}^2+f_{4g}{}^3-\cdots=f_g, \tag{90}$$

according to Eq. (81).

The change of f_{4g} to f_g in Eq. (80) is equivalent to simply omitting the term $F_{\mu 4}F_{\mu 4}$ in Eqs. (24) and (58). Therefore, the conclusion of this discussion of derivative interactions is to justify the rules which were formulated in Sec. V for the evaluation of $M(G)$. In fact, the formulas (67) and (75), (76) were obtained just by

translating the term $F_{\mu\nu}F_{\mu\nu}$ in (24) into a momentum-space integral, ignoring all singular contributions and at the same time the term in $F_{\mu 4}F_{\mu 4}$.

VII. GENERAL SURVEY OF DIVERGENCES

All the divergences which can arise in the present theory make their appearance in momentum-space integrals of the form (72). The integrals (72) are derived from Eq. (61) by the replacements described in Sec. V, replacements which change neither the general form of the integrals nor their convergence or divergence at large momenta. The integrals (61) are of the kind already discussed in the analysis of the S-matrix. The introduction of new vertices at which the interactions (24), (29), (30) operate does not introduce any new types of divergent integral. Therefore, the possible divergences in the present theory are precisely the same as those which were studied in the theory of the S-matrix developed in II.

Although the divergences are all of a kind which is already familiar from the S-matrix analysis, it is not at all to be expected at first sight that the methods of isolating and removing the divergences, which were adequate for making the S-matrix finite, will prove adequate also here. First, the divergent integrals now involve a multitude of parameters Λ_i, whereas before they depended upon at most two 4-vector parameters. Second, and more important, the relativistic invariance of the S-matrix, which was of decisive importance for the unambiguous separation of renormalization effects, is altogether lacking in the present calculations. However, in spite of these two serious obstacles, it is found that the elimination of divergences can be carried out for intermediate representation operators, in a way which is consistent and unambiguous. This is possible because of the close formal similarity between the integrals (72) and (61). Although Eq. (72) is not an invariant quantity, its lack of invariance lies only in the fact that the vectors Γ_j are of the special form (54). Considered as a function of the vectors Γ_j, Eq. (72) is a formally covariant expression; and therefore the method of separating divergent parts from covariant integrals can still be applied to it. In this way the lack of real invariance in Eq. (72) turns out to be unimportant. The treatment of divergences in Eq. (72) is more complicated in detail than the treatment of the S-matrix because of the greater number of parameters, but no essentially new problems are encountered.

The divergences in a particular $M(G, \Lambda)$ are associated with local features of the graph G, just as in the S-matrix. There are logarithmic divergences arising from vertex parts in G, linear divergences arising from electron self-energy parts, and quadratic divergences arising from photon self-energy parts. It is easy to verify that Furry's theorem[11] applies to integrals of the form (72). Also, the factors in Eq. (72) corresponding to "scattering of light by light" processes[12] give rise only to logarithmic divergences which cancel identically after the contributions from different G are added together. Therefore, the real divergences which have to be isolated and removed from $M(G, \Lambda)$ are limited to the three types already mentioned.

The vertices of G at which the interaction H_{gi} operates will be called "ordinary" vertices. Irreducible proper and improper vertex parts and self-energy parts are defined as in II. The irreducible vertex parts are identically the same as for the S-matrix, consisting always of a network of ordinary vertices only. Also, the irreducible electron self-energy parts are the same as before, either consisting of ordinary vertices, or consisting of a single vertex at which H_{gs} operates. The irreducible photon self-energy parts are the same as before, consisting of ordinary vertices, except that now a new photon self-energy part is added consisting of a single vertex at which either H_{gp}, H_{ga}, or H_{gb} operates. Vertex and self-energy parts, in which all the vertices are ordinary, will themselves be called "ordinary." Thus, all irreducible vertex and self-energy parts, with the exception of the single-vertex self-energy parts, are ordinary. When x_i is an ordinary vertex, the index m_i defined by Eq. (46) is unity, and the vector Λ_i associated with x_i reduces to a single Γ_i.

In the following three sections, it will be proved that all the divergences arising from ordinary irreducible vertex and self-energy parts are canceled identically by counter-terms arising from H_{gs}, H_{gp}, H_{ga}, H_{gb} operating at the single-vertex self-energy parts. The arguments will apply verbatim not only to ordinary irreducible parts, but also to all ordinary proper reducible vertex and self-energy parts, assuming that the divergences arising from sources interior to these parts have been canceled out previously. By application of the cancellation first to irreducible parts, and then in succession to more and more complicated reducible parts, all the divergences arising from ordinary vertex and self-energy parts will ultimately be canceled by the contributions from the single-vertex parts. Used in this way, the methods of the following three sections are sufficient to eliminate all the divergences from the theory.

It is one of the great advantages of the Gupta formalism that all mass and charge renormalization effects are canceled automatically whenever they appear in the course of the calculations. This is in contrast to the method used in II. There, the mass renormalization terms were removed by an automatic cancellation; but the charge renormalization effects were handled by a much more complicated procedure, being retained through the calculations and finally collected into the coefficients Z_1, Z_2, Z_3, which renormalized the charge explicitly according to Eq. (86) of II. The Gupta formalism works from the beginning with the renormalized charge e_1, and makes unnecessary the elaborate

[11] W. H. Furry, Phys. Rev. **51**, 125 (1937). See Sec. IV of II.

[12] See Sec. VI of II.

discussion in Sec. VII of II which justifies the step-by-step removal of divergences arising from reducible graphs. The program of successive removal of divergences is now justified simply by the fact that each divergence from an ordinary self-energy part appears at every stage of the calculations accompanied by a compensating term from a single-vertex part.

The Gupta formalism would not have simplified the renormalization problem so greatly, if Ward[13] had not previously proved that in the S-matrix the renormalization coefficients Z_1 and Z_2 are equal and exactly compensate each other, so that true charge renormalization arises only from Z_3. In consequence of Ward's work, Gupta needed to introduce an explicit counterterm only for the charge-renormalization effects produced by photon self-energy parts. The spurious charge-renormalization effects from vertex parts and electron self-energy parts cancel each other out. In the following sections, it will be proved that this cancellation of divergences without counterterms continues to be valid for intermediate representation operators. In the proof, Ward's identity plays an essential part.

VIII. REMOVAL OF DIVERGENCES ARISING FROM PHOTON SELF-ENERGY PARTS

Let W be an ordinary proper photon self-energy part, irreducible or reducible. Let G be a graph in which W occurs, $M(G)$ and $M(G,\Lambda)$ the coefficients derived from G as explained in Secs. IV–V. If W is reducible, it is supposed that these coefficients have been freed of all divergences arising from internal pieces of W, the internal divergences having been canceled out at an earlier stage of the calculation, as described in Sec. VII. Let λ_1 and λ_2 be the two photon lines adjacent to W in G. Since the special vertex x cannot lie inside W, at least one of the two lines, say, λ_1, must be internal and doubled. Then λ_2 may be either doubled, or internal and undoubled, or external. Let x_1 and x_2 be the vertices at which λ_1 and λ_2 meet W. The line-doubling in G can always be arranged so that a continuous path P of doubled lines runs from x_1 to x_2 within W. Without loss of generality, it will be assumed that the "down" direction runs from x_2 along P to x_1, then along λ_1 and other doubled lines outside W to x. Also, the direction of the arrows is chosen in λ_2 towards x_2, and in λ_1 away from x_1. Then by Eq. (69), the line λ_1 introduces into the integrand of Eq. (72) a factor (74) with

$$R_1=l+i\Delta_1. \tag{91}$$

If λ_2 is an internal line, there is also a factor (74) in the integrand of Eq. (72) with

$$R_2=l+i\Delta_2. \tag{92}$$

Here, we have $\Delta_2=0$ if λ_2 is undoubled. If λ_2 is external, we have $\Delta_2=0$ by definition and l is the momentum carried by the particle corresponding to λ_2. In all cases, we have

$$\Delta_1-\Delta_2=\Gamma_1+\Gamma_2+\cdots+\Gamma_d, \tag{93}$$

where $\Gamma_1,\cdots,\Gamma_d$ are the Γ-vectors associated with the d vertices of W.

Now consider the product Π_W of all factors in the integrand of Eq. (72) associated with the lines and vertices of W. Apart from fundamental constants, Π_W depends only upon the following quantities: $\Gamma_1,\cdots,\Gamma_d$, $(l+i\Delta_2)$, and the momentum variables $k_a{}^W$ corresponding to undoubled lines in W. The variables $k_a{}^W$ do not appear in any factors of Eq. (72) except Π_W. Also, Π_W is a second-rank tensor with two suffixes α, β, derived from the Dirac matrices γ_α, γ_β operating at the vertices x_1 and x_2. Thus, the integral

$$I^W=I_{\alpha\beta}{}^W(l+i\Delta_2,\Gamma_1,\cdots,\Gamma_d)=\int\Pi_W dk_a{}^W \tag{94}$$

appears as a factor in Eq. (72), and depends only on the parameters written explicitly in Eq. (94).

The integrals (72) and (94) are to be evaluated as Feynman integrals before any process of analytic continuation is applied to them. While Eq. (94) is in the form of a Feynman integral, the method of Sec. VI of II can be used in order to separate it into a convergent part and a divergent part, the divergent part having a very simple form determined by considerations of covariance. The divergence of Eq. (94) is at most quadratic, and divergences arising from integration over subsets of the variables $k_a{}^W$ are supposed already compensated. Therefore, we obtain

$$I^W=I_F{}^W+I_D{}^W, \tag{95}$$

$$\begin{aligned}I_F{}^W=\int dk_a{}^W\{&\Pi_W-(\Pi_W)_0-\lambda^2(\partial/\partial\lambda^2)(\Pi_W)_0\\&-[(l_\mu+i\Delta_{2\mu})(\partial/\partial l_\mu)+\Gamma_{1\mu}(\partial/\partial\Gamma_{1\mu})+\cdots\\&+\Gamma_{d\mu}(\partial/\partial\Gamma_{d\mu})](\Pi_W)_0\\&-\tfrac{1}{2}[(l_\mu+i\Delta_{2\mu})(l_\nu+i\Delta_{2\nu})(\partial^2/\partial l_\mu\partial l_\nu)\\&+2(l_\mu+i\Delta_{2\mu})\Gamma_{1\nu}(\partial^2/\partial l_\mu\partial\Gamma_{1\nu})+\cdots\\&+\Gamma_{1\mu}\Gamma_{1\nu}(\partial^2/\partial\Gamma_{1\mu}\partial\Gamma_{1\nu})+\cdots\\&+2\Gamma_{1\mu}\Gamma_{2\nu}(\partial^2/\partial\Gamma_{1\mu}\partial\Gamma_{2\nu})+\cdots](\Pi_W)_0\},\end{aligned} \tag{96}$$

this $I_W{}^F$ being an absolutely convergent integral. The notation $(\Pi_W)_0$ means that the parameters λ^2, $(l+i\Delta_2)$, $\Gamma_1,\cdots,\Gamma_d$, but not the variables $k_a{}^W$, are all to be put equal to zero in Π_W and its derivatives, after carrying out the differentiations. The divergent subtracted term $I_D{}^W$ is of the form:

$$\begin{aligned}I_D{}^W=&T_{\alpha\beta}+\lambda^2T_{\alpha\beta}{}'+T_{\alpha\beta\mu}(l_\mu+i\Delta_{2\mu})+\textstyle\sum_r T_{\alpha\beta\mu}{}^r\Gamma_{r\mu}\\&+T_{\alpha\beta\mu\nu}(l_\mu+i\Delta_{2\mu})(l_\nu+i\Delta_{2\nu})+\textstyle\sum_r T_{\alpha\beta\mu\nu}{}^r(l_\mu+i\Delta_{2\mu})\Gamma_{r\nu}\\&+\textstyle\sum_r\sum_s T_{\alpha\beta\mu\nu}{}^{rs}\Gamma_{r\mu}\Gamma_{s\nu}.\end{aligned} \tag{97}$$

[13] J. C. Ward, Phys. Rev. 78, 182 (1950).

Here the $T_{\alpha\beta}$, $T_{\alpha\beta\mu}$, etc., are dimensionless numerical tensors whose components depend only on the shape of W, multiplied by such powers of the fundamental units, $\hbar$, c, and m, as are required to give the correct dimensions to I^W. Since there exist no numerical tensors of third rank, the terms in Eq. (97) linear in l_μ and $\Gamma_{r\mu}$ vanish identically. The only possible form for a numerical tensor of second rank is

$$T_{\alpha\beta}=A\delta_{\alpha\beta}, \tag{98}$$

where A is a scalar. The fourth-rank tensors in Eq. (97) must each be of the form:

$$T_{\alpha\beta\mu\nu}=B\delta_{\alpha\beta}\delta_{\mu\nu}+B'\delta_{\alpha\mu}\delta_{\beta\nu}+B''\delta_{\alpha\nu}\delta_{\beta\mu}, \tag{99}$$

where B, B', and B'' are scalars. When Eqs. (98) and (99) are substituted into Eq. (97), the result may be written

$$I_D{}^W=\delta_{\alpha\beta}[A+A'\lambda^2+Bq^2+\textstyle\sum_r B_r q\cdot\Gamma_r +\sum_r\sum_s B_{rs}\Gamma_r\cdot\Gamma_s]+B'q_\alpha q_\beta+\sum_r\{B_r'q_\alpha\Gamma_{r\beta} +B_r''q_\beta\Gamma_{r\alpha}\}+\sum_r\sum_s B_{rs}'\Gamma_{r\alpha}\Gamma_{s\beta}, \tag{100}$$

with

$$q=l+i\Delta_2+i\Gamma_2. \tag{101}$$

In order to simplify Eq. (100), it is necessary to group the self-energy parts W into mutually exclusive classes C and to add together the integrals I^W derived from W belonging to a given C. Let I^C be the sum of these I^W. The definition of the classes C is as follows. Given a particular W, the vertex x_2 belongs to a closed loop L of r electron lines in W; the C to which W belongs is the class of self-energy parts obtained from W by moving x_2 to each of the $(r-1)$ possible positions which x_2 can occupy on L, leaving the order of the other vertices on L unchanged. If G is any graph containing W, then G_C is defined to be the class of graphs obtained by substituting for W in G the various members of C. Clearly, all graphs of G_C contribute together to every coefficient M to which G contributes. Therefore, it is convenient to consider the sums $M(G_C)$ formed by adding together the $M(G)$ derived from graphs G in G_C. The removal of divergences will be performed for the sum $M(G_C)$ as a unit; it is unnecessary to cancel divergences in each $M(G)$ separately. Since the factors of Eq. (72), other than Π_W, arise only from the part of G outside W, the function $M(G_C)$ is formed from $M(G)$ by replacing I^W by I^C.

Consider the sum

$$q_\beta I_{\alpha\beta}{}^W. \tag{102}$$

The factors in Π_W arising from the vertex x_2 and the two adjacent electron lines λ_i, λ_j are

$$[(l_{i\mu}+i\eta_i\Delta_{i\mu})\gamma_\mu-\mathrm{im}]^{-1}\gamma_\beta[(l_{j\mu}+i\eta_j\Delta_{j\mu})\gamma_\mu-\mathrm{im}]^{-1}, \tag{103}$$

where

$$l_i-l_j=l,\quad \eta_i\Delta_i-\eta_j\Delta_j=\Delta_2+\Gamma_2, \tag{104}$$

Eq. (104) being a consequence of the structure of W. By Eq. (101), the sum (102) will be obtained if one replaces the factors (103) in I^W by

$$[(l_{j\mu}+i\eta_j\Delta_{j\mu})\gamma_\mu-\mathrm{im}]^{-1}-[(l_{i\mu}+i\eta_i\Delta_{i\mu})\gamma_\mu-\mathrm{im}]^{-1}. \tag{105}$$

Therefore, one obtains

$$q_\beta I_{\alpha\beta}{}^W=E_i{}^W-E_j{}^W, \tag{106}$$

where $E_i{}^W$ is the expression obtained by omitting from I^W the factors arising from the line λ_i and the vertex x_2. Let the vertices of the closed loop L be in order x_2, x_a, x_b, $\cdots$, x_t. Consider the self-energy part W', belonging to C, in which x_2 stands between x_a and x_b instead of between x_t and x_a. Let the two electron lines adjoining x_2 in W' be λ_k, λ_l. Then, we have

$$q_\beta I_{\alpha\beta}{}^{W'}=E_k{}^{W'}-E_l{}^{W'}, \tag{107}$$

in analogy with Eq. (106). But it is easily verified that the factors in the integrand of $E_j{}^W$ are identical with those of $E_k{}^{W'}$, apart from possible differences of notation arising from a different choice of doubled lines in W and W'. The value of $E_j{}^W$, evaluated as a Feynman integral, depends only on W and is independent of the way in which the doubled lines have been chosen. Therefore, $E_j{}^W$ and $E_k{}^{W'}$ are identical. Similarly, when Eq. (106) is summed over all the W in C, each term $E_r{}^W$ appears exactly twice, once with positive and once with negative sign. This leads to the identity

$$q_\beta I_{\alpha\beta}{}^C=0. \tag{108}$$

The above proof of Eq. (108) is not altogether rigorous, since it involves some rearrangements of divergent integrals. The physical significance of Eq. (108) is to express the conservation of the charges and currents occurring as a result of the polarization of the vacuum by an applied field. Mathematically, Eq. (108) is the direct generalization to intermediate representation operators of the identity

$$(\partial/\partial x_\mu)G_{\mu\nu}(x)=0 \tag{109}$$

expressing the same conservation principle in the Schwinger theory.[14] Schwinger's proofs of Eq. (109) have given rise to much discussion;[15] they are nonrigorous from a strict mathematical point of view, but there seems to be no doubt that Eq. (109) is to be accepted as a correct equation, expressing a formal property which the theory must possess in order to be physically consistent. The dubious features in the proof of Eq. (108) are similar to those which occurred in Schwinger's work. Therefore, it is reasonable to accept Eq. (108) as formally correct, just as Eq. (109) is accepted, without here enquiring more deeply into the mathematical difficulties of the proof.

The grouping together of self-energy parts into classes C, by moving the vertex x_2 round a closed loop,

[14] J. Schwinger, Phys. Rev. **76**, 790 (1949), Eq. (A.16) in the Appendix.

[15] See, for example, W. Pauli and F. Villars, Revs. Modern Phys. **21**, 434 (1949).

can be extended by allowing both the vertices x_1 and x_2 to move independently round the loops on which they respectively lie. It will be supposed henceforth that the classes C are enlarged in this way. Then Eq. (108) will still be valid, and in addition the identity

$$q_\alpha' I_{\alpha\beta}{}^C = 0 \tag{110}$$

will hold, with

$$q' = l + i\Delta_1 - i\Gamma_1. \tag{111}$$

Let I^C be divided into its finite and divergent parts $I_F{}^C$, $I_D{}^C$ by summing Eq. (95) with respect to W. $I_D{}^C$ is formally the sum of all terms of degrees 0, 1, 2 in a power-series expansion of I^C in the variables λ, $(l+i\Delta_2)$, $\Gamma_1, \cdots, \Gamma_d$; and the identities (108), (110) are homogeneous in these variables. Therefore, Eqs. (108), (110) hold not only for I^C but also for the parts $I_F{}^C$ and $I_D{}^C$ separately. Since $I_D{}^C$ satisfies Eq. (108) identically, the coefficients when $I_D{}^C$ is written in the form (100) are mostly zero; in fact, we may write

$$A = A' = 0,\ B + B' = 0,\ B_r + B_r' = 0,\ B_r'' = B_{rs} = B_{rs}' = 0.$$

The condition $A = 0$ states that the self-energy of the photon is zero, as in the Schwinger theory. Therefore, we have

$$I_D{}^C = B(q^2\delta_{\alpha\beta} - q_\alpha q_\beta) + \textstyle\sum_r B_r(q\cdot\Gamma_r\delta_{\alpha\beta} - q_\alpha\Gamma_{r\beta}). \tag{112}$$

When we apply Eq. (110) to Eq. (112), $I_D{}^C$ reduces finally to the compact form:

$$I_D{}^C = B(q\cdot q'\delta_{\alpha\beta} - q_\alpha q_\beta'), \tag{113}$$

depending on the single scalar coefficient B. When λ and all the Γ_i are put equal to zero, Eq. (113) agrees with the form in which charge-renormalization effects appear in the calculation of the S-matrix. Therefore, B is to be identified with a certain contribution to the coefficient Z_3 of charge-renormalization defined in II.

Let I^d be the sum of all I^W derived from ordinary proper photon self-energy parts W with d vertices. The divergent part of I^d is

$$I_D{}^d = -B_d(q\cdot q'\delta_{\alpha\beta} - q_\alpha q_\beta'), \tag{114}$$

where B_d is defined as a coefficient in the expansion,

$$f = 1 - Z_3 = \sum_{d=2}^{\infty} B_d e_1{}^d, \tag{115}$$

and is therefore identical with the B_d appearing in Eq. (26). Associated with I^d there is an expression $I_0{}^d$ which describes the effects of the terms of degree $e_1{}^d$ in H_{gp}, H_{ga}, H_{gb}, operating at a single-vertex photon self-energy part W_0. By Eqs. (26) and (75), (76), the contribution to $I_0{}^d$ from H_{gp} is

$$I_p{}^d = B_d[(l+i\Delta_1)\cdot(l+i\Delta_2)\delta_{\alpha\beta} - (l+i\Delta_2)_\alpha(l+i\Delta_1)_\beta]. \tag{116}$$

By Eqs. (29) and (48), the contribution from H_{ga} is

$$I_a{}^d = B_d[(d-2)\Gamma_{i0}\Gamma_{j0} + \Gamma_{i0}{}^2](\delta_{\alpha\beta} - \delta_{\alpha 4}\delta_{\beta 4}). \tag{117}$$

By Eqs. (30) and (47), the contribution from H_{gb} is

$$I_b{}^d = -iB_d(\Gamma_{i\alpha}l_\beta - \Gamma_{j\beta}l_\alpha). \tag{118}$$

In Eqs. (117) and (118), not only the factor arising in Eq. (72) from W_0 is included, but also the factor contributed by W_0 to the product E_Γ, which appears in $M(G)$ according to Eq. (59). As explained in Sec. IV, the vectors $\Gamma_1, \cdots, \Gamma_d$ are all associated with the single vertex W_0; Γ_i and Γ_j in Eqs. (117) and (118) are any two of these vectors chosen arbitrarily. Since $M(G, \Lambda)$ involves $\Gamma_1, \cdots, \Gamma_d$ only in the combination $(\Gamma_1 + \cdots + \Gamma_d)$, the value of Eq. (59) is independent of the choice of Γ_i and Γ_j. In particular, Eqs. (117) and (118) may be replaced by the expressions,

$$I_a{}^d = B_d(\delta_{\alpha\beta} - \delta_{\alpha 4}\delta_{\beta 4})[\Gamma_{20}(\Gamma_{10} + \Gamma_{20} + \cdots + \Gamma_{d0}) - \Gamma_{20}\Gamma_{10} + (\Gamma_{20} - \Gamma_{10})\Delta_{20}], \tag{119}$$

$$I_b{}^d = -iB_d[\Gamma_{2\alpha}l_\beta - \Gamma_{1\beta}l_\alpha + l\cdot(\Gamma_1 - \Gamma_2)\delta_{\alpha\beta}]. \tag{120}$$

Adding together Eqs. (114), (116), (119), and (120), one finds

$$I_D{}^d + I_0{}^d = I_D{}^d + I_p{}^d + I_a{}^d + I_b{}^d = 0. \tag{121}$$

Now consider the coefficient M defined at the beginning of Sec. IV. It is a sum of contributions $M(G)$ from various G. Suppose that G is any graph which contains the ordinary proper photon self-energy part W consisting of d vertices. Contributions to M will also be obtained from all graphs G' in which W is replaced, either by any other ordinary proper photon self-energy part with d vertices, or by the single-vertex part W_0 at which the terms of order $e_1{}^d$ in H_{gp}, H_{ga}, and H_{gb} are supposed to operate. The $M(G')$ derived from all these graphs, as defined by Eqs. (59) and (72), will differ from $M(G)$ only by the substitution of $I^{W'}$ or $I_0{}^d$ for I^W; the factors in $M(G)$ derived from all parts of G outside W are retained unaltered in $M(G')$. Let M_d be the sum of all the $M(G')$ including $M(G)$. Then, M is obtained from $M(G)$ by writing in place of I^W

$$I^d + I_0{}^d = I_F{}^d; \tag{122}$$

by Eq. (121) this $I_F{}^d$ is simply the absolutely convergent integral separated from I^d according to Eq. (96). That is to say, in M_d all the divergences arising from the photon self-energy parts W' have been eliminated. Since this elimination of divergences proceeds independently for self-energy parts situated at all possible places in G, the coefficient M finally involves the integrals I^W only in the convergent combinations $I_F{}^d$. Therefore, M is free of divergences arising from photon self-energy parts. Since M is any coefficient in the expansion of $A_{\mu\nu}(p)$, the absence of photon self-energy and vacuum polarization divergences in $A_{\mu\nu}(p)$ is now proved.

In conclusion, some explanation should be made of the role of the analytic continuation process in the above arguments. The $M(G, \Lambda)$ were defined in Sec. V as the analytic continuation of certain Feynman inte-

grals, originally defined for large values of m and λ and imaginary values of Γ_{i0}, into the region of real Γ_{i0} and small m and λ. The proof that the $M(G, \Lambda)$ are analytic functions of this kind was given in HO.I, but the proof of analyticity is certainly invalid and meaningless when the $M(G, \Lambda)$ are divergent integrals. Here is a defect in the argument, which must be remedied as follows. The proof of analyticity will apply correctly to the convergent part of an integral defined by the subtraction procedure (96), provided that the subtracted divergent terms are manifestly and formally analytic functions of m, λ, and the Γ_{i0}. Now in Eq. (96) the subtracted terms are formally analytic; they are quadratic functions of λ and the Γ_{i0}, and they can depend on m only as a constant power m^r because m is the only other parameter involved in them with the dimensions of a mass. Therefore, the convergent parts of all the integrals $M(G, \Lambda)$ are correctly defined by the analytic continuation method. Since eventually only the convergent parts of the integrals appear in $A_{\mu\varrho}(p)$, the use of analytic continuation in the analysis of the coefficients M is fully justified.

IX. SEPARATION OF DIVERGENCES ARISING FROM ELECTRON SELF-ENERGY PARTS AND VERTEX PARTS

In this section the divergences from electron self-energy parts and from vertex parts will be treated together. Let V be an ordinary proper vertex part, irreducible or reducible, with $(d+1)$ vertices. Let W be an ordinary proper electron self-energy part with d vertices. Let G be a graph in which either V or W occurs; let $M(G)$ and $M(G, \Lambda)$ be the coefficients derived from G. As before, it is supposed that divergences arising from internal parts of V or W have already been canceled.

Let λ_1 and λ_2 be the two electron lines adjacent to V or W in G, meeting V or W at the vertices x_1, x_2, with the arrow running in λ_2 towards x_2 and in λ_1 away from x_1. Let λ_3 be the photon line adjacent to V at the vertex x_3. The line-doubling in G can be arranged so that x_1 and x_2 are connected in W or V by a continuous path P of doubled electron lines. Also, x_3 is connected in V to P by a continuous path of doubled lines; but it is no longer possible to arrange that λ_1 is necessarily doubled. The "down" direction may lead out of W to x through either λ_1 or λ_2, and out of V through either λ_1, λ_2, or λ_3. The factors arising from V or W in $M(G)$ will in general depend strongly upon which of these alternative routes the down direction takes. Let Π_V or Π_W be the product of the factors in the integrand of Eq. (72) arising from V or W. Let

$$I^V=\int \Pi_V dk_a{}^V, \quad I^W=\int \Pi_W dk_a{}^W, \tag{123}$$

where the $k_a{}^V$ or $k_a{}^W$ are the momentum variables arising from undoubled lines in V or W. In $M(G)$ the integral (123) appears as a factor, since the variables $k_a{}^V$, $k_a{}^W$ occur only in Π_V, Π_W.

Consider now the parameters upon which I^V can depend. These are indicated by

$$I^V=I_\mu{}^V(R_1, R_2, \Gamma_1, \cdots, \Gamma_{d+1}). \tag{124}$$

Here μ is a vector suffix arising from the operator γ_μ at the vertex x_3. Not shown explicitly in Eq. (124) is the fact that I^V is also a Dirac matrix, with two spinor indices which combine with the factors in $M(G)$ arising from λ_1 and λ_2. The vectors R_1, R_2 are given by Eq. (69) if λ_1 and λ_2 are internal lines; if λ_1 or λ_2 is external, the corresponding R_1 or R_2 is simply the momentum carried by λ_1 or λ_2. The vectors $\Gamma_1, \cdots, \Gamma_{d+1}$ are those associated with the vertices of V. In addition to the parameters already mentioned, I^V depends on m and λ.

Since I^V is only logarithmically divergent,

$$I^V=I_{Dt}{}^V+I_{Ft}{}^V, \tag{125}$$

$$I_{Ft}{}^V=\int dk_a{}^V\{\Pi_V-(\Pi_V)_t\}, \tag{126}$$

this $I_{Ft}{}^V$ being absolutely convergent. The subtracted term $(\Pi_V)_t$ is obtained from Π_V by substituting for $(R_1, R_2, \Gamma_1, \cdots, \Gamma_{d+1}, \lambda)$ the special values $(t, t, 0, \cdots, 0, \lambda_0)$, where t is an arbitrary 4-vector satisfying

$$t^2=-m^2, \tag{127}$$

and λ_0 is an arbitrary fixed positive mass. The divergent $I_{Dt}{}^V$, being a vector Dirac matrix depending only on t, must be of the form,

$$I_{Dt}{}^V=L\gamma_\mu+B\gamma_\mu(t_\alpha\gamma_\alpha-\mathrm{im})+B'(t_\alpha\gamma_\alpha-\mathrm{im})\gamma_\mu +C(t_\alpha\gamma_\alpha-\mathrm{im})\gamma_\mu(t_\beta\gamma_\beta-\mathrm{im}), \tag{128}$$

where L, B, B', C are scalar constants independent of t after Eq. (127) is used. Now the difference between the $I_{Dt}{}^V$ defined with two different vectors t is an absolutely convergent integral. Therefore, the coefficients B, B', C are finite, and only L is divergent. Let $I_D{}^V$ be defined by

$$I_D{}^V=L\gamma_\mu, \tag{129}$$

and write

$$I^V=I_D{}^V+I_F{}^V. \tag{130}$$

Then $I_F{}^V$ is finite and independent of t, and is by definition the finite part of I^V. The above definition of $I_F{}^V$ in two stages is similar to the two-stage definition of $\Lambda_{\mu c}$ in Sec. VI of II. Note, however, that L in (129) is still a function of λ_0. It is necessary to retain a non-zero λ_0 in L in order to avoid introducing a spurious infrared divergence into $I_F{}^V$.

Although I^V depends on the route along which the "down" direction runs out of V, the $I_{Dt}{}^V$ does not. Therefore, L depends only on the shape of V and is independent of the situation in which V appears in G. When $\lambda_0=0$, the constant L becomes the contribution from V to the vertex-renormalization constant Z_1

defined in Sec. VII of II. When $\lambda_0 \neq 0$, the constant L is a contribution to the corresponding coefficient $Z_1(\lambda_0)$, which appears in the calculation of the S-matrix of quantum electrodynamics when a neutral vector field of mass λ_0 replaces the maxwell field. Let G be any graph in which V occurs, so that $M(G)$ contains I^V as a factor. In M, together with $M(G)$, there appear all the $M(G')$ from graphs G' obtained from G by replacing V by any other ordinary proper vertex part with $(d+1)$ vertices. The sum of these $M(G')$ is denoted by M_d.

M_d is obtained from $M(G)$ when I^V is replaced by a certain sum I^d. This I^d will depend not only on d but also on the way in which the "down" direction leads out of V in G. However, in all cases one obtains

$$I^d = I_F{}^d + I_D{}^d, \tag{131}$$

where $I_F{}^d$ is convergent and

$$I_D{}^d = L_d \gamma_\mu. \tag{132}$$

Here, L_d is defined as a coefficient in the expansion,

$$Z_1(\lambda_0) = 1 - \sum_{d=2}^{\infty} L_d e_1{}^d, \tag{133}$$

and is independent of the situation of V.

Next consider I^W. If the down direction leaves W along λ_1, then λ_1 is a doubled line and $\eta_1 = +1$. In the opposite case, λ_2 is doubled and $\eta_2 = -1$. Let $\eta_W = +1$ in the first case and $\eta_W = -1$ in the second. In both cases, the vectors R_1, R_2 according to Eq. (69) are given by

$$R_1 = l + i\eta_W \Delta_1, \quad R_2 = l + i\eta_W \Delta_2, \tag{134}$$

with

$$\eta_W(\Delta_1 - \Delta_2) = \Gamma_1 + \cdots + \Gamma_d. \tag{135}$$

Here, $\Gamma_1, \cdots, \Gamma_d$ are the vectors associated with the vertices of W. The parameters upon which I^W depends are indicated by

$$I^W = I^W(l + i\eta_W \Delta_2, \Gamma_1, \cdots, \Gamma_d). \tag{136}$$

Note, however, that I^W is not the same function of these parameters in the two cases $\eta_W = \pm 1$. Also, I^W is a Dirac matrix.

Since I^W is linearly divergent, one obtains

$$I^W = I_{Dt}{}^W + I_{Ft}{}^W, \tag{137}$$

$$I_{Ft}{}^W = \int dk_a{}^W \{\Pi_W - (\Pi_W)_t - [\Gamma_{1\mu}(\partial/\partial\Gamma_{1\mu}) + \cdots + \Gamma_{d\mu}(\partial/\partial\Gamma_{d\mu}) + (l_\mu + i\eta_W \Delta_{2\mu} - t_\mu)(\partial/\partial l_\mu)](\Pi_W)_t\}, \tag{138}$$

this $I_{Ft}{}^W$ being absolutely convergent. Here $(\Pi_W)_t$ is obtained by substituting for $(l + i\eta_W \Delta_2, \Gamma_1, \cdots, \Gamma_d, \lambda)$, in Π_W and in the derivatives of Π_W after differentiation, the special values $(t, 0, \cdots, 0, \lambda_0)$. The divergent $I_{Dt}{}^W$, being a scalar Dirac matrix depending linearly on the vectors $(l + i\eta_W \Delta_2, \Gamma_1, \cdots, \Gamma_d)$, must be of the form:

$$\begin{aligned} I_{Dt}{}^W = {} & X + Y(t_\alpha \gamma_\alpha - \mathrm{im}) \\ & + \sum_r \Gamma_{r\mu}[E^r \gamma_\mu + B^r(t_\alpha \gamma_\alpha - \mathrm{im})\gamma_\mu \\ & + B^{r\prime}\gamma_\mu(t_\alpha \gamma_\alpha - \mathrm{im}) + C^r(t_\alpha \gamma_\alpha - \mathrm{im})\gamma_\mu(t_\beta \gamma_\beta - \mathrm{im})] \\ & + (l_\mu + i\eta_W \Delta_{2\mu} - t_\mu)[E^0 \gamma_\mu + B^0(t_\alpha \gamma_\alpha - \mathrm{im})\gamma_\mu \\ & + B^{0\prime}\gamma_\mu(t_\alpha \gamma_\alpha - \mathrm{im}) \\ & + C^0(t_\alpha \gamma_\alpha - \mathrm{im})\gamma_\mu(t_\beta \gamma_\beta - \mathrm{im})]. \end{aligned} \tag{139}$$

Suppose temporarily that t is an arbitrary vector, not necessarily satisfying Eq. (127). Then, the coefficients X, Y, E^r, etc., in Eq. (139) are scalar functions of t^2. Let X', Y' be the derivatives of X and Y with respect to t^2. From the identity,

$$\int dk_a{}^W (\partial/\partial t_\mu)(\Pi_W)_t = Y\gamma_\mu + 2X' t_\mu + 2Y' t_\mu (t_\alpha \gamma_\alpha - \mathrm{im}), \tag{140}$$

there follow the relations:

$$E^0 = Y + 2\,\mathrm{im}X' = Y + \mathrm{im}(B^0 + B^{0\prime}). \tag{141}$$

Now, t being restricted to satisfy Eq. (127), the coefficients X, Y, etc., become constants depending only on m and λ_0. Let $I_D{}^W$ be defined by

$$I_D{}^W = X + \textstyle\sum_r E^r \Gamma_{r\mu}\gamma_\mu + E^0[(l_\mu + i\eta_W \Delta_{2\mu})\gamma_\mu - \mathrm{im}], \tag{142}$$

and write

$$I^W = I_D{}^W + I_F{}^W. \tag{143}$$

Integrals such as

$$\int dk_a{}^W (\partial/\partial\Gamma_{1\mu})(\Pi_W)_t \tag{144}$$

are only logarithmically divergent, and the difference between two integrals (144) constructed with two different vectors t is convergent. From this it follows that the coefficients B^r, $B^{r\prime}$, C^r, B^0, $B^{0\prime}$, C^0 in Eq. (139) are all finite. Comparing Eq. (143) with Eq. (137), Eq. (142) with Eq. (139), and using Eq. (141), one deduces that $I_F{}^W$ is finite. Being independent of t, $I_F{}^W$ is by definition the finite part of I^W. This definition is similar to the two-stage definition of $\sum_c$ in Sec. VI of II. The λ_0-dependent coefficients X, E^r, and E^0 are, however, retained in Eq. (143) in order to avoid infrared divergences.

I^W, like I^V, depends on the route along which the down direction runs out of W. The coefficients X and E^0, like L in Eq. (129), are independent of this direction and depend only on the shape of W. When $\lambda_0 = 0$, the coefficient E^0 becomes the contribution of W to the electron-line renormalization constant Z_2 defined in Sec. VII of II. When $\lambda_0 \neq 0$, the coefficient E^0 is a contribution to the corresponding coefficient $Z_2(\lambda_0)$

appearing in the S-matrix with a neutral meson field of mass λ_0. When $\lambda_0=0$, the coefficient X becomes the contribution of W to the electron self-energy denoted by $(1/2\pi)A=iZ_2\delta\kappa_0$ in Eqs. (72) and (94) of II. When $\lambda_0\neq 0$, the coefficient X is a contribution to the self-energy

$$(1/2\pi)A(\lambda_0)=iZ_2(\lambda_0)\delta m(\lambda_0) \tag{145}$$

of an electron in a neutral meson field of mass λ_0. However, in Eq. (142) the coefficients $E^1, \cdots, E^d$ will in general depend on the down direction in W. It is necessary to use a special symmetrization procedure, explained in the following paragraph, to eliminate these coefficients.

Consider the electron self-energy part W^*, obtained by reversing the direction of all arrows in W and at the same time reversing the direction "down" in the path P of doubled lines joining x_1 to x_2. If G is any graph in which W occurs, there exists a graph G^* obtained from G by substituting W^* for W. To pass from G to G^*, it is necessary only to perform a "reflection" of W which interchanges the two ends x_1 and x_2. The factors in Π_W are of three kinds, arising from undoubled lines, doubled lines not in P, and lines in P, respectively. Factors from undoubled lines appear unchanged in $\Pi_W{}^*$. Factors from doubled lines not in P appear in $\Pi_W{}^*$ with a vector $(q+i\eta\Delta)$ changed to $(q-i\eta\Delta)$, where q is a certain linear combination of the $k_a{}^W$, and Δ is the sum of a certain subset of $(\Gamma_1, \cdots, \Gamma_d)$. Factors from lines in P appear in $\Pi_W{}^*$ with a vector $(l+q+i\eta_W(\Delta_2+\Delta))$ changed to $(l+q+i\eta_W(\Delta_1-\Delta))$, the sign of η_W being unchanged because both the down direction in P and the direction of arrows are reversed. Also, the order of all factors in Dirac matrix products is reversed in passing from Π_W to $\Pi_W{}^*$. Therefore, we have

$$I^{W*}(l+i\eta_W\Delta_2, \Gamma_1, \cdots, \Gamma_d) = I^W(l+i\eta_W\Delta_1, -\Gamma_1, \cdots, -\Gamma_d)_R, \tag{146}$$

where the suffix R denotes the reversal of the order of factors. By Eq. (142), the divergent part of I^{W*} is

$$I_D{}^{W*}=X-\textstyle\sum_r E^r\Gamma_{r\mu}\gamma_\mu + E^0[(l_\mu+i\eta_W\Delta_{1\mu})\gamma_\mu-\mathrm{im}]. \tag{147}$$

Now G and G^* always contribute together to any coefficient M. Therefore, the value of M is unchanged if one replaces every I^W by the symmetrized expression $\frac{1}{2}(I^W+I^{W*})$. Instead of Eq. (142), the symmetrized $I_D{}^W$ has the simple form,

$$I_D{}^W=X+E^0[(l_\mu+i\bar{\Delta}_\mu)\gamma_\mu-\mathrm{im}], \tag{148}$$

with

$$\bar{\Delta}=\tfrac{1}{2}\eta_W(\Delta_1+\Delta_2). \tag{149}$$

The dependence of $I_D{}^W$ on the down direction in W has now disappeared.

Let G be any graph in which W occurs, and G' a graph obtained from G by replacing W by any ordinary proper electron self-energy part with d vertices. The sum of the $M(G')$ is denoted by M_d; and M_d is obtained from $M(G)$ when I^W is replaced by a sum

$$J^{d'}=J_F{}^d+J_D{}^{d'}, \tag{150}$$

where $J_F{}^d$ is convergent and

$$J_D{}^{d'}=iK_dm+E_d[(l_\mu+i\bar{\Delta}_\mu)\gamma_\mu-\mathrm{im}]. \tag{151}$$

Note that the vector $(l+i\bar{\Delta})$ depends only on the part of G outside W, and is the same for all G'. In Eq. (151), K_d and E_d are coefficients in the expansions,

$$Z_2(\lambda_0)\left[\frac{\delta m(\lambda_0)}{m}\right]=\sum_{d=2}^{\infty}K_de_1{}^d, \tag{152}$$

$$Z_2(\lambda_0)=1+\sum_{d=2}^{\infty}E_de_1{}^d, \tag{153}$$

in accordance with Eqs. (93) and (94) of II.[16] They are independent of G and of the situation of W in G.

There will also be a contribution to M from the graph G_0 obtained from G by replacing W by a single-vertex electron self-energy part W_0, at which the term of degree $e_1{}^d$ in the interaction H_{gs} is supposed to operate. The addition to $J^{d'}$ from W_0 is

$$J_0{}^d=-iA_dm, \tag{154}$$

A_d being the coefficient appearing in Eq. (25). Note that this does not precisely cancel the term K_d in Eq. (151). The sum J^d of integrals I^W, from proper self-energy parts with d vertices and from W_0, is

$$J^d=J_D{}^d+J_F{}^d, \tag{155}$$

$$J_D{}^d=\mathrm{im}(K_d-A_d)+E_d[(l_\mu+i\bar{\Delta}_\mu)\gamma_\mu-\mathrm{im}]. \tag{156}$$

By Ward's identity,[13] which holds for neutral vector meson fields as well as for the ordinary maxwell field, the renormalization factors $Z_1(\lambda_0)$ and $Z_2(\lambda_0)$ in Eqs. (133) and (153) are equal. Therefore, one obtains

$$E_d=-L_d. \tag{157}$$

To summarize the results of the analysis so far, all the remaining divergences of the theory are comprised in Eqs. (132) and (156), arising, respectively, from proper vertex parts and from proper electron self-energy parts. The divergent coefficients are defined by Eqs. (25), (133), (152), and (153). It remains only to show in the next section how Eqs. (132) and (156) cancel each other exactly in the calculation of M. The physical reason for the cancellation is that these terms are not true renormalization effects but only "wave function renormalization terms" of a kind which are familiar in elementary perturbation theory. In the present paper, thanks to the complex denominators which never

[16] The factors 2π which appeared in II are now dropped because all the integrals $M(G, \Lambda)$ are expressed in terms of the functions $2\pi D_F$, $2\pi S_F$. The inconvenient factors 2π arose originally from Eqs. (44) and (45) of II.

vanish or become indeterminate, the treatment of such wave function renormalization effects is free from the difficulties and ambiguities which arose in earlier discussions.[17]

X. REMOVAL OF REMAINING DIVERGENCES

In this section the divergences (132) and (156) are to be removed. The strategy of the removal is as follows. A supplementary interaction H_q is introduced, which plays the same role for the electron-positron field as the supplementary interaction H_p of the Gupta formalism played for the electromagnetic field. H_p was a real interaction which was introduced into the lagrangian of quantum electrodynamics to take account of the renormalization of electromagnetic fields. But there is no actual renormalization of the electron-positron field, in consequence of Ward's identity, which is essentially a statement of the law of conservation of charge in radiative processes. Therefore, H_q is not a real interaction, but must be introduced artificially into the formalism. This is done by adding to the hamiltonian H^I given by Eq. (11) a term (H_q-H_q). To the interaction (28) there is added a corresponding term $(H_{gq}-H_{gq})$. The definitions of H_q and H_{gq} are

$$H_q(x)=[1-Z_2^{-1}(\lambda_0)]L_D(x),$$
$$H_{gq}=[1-Z_{2g}^{-1}(\lambda_0)]L_D(x'), \tag{158}$$

where

$$L_D(x)=-\hbar c\{\tfrac{1}{2}[\bar\psi\gamma_\mu(\partial\psi/\partial x_\mu) - (\partial\bar\psi/\partial x_\mu)\gamma_\mu\psi]+m\bar\psi\psi\}, \tag{159}$$

$$Z_{2g}(\lambda_0)=1+\sum_{d=2}^{\infty}E_d[e_1g(t-t')]^d, \tag{160}$$

$$Z_{2g}^{-1}(\lambda_0)=1-\sum_{d=2}^{\infty}C_d[e_1g(t-t')]^d. \tag{161}$$

The additional interactions H_{gq} and $(-H_{gq})$ are treated in just the same way as the other terms in H_g. Thus, the possible graphs G giving contributions $M(G)$ to M are made more numerous by allowing two new types of vertex to appear, which will be called P-vertices and N-vertices. Both P- and N-vertices may be inserted in all possible combinations in all the electron lines of every graph. At each P- or N-vertex there are two electron lines and no photon line incident. At each P-vertex one term, of a particular order e_1^d in the series expansion of the interaction H_{gq}, is supposed to operate; at each N-vertex one term in the series expansion of $(-H_{gq})$.

[17] For example, J. Schwinger, Phys. Rev. **76**, 790 (1949), especially p. 795; and Sec. VII of II, Eq. (99). The best published treatment is that of R. Karplus and N. M. Kroll, Phys. Rev. **77**, 536 (1949), p. 542.

Every P- and N-vertex is a single-vertex electron self-energy part which may be denoted by W. The two lines adjacent to W will carry momentum vectors R_1 and R_2 given by Eqs. (134) and (135). Let G be any graph in which W appears. They by Eq. (159) the factor contributed by W to $M(G)$ is for a P-vertex

$$I_P{}^W=C_d[(l_\mu+i\bar\Delta_\mu)\gamma_\mu-\mathrm{im}], \tag{162}$$

and for an N-vertex

$$I_N{}^W=-C_d[(l_\mu+i\bar\Delta_\mu)\gamma_\mu-\mathrm{im}]. \tag{163}$$

Strictly speaking, the time-derivatives in Eq. (159) will introduce singular terms in addition to Eqs. (162) and (163), similar to the singular terms discussed in Sec. VI. But the singular terms produced by H_{gq} and by $(-H_{gq})$ are equal and opposite, and may therefore be dropped without further argument.

Eventually, the P-vertices will cancel the divergences (132), and the N-vertices will cancel Eq. (156). But the cancellation is not immediate, and careful consideration must be given to the reducible vertex parts and self-energy parts which contribute to Eqs. (132) and (156) only after the removal of their internal divergences. It is convenient first of all to make an intensive study of the effects of P-vertices in isolation. This study will occupy the following nine paragraphs.

Consider any graph G_0 without P-vertices. A class C of graphs G may be derived by inserting any number of P-vertices independently into each electron line of G_0. It is desired to find the relation between $M(G_0)$ and the sum $M(C)$ of the $M(G)$ with G belonging to C. Consider first the effect of inserting P-vertices into a single external line λ_E of G_0. Suppose that λ_E is incident at the vertex y of G_0, and let $\eta=\pm1$ according as the arrow in λ_E points towards or away from y. Let l be the momentum carried by λ_E. Since l is the momentum of a real particle whose spin-function appears as a factor in Eq. (41), the Dirac equation,

$$l_\mu\gamma_\mu-\mathrm{im}=0, \tag{164}$$

holds whenever the left side of Eq. (164) operates directly on Eq. (41). Let G be a typical graph derived from G_0 by inserting into λ_E the P-vertices $y_1, \cdots, y_r$, in order reading inwards towards y. At each y_i the term of order $e_1^{d_i}$ in H_{gq} is operating. Let Λ_i be the sum of d_i vectors Γ_j associated with y_i according to Eq. (53). Let

$$\Delta_i=\Lambda_1+\cdots+\Lambda_i, \quad \Delta_0=0. \tag{165}$$

Then, $M(G)$ is obtained from $M(G_0)$ by making two alterations. First, an extra Δ_r is added to the vectors Δ in the denominators of all factors in $M(G_0)$ corresponding to doubled lines of G_0 on the route from y to x. Second, additional factors are inserted in $M(G_0)$,

namely,

$$F=[(l_\mu+i\eta\Delta_{r\mu})\gamma_\mu-\mathrm{im}]^{-1}C_{d_r}\times[(l_\mu+\tfrac{1}{2}i\eta(\Delta_{r\mu}+\Delta_{r-1,\mu}))\gamma_\mu-\mathrm{im}]\times[(l_\mu+i\eta\Delta_{r-1,\mu})\gamma_\mu-\mathrm{im}]^{-1}\cdots\times C_{d_2}[(l_\mu+\tfrac{1}{2}i\eta(\Delta_{2\mu}+\Delta_{1\mu}))\gamma_\mu-\mathrm{im}]\times[(l_\mu+i\eta\Delta_{1\mu})\gamma_\mu-\mathrm{im}]^{-1}C_{d_1}\times[(l_\mu+\tfrac{1}{2}i\eta\Delta_{1\mu})\gamma_\mu-\mathrm{im}]. \quad (166)$$

The order of factors in Eq. (166) is as written when $\eta=+1$, and is reversed when $\eta=-1$. Let F'' denote the product of the numerical factors C_{d_i} in Eq. (166), and let F' be the product of the remaining factors.

Let $\sum_j F'$ denote the sum of the $(j!)$ products obtained by permuting $\Lambda_1, \cdots, \Lambda_j$ in F'. In $\sum_1 F'=F'$, the last factor of Eq. (166) may be replaced, using Eq. (164), by

$$[(\tfrac{1}{2}l_\mu+\tfrac{1}{2}i\eta\Delta_{1\mu})\gamma_\mu-\tfrac{1}{2}\mathrm{im}]. \quad (167)$$

This combines with the previous factor to give (1/2) simply. Next, in $\sum_2 F'$ the third from last factor of Eq. (166) becomes

$$[(2l_\mu+\tfrac{3}{2}i\eta\Delta_{2\mu})\gamma_\mu-2\,\mathrm{im}], \quad (168)$$

and using Eq. (164) this may be replaced by

$$[(\tfrac{3}{2}l_\mu+\tfrac{3}{2}i\eta\Delta_{2\mu})\gamma_\mu-\tfrac{3}{2}\,\mathrm{im}] \quad (169)$$

which combines with the previous factor to give (3/2) simply. Continuing in this way, in $\sum_j F'$ the $(2r-2j+2)$th factor of Eq. (166) becomes

$$[(jl_\mu+(j-\tfrac{1}{2})i\eta\Delta_{j\mu})\gamma_\mu-j\,\mathrm{im}], \quad (170)$$

which may be replaced, using Eq. (164), by

$$(j-\tfrac{1}{2})[(l_\mu+i\eta\Delta_{j\mu})\gamma_\mu-\mathrm{im}]. \quad (171)$$

This combines with the $(2r-2j+1)$th factor to give $(j-\frac{1}{2})$ simply. Therefore, finally

$$\sum_r F'=\left[\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{2r-1}{2}\right]=(r!)f_r, \quad (172)$$

where

$$f_k=\left[\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2k-1}{2k}\right], \quad f_0=1. \quad (173)$$

Suppose that n_1 of the d_i are equal to 1, n_2 are equal to 2, and so on. Consider all the graphs G obtained by inserting the P-vertices $y_1, \cdots, y_r$ into λ_E with the given set of indices $d_1, \cdots, d_r$ permuted in all possible ways. Let $\sum M(G)$ be the sum of the $M(G)$ derived from these graphs. In $\sum M(G)$ there appears instead of the factor F the expression,

$$\sum F=(n_1!n_2!\cdots)^{-1}F''\sum_r F'=[r!/(n_1!n_2!\cdots)]f_r C_1^{n_1}C_2^{n_2}\cdots, \quad (174)$$

which is a pure number.

Let

$$d=\sum d_i=\sum jn_j. \quad (175)$$

Consider now all the graphs G obtained by inserting any number of P-vertices with any indices d_i into λ_E, only the value of d being fixed. The sum $\sum_d M(G)$ summed over all such G is obtained by inserting in $M(G_0)$ instead of Eq. (166) the numerical factor F_d, which is the coefficient of $e_1{}^d$ in the expansion,

$$\sum_{n_1}\sum_{n_2}\cdots[(n_1+n_2+\cdots)!/(n_1!n_2!\cdots)]\times f_{(n_1+n_2+\cdots)}(C_1e_1)^{n_1}(C_2e_1^2)^{n_2}\cdots. \quad (176)$$

By the multinomial theorem and Eq. (161), this expansion is identically

$$1+\sum F_d e_1{}^d=[1-\sum C_d e_1{}^d]^{-\frac{1}{2}}=Z_2{}^{\frac{1}{2}}(\lambda_0). \quad (177)$$

In every term of $\sum_d M(G)$ there is, in addition to the factor F, also the extra Δ_r to be inserted into the denominators of factors of $M(G_0)$. This Δ_r is the sum of d vectors Γ_j, and is the same for all terms in $\sum_d M(G)$. Such an added Δ_r would be introduced into $M(G_0)$, in precisely the same way, if the interaction operating at the vertex y of G_0 were multiplied by $[g(t-t')]^d$. Let then $M(G)$ be summed over all graphs G derived from G_0 by inserting all combinations of P-vertices into λ_E, the result being $M(G_0, \lambda_E)$. To obtain $M(G_0, \lambda_E)$ from $M(G_0)$, it is only necessary to multiply the interaction operating at y in G_0 by the factor,

$$1+\sum F_d e_1{}^d[g(t-t')]^d=Z_2{}^{\frac{1}{2}}(\lambda_0), \quad (178)$$

in virtue of Eqs. (161) and (177). Incidentally, the above derivation of Eq. (178), with the square root appearing as the sum of a multinomial expansion, indicates the lines along which a rigorous mathematical derivation of the square-root renormalization factors in Eq. (99) of II can be constructed.

Next consider the effect of inserting P-vertices into a single internal line λ_s of G_0. Let the ends of λ_s be y and z. The line-doubling in G_0 can always be arranged so that λ_s is doubled. It is supposed that the down direction in λ_s runs from z to y. Let

$$R_s=l+i\eta\Delta \quad (179)$$

be the vector appearing in the factor (73) contributed by λ_s to $M(G_0)$. Let G be a typical graph derived from G_0 by inserting in λ_s the P-vertices $y_1, \cdots, y_r$, in order reading from z to y. Let d_i, Λ_i, and Δ_i be defined for each y_i as before. Then $M(G)$ is obtained from $M(G_0)$ by making two alterations. First, an extra Δ_r is inserted in some factors of $M(G_0)$ precisely as before. Second, new factors are inserted in $M(G_0)$, namely, those obtained by replacing l in Eq. (166) by $(l+i\eta\Delta)$. That is to say, the factor (73) in $M(G_0)$ is replaced by

$$F_\Delta=F''F_\Delta',$$

with

$$F_\Delta' = [(l_\mu + i\eta(\Delta_\mu + \Delta_{r\mu}))\gamma_\mu - \mathrm{im}]^{-1} \times [(l_\mu + i\eta(\Delta_\mu + \tfrac{1}{2}\Delta_{r\mu} + \tfrac{1}{2}\Delta_{r-1,\mu}))\gamma_\mu - \mathrm{im}]\cdots \times [(l_\mu + i\eta(\Delta_\mu + \Delta_{1\mu}))\gamma_\mu - \mathrm{im}]^{-1} \times [(l_\mu + i\eta(\Delta_\mu + \tfrac{1}{2}\Delta_{1\mu}))\gamma_\mu - \mathrm{im}] \times [(l_\mu + i\eta\Delta_\mu)\gamma_\mu - \mathrm{im}]^{-1}. \quad (180)$$

The order of factors in F_Δ' is again to be reversed if $\eta = -1$.

Let $\sum_r F_\Delta'$ be defined as the sum of the products obtained by permuting $\Lambda_1, \cdots, \Lambda_r$ in F_Δ'. The following identity, which is a generalization of Eq. (172), will be proved by induction on r.

$$\sum_r F_\Delta' = \sum_{j=0}^{r} f_j f_{r-j} \sum_r [(l_\mu + i\eta(\Delta_\mu + \Delta_{j\mu}))\gamma_\mu - \mathrm{im}]^{-1}, \quad (181)$$

with f_j given by Eq. (173). The summation $\sum_r$ is over the $(r!)$ permutations of the Λ_j, on both sides of Eq. (181). To prove Eq. (181) for any value of r, assume it to hold for $(r-1)$. Then we have

$$\sum_r F_\Delta' = \sum_{j=0}^{r-1} f_j f_{r-1-j} \sum_r [(l_\mu + i\eta(\Delta_\mu + \Delta_{r\mu}))\gamma_\mu - \mathrm{im}]^{-1} \times [(l_\mu + i\eta(\Delta_\mu + \tfrac{1}{2}\Delta_{r\mu} + \tfrac{1}{2}\Delta_{r-1,\mu}))\gamma_\mu - \mathrm{im}] \times [(l_\mu + i\eta(\Delta_\mu + \Delta_{j\mu}))\gamma_\mu - \mathrm{im}]^{-1}. \quad (182)$$

The middle factor in square brackets in Eq. (182), when summed over permutations of the Λ_j, is equivalent to

$$[(2r-2j-1)/(2r-2j)][(l_\mu + i\eta(\Delta_\mu + \Delta_{r\mu}))\gamma_\mu - \mathrm{im}] + [1/(2r-2j)][(l_\mu + i\eta(\Delta_\mu + \Delta_{j\mu}))\gamma_\mu - \mathrm{im}]. \quad (183)$$

Substituting Eq. (183) into Eq. (182) gives immediately Eq. (181) with the help of the elementary identity[18]

$$\sum_{j=0}^{r-1} \frac{1}{(2r-2j)} f_j f_{r-1-j} = f_r. \quad (184)$$

This completes the proof of Eq. (181).

Let $\sum M(G)$ denote as before the sum of the $M(G)$ derived from G obtained by inserting the P-vertices $y_1, \cdots, y_r$ with the given set of indices $d_1, \cdots, d_r$ permuted in all possible ways. In $\sum M(G)$ there appears instead of the factor F_Δ the expression

$$\sum F_\Delta = (n_1! n_2! \cdots)^{-1} F'' \sum_r F_\Delta'. \quad (185)$$

Let Q denote any subset of the indices d_i, containing q_1 indices equal to 1, q_2 equal to 2, and so on. Thus, $0 \le q_i \le n_i$ for each i. Let

$$p_i = n_i - q_i, \quad j = \sum q_i, \quad r - j = \sum p_i. \quad (186)$$

[18] To prove Eq. (184), equate coefficients of x^r in the identity $(1-x)^{-\frac{1}{2}}(1-x)^{\frac{1}{2}} = 1$.

Using (181), we have for Eq. (185)

$$\sum F_\Delta = \sum_Q [j!/(q_1! q_2! \cdots)][(r-j)!/(p_1! p_2! \cdots)] \times f_j f_{r-j} [C_1^{p_1+q_1} C_2^{p_2+q_2} \cdots] \times [(l_\mu + i\eta(\Delta_\mu + \Delta_{Q\mu}))\gamma_\mu - \mathrm{im}]^{-1}, \quad (187)$$

where Δ_Q is the sum of the Λ_i corresponding to the indices d_i in Q. The part of $\sum M(G)$ arising from a particular term Q in the sum (187) will be denoted by $M_Q(G)$.

Let d' be the sum of the d_i in Q, d'' the sum of the d_i not in Q, and $d = d' + d''$. Let $M_Q(G)$ now be summed over all graphs G obtained by inserting any number of P-vertices into λ_s with any set of indices $d_1, \cdots, d_r$, and over all subsets Q of the d_i, with G and Q being restricted only by the condition that d' and d'' are to have fixed values. Let the double sum be denoted by $\sum_d M_Q(G)$. The Δ_Q appearing in Eq. (187) is a sum of d' vectors Γ_i. Since these Γ_i are equivalent integration variables in $M(G)$ according to Eq. (59), Δ_Q is effectively only dependent on d' and is the same in all terms of the sum $\sum_d M_Q(G)$. Therefore, $\Delta_{d'}$ will be written instead of Δ_Q. To obtain $\sum_d M_Q(G)$ from $M(G_0)$, the factor

$$[(l_\mu + i\eta\Delta_\mu)\gamma_\mu - \mathrm{im}]^{-1}$$

is replaced by

$$F_{d'd''}[(l_\mu + i\eta(\Delta_\mu + \Delta_{d'\mu}))\gamma_\mu - \mathrm{im}]^{-1}. \quad (188)$$

The numerical factor $F_{d'd''}$ is by Eq. (187) the coefficient of $(e_1')^{d'}(e_1'')^{d''}$ in the expansion

$$\sum_{p_1} \sum_{p_2} \cdots \sum_{q_1} \sum_{q_2} \cdots [(p_1 + p_2 + \cdots)!/(p_1! p_2! \cdots)] \times [(q_1 + q_2 + \cdots)!/(q_1! q_2! \cdots)] f_{(p_1+p_2+\cdots)} \times f_{(q_1+q_2+\cdots)} (C_1 e_1')^{q_1} (C_2 e_1'^2)^{q_2} \cdots \times (C_1 e_1'')^{p_1} (C_2 e_1''^2)^{p_2} \cdots, \quad (189)$$

which is the multinomial expansion of

$$[1 - \sum C_d (e_1')^d]^{-\frac{1}{2}} [1 - \sum C_d (e_1'')^d]^{-\frac{1}{2}} = Z_2^{\frac{1}{2}}(\lambda_0, e_1') Z_2^{\frac{1}{2}}(\lambda_0, e_1''). \quad (190)$$

Let $M(G_0, \lambda_s)$ be the sum of the $M(G)$ obtained from all G derived from G_0 by inserting P-vertices in λ_s in any way whatever. The same argument that was used for $M(G_0, \lambda_E)$ can be applied to $M(G_0, \lambda_s)$, starting from Eqs. (188) and (190). The conclusion is the following. To obtain $M(G_0, \lambda_s)$ from $M(G_0)$, it is necessary only to multiply the interactions operating both at z and at y in G_0 by the same factor (178).

Returning finally to the class C of graphs obtained by inserting P-vertices into all electron lines of G_0, the sum $M(C)$ will be obtained as follows. Let an E-vertex be defined as a vertex at which electron lines are incident. At every E-vertex of G_0 there are incident exactly two electron lines, internal or external, in each

of which P-vertices may be independently inserted. Therefore, $M(C)$ is derived from $M(G_0)$ by multiplying every interaction operating at every E-vertex of G_0 by the factor

$$Z_{2g}(\lambda_0). \tag{191}$$

This completes the discussion of the effects of P-vertices.

To summarize the results, it has been shown that all the effects of P-vertices are exactly taken into account, if the P-vertices themselves are dropped and the interactions operating at all other E-vertices are multiplied by (191).

There are three types of E-vertex in G_0, namely, ordinary vertices at which H_{gi} operates, S-vertices at which H_{gs} operates, and N-vertices. Consider first the effect of the multiplying factor (191) at an ordinary vertex y. By Eqs. (157) and (160), the term of order d in (191) will precisely cancel the divergent term (132) arising from the insertion of vertex parts with $(d+1)$ vertices into G_0 at y. The cancellation is to be made step by step, first the term $d=2$ in (191) canceling the term $d=2$ in Eq. (132), next the terms $d=4$ canceling, and so on. In this way it is clear that each vertex part is freed from its internal divergences before its own contribution to Eq. (132) is canceled. Therefore, it was correct to calculate L_d in Eq. (132) assuming the internal divergences to have been previously removed.

Next consider the effect of the multiplying factor (191) at an N-vertex. By Eq. (158), the interaction operating at N-vertices is thereby changed into

$$[1-Z_{2g}(\lambda_0)]L_D(x')=-\sum_{d=2}^{\infty}E_d[e_1g(t-t')]^dL_D(x'). \tag{192}$$

Let W be any N-vertex in G_0 at which the term of order ${e_1}^d$ in Eq. (192) is operating. Then the factor contributed by W to $M(C)$ is

$$-E_d[(l_\mu+i\bar{\Delta}_\mu)\gamma_\mu-\mathrm{im}], \tag{193}$$

which now replaces Eq. (163). The multiplying factor (191) does not change the contribution (150) to $M(C)$ arising from ordinary proper electron self-energy parts W'; in fact, the effects of (191) were exactly used up in canceling the internal vertex-part divergences from each W' before Eq. (150) was calculated. The contributions (150) and (193) are both correct after the multiplication by (191), and are to be added together as they stand in calculating $M(C)$. Therefore, the term in E_d in Eq. (156) is precisely canceled by the contribution from N-vertices.

Finally, consider the effect of (191) at an S-vertex. By Eqs. (23), (25), and (152), the interaction at S-vertices is changed into

$$-Z_{2g}(\lambda_0)\delta_g mc^2\bar{\psi}\psi(x')$$

$$=-\sum_{d=2}^{\infty}X_d[e_1g(t-t')]^dmc^2\bar{\psi}\psi(x'), \tag{194}$$

where X_d is a numerical coefficient depending on λ_0, which becomes formally identical with K_d as λ_0 tends to zero. Therefore, the contribution (154) to J_d is changed into

$$-iX_dm. \tag{195}$$

The term (iK_dm) in Eq. (156) is, however, not changed by the multiplication by (191), since this term arose from ordinary self-energy parts in the same way as the other term in Eq. (151) which was discussed previously. The total effect of the multiplication by (191) is thus to replace Eq. (156) by

$$\mathrm{im}(K_d-X_d). \tag{196}$$

If, at the end of the calculation of M, the auxiliary mass λ_0 is put equal to zero, then (196) vanishes. Unfortunately, putting λ_0 equal to zero will introduce infrared divergences into the convergent integrals ${I_F}^d$ and ${J_F}^d$. The best solution of this difficulty is to insert a small photon mass λ_0 into the formalism from the very beginning, so that the self-energy $\delta m(\lambda_0)$ appears already in Eq. (13). Then, according to Eqs. (152) and (194), K_d and X_d are equal, and (196) vanishes without creating infrared divergences anywhere.

All the divergent terms (132) and (156) are now eliminated from M. This completes the proof, which has occupied Secs. IV–X, that the operator $A_{\mu g}(p)$ is divergence-free.

XI. NOTE ON MATTER-FIELD OPERATORS

Hitherto, attention has been concentrated exclusively on the electromagnetic potentials $A_{\mu g}(p)$. This was done not because the $A_{\mu g}(p)$ are in themselves of any special importance, but because they served as a concrete example to illustrate the technique of carrying through the renormalization program in the intermediate representation. The methods which have been developed in Secs. IV–X can be applied with only minor changes to the analysis of any intermediate representation operators. The most important application of these methods will be to the hamiltonian operator $H'(t)$ of the intermediate representation, defined in Eqs. (6) and (7) of RM. The detailed discussion of $H'(t)$ will be published in a separate paper. Here only the results of the renormalization program applied to the matter-field operators $\psi_g(p)$, $j_{\mu g}(p)$ will be described.

The graphs which contribute to the normal constituents of $\psi_g(p)$ are similar to those contributing to $A_{\mu g}(p)$, except that a single electron line, instead of a single photon line, is incident at the special vertex x. The whole analysis of Secs. IV to X applies unchanged to $\psi_g(p)$, except that in Sec. X the vertex x appears as an additional E-vertex at which the multiplying factor (178) will occur. At x there are no vertex part divergences to compensate Eq. (178). Moreover, at x the two times t and t' in Eq. (178) are equal, and by Eq. (17) the factor (178) reduces to the numerical constant

$Z_2^{\frac{1}{2}}(\lambda_0)$, which becomes $Z_2^{\frac{1}{2}}$ when λ_0 is put equal to zero. Hence, one obtains the result that the operator $\psi_g(p)$ is a divergence-free expression multiplied by the renormalization factor $Z_2^{\frac{1}{2}}$. The same is true of $\bar{\psi}_g(p)$.

The graphs contributing to $j_{\mu g}(p)$ have the property that two electron lines are incident at x. In this case the analysis of Secs. IX–X applies without change, for x behaves like an ordinary E-vertex at which vertex part divergences cancel the multiplying factor (178). However, the cancellation of Sec. VIII fails for the divergences produced by a proper photon self-energy part W one of whose end-vertices is x. The divergence (113) arising from W is not removed by any compensating terms (116), (117), (118). In the intermediate representation operator $j_{\mu g}(p)$, the divergences (113) occur in a rather complicated way. This operator is in general not made finite by renormalization.

However, in the special case of the Heisenberg representation, $g(t)=1$ and all the Γ_i tend to zero. The Heisenberg charge-current operator $\mathbf{j}_\mu(x)$ is defined in the Gupta formalism by

$$\mathbf{j}_\mu(x)=ie_1c\bar{\psi}(x)\gamma_\mu\psi(x). \tag{197}$$

In $\mathbf{j}_\mu(p)$, the divergence (113) arising from a photon self-energy part W incident at x takes the form

$$I_D{}^C=-cB(p^2\delta_{\mu\nu}-p_\mu p_\nu). \tag{198}$$

In $\mathbf{A}_\mu(p)$, the photon line incident at x contributes a factor $(p^2)^{-1}$, and the remaining factor is what remains of $[(1/c)\mathbf{j}_\mu(p)]$ after all the divergences (198) are removed. Also, the divergences (198) in $\mathbf{j}_\mu(p)$ multiply an expression which is identical with $[-c\mathbf{A}_\mu(p)]$. Therefore, altogether, using Eq. (26), one obtains

$$\begin{aligned}(1/c)\mathbf{j}_\mu(p)&=p^2\mathbf{A}_\mu(p)-[\textstyle\sum B_d e_1{}^d](p^2\delta_{\mu\nu}-p_\mu p_\nu)\mathbf{A}_\nu(p)\\&=p^2\mathbf{A}_\mu(p)-f(p^2\delta_{\mu\nu}-p_\mu p_\nu)\mathbf{A}_\nu(p).\end{aligned} \tag{199}$$

This shows that $\mathbf{j}_\mu(p)$ is divergence-free except for the single constant f appearing in Eq. (199). The above derivation of Eq. (199) confirms the consistency of the renormalization method because Eq. (199) is identical with the field-equation for $\mathbf{A}_\mu$ obtained from the lagrangian (3). When the supplementary condition $(p_\nu\mathbf{A}_\nu(p)=0)$ is imposed, Eq. (199) becomes

$$(1/c)\mathbf{j}_\mu(p)=Z_3p^2\mathbf{A}_\mu(p), \tag{200}$$

so that $\mathbf{j}_\mu(p)$ is a divergence-free expression multiplied by the constant $Z_3=(e_1/e)^2$.

XII. THE TRANSITION TO HEISENBERG OPERATORS. CONCLUDING REMARKS

The transition from the intermediate to the Heisenberg representation is to be made by letting the function $g(t)\to 1$. To define the limiting process precisely, it is convenient to replace $g(t)$ by $g(t/T)$, where T is a parameter, and then let $T\to\infty$. The derivatives g' and g'' in Eqs. (29) and (30) tend uniformly to zero as $T\to\infty$. The interaction $H_g(x,x')$ defined by Eq. (28) tends uniformly to $H^I(x')$ given by Eq. (11). Therefore, the transformation operator (35) becomes in the limit identical with Eq. (6) of HO.I, the operator leading from the interaction representation to the Heisenberg representation. Heisenberg operators are limiting cases of intermediate representation operators; this is the exact meaning of the remarks made in Sec. IV of HO.I concerning the relation of Eq. (32) to Eq. (33). The limiting process introduces no difficulties into the formal separation and cancellation of divergences, since all the divergent coefficients are constants independent of $g(t)$. However, it is important to inquire, under what conditions the convergent expressions remaining after renormalization will tend to well-defined limits as $g(t)\to 1$, independent of the precise way in which the limiting process is performed. Only when such conditions are satisfied will it be permissible to represent Heisenberg operators by power-series in the interaction hamiltonian, as it is done in HO.I.

When $g(t)$ is a function which is equal to 1 at all recent times t and tends to zero only in the remote past, the intermediate representation operators are equivalent to the Heisenberg operators defined for a theory in which the charge e_1 rises adiabatically from the value zero in the remote past to its actual value at the present time. The power-series expansions of Heisenberg operators in HO.I are by definition the limits of the expansions of Heisenberg operators with the adiabatically varying e_1, supposing the limits to exist as the rate of variation of e_1 is made infinitely slow. The whole discussion of Heisenberg operators has a meaning only when these limits exist. Physically speaking, the limits will exist only if the actual state of the system can be deduced from the state of the system existing in the remote past before the adiabatic switching-on of e_1 was begun, independently of the details of the switching-on process. That is to say, the state of the system must be such that at some time in the past only separate free particles existed; then, each free particle can be considered to have been independently formed from an equivalent "bare" particle when e_1 was switched on. The switching-on process must be finished before the separate free particles converge and interact and give rise to the actual state of the system. The state of the system in the remote past must be free from any groups of particles bound together into stable composite structures by the radiation interaction.

The definition of Heisenberg operators as limits of intermediate representation operators, or the equivalent definition by a limiting process involving the adiabatic switching on of e_1, will thus be meaningful only under the following conditions. Tracing the history of the system back into the past, the radiation interaction must have operated only a finite number of times as a true interaction between different particles; further back than this, the system was in a dissociated state

where the radiation interaction produced only the self-fields of individual free particles.

From the last statement of the condition for the validity of the limiting process there follows an interesting consequence. In the power series expansions of Heisenberg operators, after the cancellation of divergences due to renormalization effects, the terms of order e_1^n represent the processes in which the radiation interaction has operated as a true interaction n times during the whole previous history of the system. If the radiation interaction has operated in this way only a finite number of times, then the terms of very high order in e_1 will make a negligible contribution to the operators. That is to say, the series expansions of Heisenberg operators, after renormalization, will themselves be convergent power series. And conversely, if the previous history of the system involves bound states, so that the radiation interaction has operated an infinite number of times, then the power series will not converge.

The convergence of the power series after renormalization has in no case yet been mathematically proved. The above physical argument only makes it plausible that, under the restricted conditions in which the definition of the series expansions of Heisenberg operators by a limiting process is meaningful, the series will always converge.

It needs to be stressed that the conditions under which the series expansions of Heisenberg operators can be defined are very restrictive. In almost all physical situations, bound states of some kind are involved either actually or virtually, and the expansions are either meaningless or misleading. The same remarks apply *a fortiori* to the series expansion of the S-matrix, which requires the absence of bound states not only in the past but also in the future. Just for this reason, the intermediate representation extends the technique of renormalization to a wide range of problems which could not be satisfactorily treated before.[19,20]

In the power series expansions of intermediate representation operators after renormalization, the terms of order e_1^n represent processes in which the radiation interaction has operated as a true interaction n times during the finite interval of time in which $g(t)$ is appreciable. It is plausible, but not proved, that the total probability of such processes will tend to zero rapidly as n tends to infinity, in all circumstances, whether bound states are involved in the system or not. Thus, not only does the use of the intermediate representation overcome the restrictions on the definability of the series expansions of the S-matrix and of Heisenberg operators, but also it may be hoped with some confidence that intermediate representation operators have power series expansions which are always convergent after renormalization.

[19] Another method of treating radiation problems has recently been published by E. C. G. Stueckelberg, Phys. Rev. **81**, 130 (1951), which is similar in its basic idea to the intermediate representation method. Both methods begin by replacing the customary integration over an infinite time-interval by an integration over a finite time-interval. But the two methods are so dissimilar in their subsequent development that a direct comparison between them is hardly possible. See also E. C. G. Stueckelberg and T. A. Green, Helv. Phys. Acta **24**, 153 (1951).

[20] A clear discussion of the limitations of the old series expansion methods is given by B. Ferretti, Nuovo cimento **8**, 108 (1951).