The Relativistic Theory of Electro-Magneto-Ionic Waves

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The relativistic equations governing an ionized gas pervaded by static electric and magnetic fields and the corresponding equations for small perturbations, are derived. The equations for plane perturbations are then obtained and several important cases are developed in detail. Frequency bands in which growing wave-modes occur are also determined.

By means of certain rules of transformation the theory is also used to study the plane waves which can occur in interpenetrating double streams of electrons. The results obtained are formally similar to those obtained for waves in an ionized gas.

In the absence of static magnetic fields and with the effects of collisions neglected it is found that, in either an ionized gas or in interpenetrating double streams of electrons, certain waves propagated obliquely to the drift motions may both grow and possess Poynting fluxes; these fluxes are such that certain initial disturbances can lead to the escape of amplified electromagnetic energy from an ionized medium.

The exchange of momentum and energy, between the streams of electrons and ions and the growing waves, is discussed by means of the momentum-energy tensors of the charged particles and of the electromagnetic field.

The results of the relativistic theory are then used to discuss briefly the problem of the origin of cosmic noise and of "isolated bursts" and "outbursts" of solar noise. It is concluded that both theory and observation lend support to the hypothesis suggested previously that a notable part of cosmic noise and strong solar noise originates as electro-magneto-ionic waves in magnetized ionized regions. It would then follow that such regions occur in our Galaxy and in the Great Nebula in Andromeda. It is suggested that all "point sources" of cosmic noise be examined for at least transient traces of the Zeeman effect and an excess of elliptically polarized noise.

I. INTRODUCTION

In the original publications[1–3] on plane waves in an ionized gas pervaded by static electric and magnetic fields the equations of motion of the electrons and positive ions were taken in their classical, nonrelativistic forms. The resulting nonrelativistic theory was found to offer a simple explanation of the spontaneous generation of strong high frequency noise in discharge tubes subjected to magnetic fields and in sunspots.

In order to examine the generation of solar noise under the conditions envisaged by Giovanelli, in which the electrons attain drift velocities approaching that of light, it became necessary to develop the theory in a relativistic form.[3,4] The need for this development is reinforced by the fact, independently pointed out by Walker,[4] that in the absence of magnetic fields and electron temperatures the nonrelativistic theory may incorrectly lead to certain wave amplification.

In the present paper this development is carried out in detail for an ionized gas in which the effects of gradients of partial pressures are neglected. This covers many fields of application. The relativistic consideration of the effects of partial pressure gradients is not straightforward and for this reason will be postponed to another occasion.

In order to make the theory also easily available for the discussion of plane waves in interpenetrating double streams of electrons the ratio of the static (or mean) densities $\bar{N}$ and $\bar{N}'$ of the electrons and ions, respectively, is taken as having any value.

The principal symbols used are defined in the following Table of Notation.

TABLE OF NOTATION

β	$=(1-V^2/c^2)^{-\frac{1}{2}}$, $\beta'=(1-V'^2/c^2)^{-\frac{1}{2}}$.
e	charge on an electron $=-4.80\times10^{-10}$ esu.
F_{ij}	total electromagnetic field tensor.
f_{ij}	variable part of the electromagnetic field tensor.
$\mathbf{H}$	static magnetic field.
k	$=e/m_0c$, $k'=-e/m_0'c$.
l_i	four-vector of wave number and wave frequency.
l_4	$=\omega/ic$.
ν_0, ν_0'	collision frequencies of an electron and positive ion, respectively.
m_0, m_0'	rest masses of an electron and positive ion, respectively.
N, N'	number-densities, in their proper frames, of the electrons and positive ions, respectively.
n, n'	variable parts of these number-densities.
P	$=m_0/4\pi\bar{N}e^2$.
p_0^2	$=4\pi\bar{N}e^2/m_0$.
R	$=\beta^{-1}\bar{U}_jl_j$ or $Vl_1+\omega$ or $\omega-V_1L$.
S	$=-l_1-\omega V/c^2$.
S_i	four-vector of total current-density and charge density.
s_i	variable part of S_i.
σ	$=\bar{N}'/\bar{N}$.
U_i, U_i'	velocity four-vectors of electrons and ions, respectively.
u_i, u_i'	variable parts of U_i and U_i', respectively.
V_1, V_2, V_3	the components of electron drift velocity.
V, V'	drift velocities of electrons and ions, respectively.
W_i, W_i'	four-vectors defined under (9).
X	$=\beta^2R^2-p_0^2-i\beta^2\nu_0R$.
Y	$=\beta R(Z+p_0^2)-i\beta\nu_0Z$.
Z	$=c^2l_jl_j$ or $c^2(l_1^2+l_2^2)-\omega^2$ or $c^2L^2-\omega^2$.
ω	angular wave frequency.
$\mathbf{\Omega}$, $\mathbf{\Omega}'$	angular gyro-frequency vectors of an electron and ion, respectively i.e., $\mathbf{H}(-e/\beta m_0c)$ and $\mathbf{H}(-e/\beta'm_0'c)$.
i, j, k	indices running through the values 1, 2, 3, 4.

For repeated indices the summation convention is understood. Bars over symbols indicate static, or mean, values.

The theory is based on the following laws of physics:

I. Maxwell's laws of the electromagnetic field.

II. The conservation of electrons and of positive ions.

[1] V. A. Bailey, J. and Proc. Roy. Soc. N. S. W. **82**, 107 (1948).
[2] V. A. Bailey, Australian J. Sci. Res. **A1**, 351 (1948).
[3] V. A. Bailey, Phys. Rev. **78**, 428 (1950).
[4] V. A. Bailey, Phys. Rev. **77**, 418 (1950); L. R. Walker, Phys. Rev. **76**, 1721 (1949).

III. Maxwell's laws of the transfer of momentum in mixtures of different kinds of particles.

With regard to I there is a choice in the formulation: We may introduce auxiliary quantities like the vector and scalar potentials (as in the earlier publications), or we may do without them. The choice may be determined by the purpose in hand or merely by considerations of convenience. In the present paper these potentials are not used and the laws of the electromagnetic field are taken in the form:[5]

$$\partial F_{ij}/\partial x_j = (4\pi/c)S_i, \tag{A}$$

$$\partial F_{jk}/\partial x_i + \partial F_{ki}/\partial x_j + \partial F_{ij}/\partial x_k = 0. \tag{B}$$

From these equations we easily deduce the equation,

$$\partial S_i/\partial x_i = 0, \tag{C}$$

and Eq. (2) given below. But (C) expresses the conservation of electric charge which is also a consequence of the conservation laws under II. In order to avoid this redundancy we have adopted the Eqs. (1) to (4) as a joint formulation of the laws under I and II which is sufficient for our present purposes. For the study of problems involving initially prescribed conditions or boundary conditions it may be necessary to have recourse also to Eqs. (A) and (B).

The effects of collisions between charged particles of opposite signs are here neglected.

II. THE FUNDAMENTAL EQUATIONS

The fundamental equations of the electromagnetic field F_{ij} and of conservation and motion of the two kinds of charged particles involved are as follows:

$$S_i = e(NU_i - N'U_i'), \tag{1}$$

$$\Box^2 F_{ij} = \frac{4\pi}{c}\left(\frac{\partial S_i}{\partial x_j} - \frac{\partial S_j}{\partial x_i}\right), \tag{2}$$

$$\partial(NU_i)/\partial x_i = 0, \tag{3}$$

$$\partial(N'U_i')/\partial x_i = 0, \tag{4}$$

$$U_j\partial U_i/\partial x_j + \nu_0 W_i = kF_{ij}U_j, \tag{5}$$

$$U_j'\partial U_i'/\partial x_j + \nu_0' W_i' = k'F_{ij}U_j', \tag{6}$$

$$U_iU_i = -c^2, \tag{7}$$

$$U_i'U_i' = -c^2, \tag{8}$$

where, in a frame of reference at rest in the gas,

$$\left.\begin{aligned} W_i &= (-i/c)(U_1U_4,\ U_2U_4,\ U_3U_4,\ c^2+U_4^2),\\ W_i' &= (-i/c)(U_1'U_4',\ U_2'U_4',\ U_3'U_4',\ c^2+U_4'^2). \end{aligned}\right\} \tag{9}$$

The introduction of the term $\nu_0 W_i$ in (5) to represent, in relativistic form, the rate of loss of momentum through collisions with gas molecules is justified by the discussion in Appendix I.

[5] With the summation convention for repeated indices.

The indices i, j run through the values 1, 2, 3, 4. Since $U_iW_i=0$ the fourth equation in (5) follows from the first three and (7). It may therefore be omitted or regarded as an alternative to (7). Similarly the fourth equation in (6) may be omitted or regarded as an alternative to (8). Thus the system (1) to (8) contains twenty independent equations in the twenty variables S_i, N, N', U_i, U_i', F_{ij}.

Denoting the static values (or mean values in time) by means of bars over the symbols, and the perturbations by means of lower case letters we set

$$\left.\begin{aligned} N &= \bar{N}+n, & U_i &= \bar{U}_i+u_i,\\ N' &= \bar{N}'+n', & U_i' &= \bar{U}_i'+u_i',\\ S_i &= \bar{S}_i+s_i, & F_{ij} &= \bar{F}_{ij}+f_{ij}. \end{aligned}\right\} \tag{10}$$

The equations for the static, uniform state are

$$\bar{S}_i = e(\bar{N}\bar{U}_i - \bar{N}'\bar{U}_i'), \tag{1.0}$$

$$\nu_0\bar{W}_i = k\bar{F}_{ij}\bar{U}_j, \tag{5.0}$$

$$\nu_0'\bar{W}_i' = k'\bar{F}_{ij}\bar{U}_j', \tag{6.0}$$

$$\bar{U}_i\bar{U}_i = -c^2, \tag{7.0}$$

$$\bar{U}_i'\bar{U}_i' = -c^2. \tag{8.0}$$

The equations for sufficiently small perturbations are therefore as follows:—

$$s_i = e(\bar{N}u_i + \bar{U}_i n - \bar{N}'u_i' - \bar{U}_i'n'), \tag{1.1}$$

$$\Box^2 f_{ij} = \frac{4\pi}{c}\left(\frac{\partial s_i}{\partial x_j} - \frac{\partial s_j}{\partial x_i}\right), \tag{2.1}$$

$$\bar{U}_j\partial n/\partial x_j + \bar{N}\partial u_j/\partial x_j = 0, \tag{3.1}$$

$$\bar{U}_j'\partial n'/\partial x_j + \bar{N}'\partial u_j'/\partial x_j = 0, \tag{4.1}$$

$$\bar{U}_j\partial u_i/\partial x_j + \nu_0 w_i = k\bar{F}_{ij}u_j + k\bar{U}_j f_{ij}, \tag{5.1}$$

$$\bar{U}_j'\partial u_i'/\partial x_j + \nu_0' w_i' = k'\bar{F}_{ij}u_j' + k'\bar{U}_j' f_{ij}, \tag{6.1}$$

$$\bar{U}_j u_j = 0, \tag{7.1}$$

$$\bar{U}_j' u_j' = 0, \tag{8.1}$$

where

$$\left.\begin{aligned} w_i &= (-i/c)(\bar{U}_i u_4 + \bar{U}_4 u_i),\\ w_i' &= (-i/c)(\bar{U}_i' u_4' + \bar{U}_4' u_i') \end{aligned}\right\} \tag{9.1}$$

for $i=1, 2, 3, 4$.

These equations may be used to study plane or cylindrical waves.

III. PLANE PERTURBATIONS

We now consider perturbations of the form,

$$\psi = A\exp(il_jx_j), \tag{11}$$

where

$$l_j = (l_1, l_2, l_3, l_4), \quad (l_4 = \omega/ic)$$

is the four-vector of wave number and frequency, and A is a constant.

We may therefore now make the following substitutions in our equations:

$$\partial/\partial x_j = il_j, \quad \Box^2 = -Z/c^2,$$

where

$$Z = c^2 l_j l_j. \tag{12}$$

On setting also

$$\bar{U}_j l_j = \beta R, \quad \bar{U}_j' l_j = \beta' R', \tag{13}$$

where

$$\beta = (1 - V^2/c^2)^{-\frac{1}{2}}, \quad \beta' = (1 - V'^2/c^2)^{-\frac{1}{2}},$$

Eqs. (2.1) to (6.1) become

$$Z f_{ij} = 4\pi ic(l_i s_j - l_j s_i), \tag{2.2}$$

$$\beta R n + \bar{N} l_j u_j = 0, \tag{3.2}$$

$$\beta' R' n' + \bar{N}' l_j u_j' = 0, \tag{4.2}$$

$$\beta R u_i - i\nu_0 w_i = -ik(\bar{F}_{ij} u_j + \bar{U}_j f_{ij}), \tag{5.2}$$

$$\beta' R' u_i' - i\nu_0' w_i' = -ik'(\bar{F}_{ij} u_j' + \bar{U}_j' f_{ij}). \tag{6.2}$$

From (1.1), (3.2) and (4.2) we now obtain

$$s_i = e\bar{N}\left(u_i - \frac{\bar{U}_i}{\beta R} l_j u_j\right) - e\bar{N}'\left(u_i' - \frac{\bar{U}_i'}{\beta' R'} l_j u_j'\right). \tag{14}$$

Hence from (2.2) we obtain

$$\left(\frac{Z}{4\pi ice\bar{N}}\right) f_{ij} = l_i u_j - l_j u_i - \left(\frac{l_i \bar{U}_j - l_j \bar{U}_i}{\beta R}\right) l_k u_k - \sigma\left[l_i u_j' - l_j u_i' - \left(\frac{l_i \bar{U}_j' - l_j \bar{U}_i'}{\beta' R'}\right) l_k u_k'\right], \tag{15}$$

where

$$\sigma = \bar{N}'/\bar{N}. \tag{16}$$

On multiplying both sides by $\bar{U}_j$, summing over j and using (7.1), (7.0), and (13) we obtain

$$(Z/4\pi ice\bar{N})\bar{U}_j f_{ij} = -\beta R u_i + A_i l_k u_k - \sigma(l_i \bar{U}_j u_j' - \beta R u_i' - B_i' l_k u_k'),$$

where

$$\left.\begin{aligned} A_i &= (l_i c^2 + \beta R \bar{U}_i)/\beta R, \\ B_i' &= (l_i \bar{U}_j \bar{U}_j' - \beta R \bar{U}_i')/\beta' R'. \end{aligned}\right\} \tag{17}$$

Then substituting for $\bar{U}_j f_{ij}$, from this last result, in (5.2) and using (9.1) to eliminate w_i we obtain after some reduction

$$PYu_i - A_i l_k u_k - PZ\nu_0 \bar{U}_i u_4/c - iZ\phi_{ij} u_j = \sigma(\beta R u_i' - l_i \bar{U}_j u_j' + B_i' l_k u_k'), \tag{18}$$

where

$$\left.\begin{aligned} P &= 1/4\pi cek\bar{N} = m_0/4\pi\bar{N}e^2 = 1/p_0^2, \\ \phi_{ij} &= -Pk\bar{F}_{ij}, \\ Y &= \beta R(Z + p_0^2) - i\beta\nu_0 Z. \end{aligned}\right\} \tag{19}$$

By symmetry Eq. (6.2) must yield similarly

$$\beta' R' u_i - l_i \bar{U}_j' u_j + B_i l_k u_k = \sigma(P'Y'u_i' - A_i' l_k u_k' - P'Z\nu_0' \bar{U}_i' u_4'/c) + iZ\phi_{ij} u_j', \tag{20}$$

where

$$\left.\begin{aligned} P' &= -1/4\pi cek'\bar{N}' = m_0'/4\pi\bar{N}'e^2 = 1/p_0'^2, \\ Y' &= \beta' R'(Z + p_0'^2) - i\beta' \nu_0' Z, \\ A_i' &= (l_i c^2 + \beta' R' \bar{U}_i')/\beta' R', \\ B_i &= (l_i \bar{U}_j' \bar{U}_j - \beta' R' \bar{U}_i)/\beta R. \end{aligned}\right\} \tag{21}$$

It should be noted that in the third term in (18) the factor $\nu_0 \bar{U}_i$ can be replaced by an expression free from ν_0 by using the relation (5.0) and the definition (9).

For a medium which on the average is initially neutral we have $\bar{N}' = \bar{N}$, i.e., $\sigma = 1$.

The six equations in (18) and (20) which correspond to $i = 1, 2, 3$, together with Eqs. (7.1) and (8.1) constitute a system (S_u^8) of eight homogeneous linear equations in the eight velocity components u_i, u_i'. The condition necessary for (S_u^8) to possess a nonzero solution is

$$\Delta_8 = 0, \tag{22}$$

where Δ_8 is the determinant of the eighth order formed by the coefficients of (S_u^8). This is also the equation of dispersion. When it is satisfied the velocity components are proportional to the cofactors of any row of Δ_8.

The field components f_{ij} and the density variations n, n' are then given by (15), (3.2), and (4.2) respectively.

Alternatively we may proceed as follows. From (7.1) and (8.1) we obtain

$$u_4 = ic^{-1}(V_1 u_1 + V_2 u_2 + V_3 u_3), \tag{23}$$

$$u_4' = ic^{-1}(V_1' u_1' + V_2' u_2' + V_3' u_3') \tag{24}$$

and so we can eliminate u_4, u_4' from (18) and (20). This yields a system (S_u^6) of six equations in u_1, u_2, u_3, u_1', u_2', u_3' from which the equation of dispersion can be derived in the form,

$$\Delta_6 = 0, \tag{25}$$

where Δ_6 is a determinant of the sixth order.

Equation (25) corresponds to Eq. (19) given in the earlier publication[2] which was found to be of degree 12 in the quantities l_i after removing irrelevant factors. We may therefore expect Eq. (25) to be of degree 12 after removal of such factors. This is confirmed by the discussion, in Sec. VI below, of the important case in which the drift velocities, the static magnetic field and the direction of propagation are all parallel.

A detailed comparison of the results of the present theory with those of the earlier nonrelativistic theory[2] is in general not easy to make. But in the special case treated in Sec. VI the equations are in substantial agreement with the corresponding equations previously given.

Another check on the theory is provided by Eq. (44) given below, which is the form assumed by the dispersion equation when the motions of the positive ions are neglected. Equation (44) agrees exactly with that obtained previously[3] by means of a Lorentz transformation.

The present theory can also be used for the study of plane waves in interpenetrating double streams of electrons in a vacuum, with mean densities $\bar{N}$, $\bar{N}'$, if the following changes are made.

(1) In all the formulas down to (21) inclusive, except as indicated under (2) below, we replace the symbols shown in the first row of the table (26) by the symbols in the second row

$$\left.\begin{array}{cccccc} m_0', & k', & N', & \bar{N}', & n', & \sigma, \\ m_0, & k, & -N, & -\bar{N}', & -n', & -\sigma. \end{array}\right\} \quad (26)$$

(2) In the first line of (21) the expression $m_0'/4\pi\bar{N}'e^2$ is replaced by $m_0/4\pi\bar{N}'e^2$.

(3) We set $\nu_0=\nu_0'=0$.

IV. APPROXIMATION WHEN THE MOTIONS OF POSITIVE IONS ARE NEGLECTED

We will here consider the case when the ionic mass $m_0'\to\infty$. Then in the foregoing theory we take $P'\to\infty$ and so by (20) $u_i'=0$.

Accordingly (18) reduces to

$$PYu_i-A_il_ku_k-iZ\phi_{ij}u_j-PZ\nu_0\bar{U}_iu_4/c=0. \quad (27)$$

On multiplying by βRl_i, summing over i and dividing through by Z we obtain

$$PXl_ku_k-i\beta Rl_i\phi_{ij}u_j-P\nu_0\beta^2R^2u_4/c=0, \quad (28)$$

where

$$X=\beta^2R^2-p_0^2-i\beta^2\nu_0R. \quad (29)$$

This may be taken as an alternative to one of the equations under (27).

We will now take the axis Ox in the direction of electron drift and Oz perpendicular to the direction of propagation, i.e., we take

$$\bar{U}_2=\bar{U}_3=0, \quad l_3=0. \quad (30)$$

Then from (23), (13), (5.0), and (17) we obtain

$$u_4=u_1iV/c, \quad (31)$$

$$R=Vl_1+\omega, \quad (32)$$

$$P\nu_0\bar{U}_1=-ic\phi_{14},\ 0=V\phi_{21}+ic\phi_{24},\ 0=V\phi_{31}+ic\phi_{34}, \quad (33)$$

$$A_2=l_2c^2/\beta R, \quad A_3=0, \quad (34)$$

and so

$$l_ku_k=-Su_1+l_2u_2, \quad (35)$$

where

$$S=-l_1-\omega V/c^2. \quad (36)$$

By (33) and (32) we have

$$l_1\phi_{14}u_4+l_4\phi_{41}u_1=\phi_{14}Ru_4/V=-P\nu_0\beta Ru_4/ic. \quad (37)$$

Hence (28) expands into

$$PX(-Su_1+l_2u_2)-i\beta R[l_2\phi_{21}u_1+(l_1\phi_{12}+l_4\phi_{42})u_2 + (l_1\phi_{13}+l_2\phi_{23}+l_4\phi_{43})u_3+l_2\phi_{24}u_4]=0.$$

This equation has for its coefficients polynomials of lower degree than those of the equation under (27) which corresponds to $i=1$ and it will therefore be taken in place of the latter.

On taking in turn $i=2$ and 3 in (27) and using (34) we also obtain

$$PYu_2+(l_2c^2/\beta R)(Su_1-l_2u_2) -iZ(\phi_{21}u_1+\phi_{23}u_3+\phi_{24}u_4)=0,$$

$$PYu_3-iZ(\phi_{31}u_1+\phi_{32}u_2+\phi_{34}u_4)=0.$$

On eliminating u_4, ϕ_{24}, ϕ_{34} from the last three equations, by means of (31) and (33), they reduce to the following:

$$\alpha_1u_1+\alpha_2u_2+\alpha_3u_3=0, \quad (38)$$

$$\beta_1u_1+\beta_2u_2+\beta_3u_3=0, \quad (39)$$

$$\gamma_1u_1+\gamma_2u_2+\gamma_3u_3=0, \quad (40)$$

where

$$\left.\begin{array}{l} \alpha_1=-XS+iRl_2\Omega_3, \quad \alpha_2=Xl_2+i\beta^2RS\Omega_3, \\ \qquad \alpha_3=-i\beta^2R(S\Omega_2+l_2\Omega_1), \\ \beta_1=p_0^2c^2l_2S/\beta R+iZ\beta^{-1}\Omega_3, \\ \qquad \beta_2=Y-p_0^2c^2l_2^2/\beta R, \quad \beta_3=-iZ\beta\Omega_1, \\ \gamma_1=-iZ\beta^{-1}\Omega_2, \quad \gamma_2=iZ\beta\Omega_1, \quad \gamma_3=Y, \end{array}\right\} \quad (41)$$

where

$$\left.\begin{array}{l} \mathbf{\Omega}=(-e/\beta m_0c)\mathbf{H}=\beta^{-1}p_0^2(\phi_{23},\phi_{31},\phi_{12}), \\ Z=c^2(l_1^2+l_2^2)-\omega^2. \end{array}\right\} \quad (42)$$

The equation of dispersion is therefore

$$\Delta\equiv\alpha_1D_1+\alpha_2D_2+\alpha_3D_3=0, \quad (43)$$

where

$$D_1=Y^2-p_0^2c^2l_2^2Y/\beta R-Z^2\beta^2\Omega_1^2,$$

$$D_2=-Z^2\Omega_1\Omega_2-iZY\beta^{-1}\Omega_3-p_0^2c^2l_2YS/\beta R,$$

$$D_3=-Z^2\Omega_3\Omega_1+iZ\Omega_1p_0^2c^2l_2S/R +iYZ\beta^{-1}\Omega_2-iZ\beta^{-2}\Omega_2p_0^2c^2l_2^2/R.$$

On substituting for D_1, D_2, D_3, expanding Δ in powers of l_2 and using the relation

$$\beta^2S^2+l_2^2=c^{-2}(Z+\beta^2R^2)$$

we find after reduction and division by S that (43) simplifies to

$$X[Y^2-(\beta^2\Omega_1^2+\Omega_2^2+\Omega_3^2)Z^2] -[\beta^2(S\Omega_2+l_2\Omega_1)^2+c^{-2}(Z+\beta^2R^2)\Omega_3^2]Zc^2p_0^2=0. \quad (44)$$

It is easily verified that (44) can also be expressed in the form:

$$XY^2-(\beta^2\Omega_1^2+\Omega_2^2+\Omega_3^2)\beta RZY \\ +(\beta^2 S\Omega_1-l_2\Omega_2)^2 Zc^2p_0^2=0. \quad (45)$$

Equations (45)[6] and (44) agree exactly with Eqs. (1a)[6] and (2a), respectively, given in Appendix I of the earlier publication,[3] when in these we set $\tau=0$.

When (44) or (45) is satisfied the velocity components are in general given by the formulas,

$$u_1=D_1E,\quad u_2=D_2E,\quad u_3=D_3E, \quad (46)$$

where

$$E=C\exp(il_1x+il_2y+i\omega t) \quad (47)$$

and C is an arbitrary number; but in certain cases like that in which

$$l_2=0,\quad \Omega_2=\Omega_3=0,$$

it may be more convenient to derive these components directly from Eqs. (38)–(40).

The components of actual velocity are evidently u_1/β, u_2/β, u_3/β.

V. PROPAGATION ALONG THE AXIS *Ox*

In the original nonrelativistic form[1–3] of the theory with negligible ionic motions, the axis Ox was taken along the direction of propagation, i.e., the small perturbations considered all had the form:

$$A\exp(i\omega t-iLx). \quad (48)$$

The corresponding relativistic form of the theory may be derived directly from the results given here in Sec. III by taking

$$l_2=l_3=0,\quad \bar{U}_3=0,\quad m_0'\rightarrow\infty.$$

But a simpler method of derivation is to transform the equations of Sec. IV by referring them to a new system of axes K' obtained from the old system K by a right-handed rotation about the axis Oz through an angle θ such that

$$\tan\theta=l_2/l_1.$$

Since (47) transforms into (48) we have

$$l_1=-L\cos\theta,\quad l_2=-L\sin\theta, \quad (49)$$

where

$$L^2=l_1^2+l_2^2, \quad (50)$$

and so

$$Vl_1=-V_1'L,\quad Vl_2=V_2'L,\quad 0=V_3',\quad V=V', \quad (51)$$

where

$$V'=(V_1'^2+V_2'^2)^{\frac{1}{2}}. \quad (52)$$

We now obtain the following transformations from (32), (36), and (42):

$$\left.\begin{aligned} R&=-V_1'L+\omega,\\ S&=LV_1'/V'-\omega V'/c^2,\\ \beta&=(1-V'^2/c^2)^{-\frac{1}{2}},\\ Z&=c^2L^2-\omega^2. \end{aligned}\right\} \quad (53)$$

Therefore X and Y may retain the forms in (29) and (19) by which they were first defined.

Further we have the transformations,

$$\left.\begin{aligned} \Omega_1&=(\Omega_1'V_1'+\Omega_2'V_2')/V',\\ \Omega_2&=(-\Omega_1'V_2'+\Omega_2'V_1')/V',\\ \Omega_3&=\Omega_3'. \end{aligned}\right\} \quad (54)$$

$$\left.\begin{aligned} u_1&=(u_1'V_1'+u_2'V_2')/V',\\ u_2&=(-u_1'V_2'+u_2'V_1')/V',\\ u_3&=u_3'. \end{aligned}\right\} \quad (55)$$

Then (38)–(40) transform into three similar equations with the primed symbols α', β', γ', u' replacing α, β, γ, u, respectively, where

$$\left.\begin{aligned} &\alpha_1'=(\alpha_1V_1'-\alpha_2V_2')/V',\\ &\qquad\alpha_2'=(\alpha_1V_2'+\alpha_2V_1')/V',\quad \alpha_3'=\alpha_3,\\ &\beta_1'=(\beta_1V_1'-\beta_2V_2')/V',\\ &\qquad\beta_2'=(\beta_1V_2'+\beta_2V_1')/V',\quad \beta_3'=\beta_3,\\ &\text{etc.} \end{aligned}\right\} \quad (56)$$

On substituting in (56) the expressions for α_1, α_2, α_3, β_1, β_2, β_3, etc., given in (41), using the transformations (49) to (55), rationalizing all expressions and dropping all the primes we thus obtain

$$\left.\begin{aligned} \alpha_1u_1+\alpha_2u_2+\alpha_3u_3&=0,\\ \beta_1u_1+\beta_2u_2+\beta_3u_3&=0,\\ \gamma_1u_1+\gamma_2u_2+\gamma_3u_3&=0, \end{aligned}\right\} \quad (57)$$

where

$$\left.\begin{aligned} \alpha_1&=-XS_1c^2+i\beta^2R^2V_2\Omega_3,\\ \alpha_2&=X\omega V_2+i\beta^2R\Omega_3(S_1c^2-LV_2^2),\\ \alpha_3&=-i\beta^2R(\omega V_2\Omega_1+S_1c^2\Omega_2),\\ \beta_1&=-V_2(\beta RY-p_0^2LS_1c^2)+iRZV_1\Omega_3,\\ \beta_2&=V_1\beta RY-p_0^2\omega LV_2^2+iRZV_2\Omega_3,\\ \beta_3&=-i\beta^2RZ(V_1\Omega_1+V_2\Omega_2),\\ \gamma_1&=-i\beta Z[\Omega_2(1-V_1^2/c_1^2)+\Omega_1V_1V_2/c^2],\\ \gamma_2&=i\beta Z[\Omega_1(1-V_2^2/c^2)+\Omega_2V_1V_2/c^2],\\ \gamma_3&=Y, \end{aligned}\right\} \quad (58)$$

where

$$R=\omega-V_1L,\quad S_1=L-\omega V_1/c^2. \quad (59)$$

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[6] Equation (45) or (1a) was also given in Phys. Rev. 77, 419 (1950).

Similarly we can transform the equations of dispersion (44) and (45), respectively, into

$$X[Y^2-\Omega^2Z^2-\beta^2c^{-2}(V_1\Omega_1+V_2\Omega_2)^2Z^2] -[\beta^2(\Omega_2S_1c+\Omega_1\omega V_2/c)^2+(Z+\beta^2R^2)\Omega_3^2]Zp_0^2=0 \quad (60)$$

and

$$XY^2-[\Omega^2+\beta^2c^{-2}(V_1\Omega_1+V_2\Omega_2)^2]\beta RZY +[\Omega_1(S_1c-LV_2^2/c)-\Omega_2RV_2/c]^2\beta^4Zp_0^2=0. \quad (61)$$

VI. LONGITUDINAL AND TRANSVERSE WAVES

As a further check and for its own sake we shall now consider the special case of the theory of Sec. III in which the drift velocities and the magnetic field **H** are all parallel to the direction of propagation. For convenience Ox will be taken parallel to this common direction.

Accordingly we have now

$$\left.\begin{aligned}&\bar{U}_1=\beta V,\quad \bar{U}_2=\bar{U}_3=0,\quad \bar{U}_4=i\beta c,\\&\bar{U}_1'=\beta'V',\quad \bar{U}_2'=\bar{U}_3'=0,\quad \bar{U}_4'=i\beta'c,\\&\bar{F}_{23}=H,\quad \bar{F}_{31}=\bar{F}_{12}=0,\\&l_1=-L,\quad l_2=l_3=0,\quad l_4=\omega/ic.\end{aligned}\right\} \quad (62)$$

Then by (12), (13), (17), and (21) we have

$$Z=c^2L^2-\omega^2,\quad R=\omega-VL,\quad R'=\omega-V'L, \quad (63)$$

$$\left.\begin{aligned}&A_1=-\beta c^2S/R, && A_2=A_3=0,\\&B_1'=\beta c^2S'/R', && B_2'=B_3'=0,\\&A_1'=-\beta'c^2S'/R', && A_2'=A_3'=0,\\&B_1=\beta'c^2S/R, && B_2=B_3=0,\end{aligned}\right\} \quad (64)$$

where

$$S=L-\omega V/c^2,\quad S'=L-\omega V'/c^2.$$

Next from (9), (5.0), and (6.0) we obtain

$$\bar{F}_{14}=\nu_0\beta V/kic=\nu_0'\beta'V'/k'ic,\quad \bar{F}_{24}=\bar{F}_{34}=0. \quad (65)$$

This shows that the static electric field is also parallel to Ox and, when ν_0, ν_0' are not zero, also gives a relation between the drift velocities of the electrons and positive ions. Also by (19), (21), and (16) we now have

$$Pk=-\sigma P'k', \quad (66)$$

$$\left.\begin{aligned}&\phi_{23}=P\beta\Omega=P'\sigma\beta'\Omega',\quad \phi_{31}=\phi_{12}=0,\\&\phi_{14}=-P\nu_0\beta V/ic=\sigma P'\nu_0'\beta'V'/ic,\\&\phi_{24}=\phi_{34}=0,\end{aligned}\right\} \quad (67)$$

where

$$\Omega=\frac{-eH}{\beta m_0c},\quad \Omega'=\frac{-eH}{\beta'm_0'c}, \quad (68)$$

are the gyro-frequencies of an electron and ion, respectively.

Lastly, from (23) and (24) we have

$$u_4=ic^{-1}Vu_1,\quad u_4'=ic^{-1}V'u_1'. \quad (69)$$

With the help of formulas (62) to (69), Eqs. (18) and (20) for $i=1$ now reduce to

$$Xu_1/\beta^2R=-\sigma p_0^2u_1'/\beta'^2R' \quad (70)$$

and

$$X'u_1'/\beta'^2R'=-\sigma^{-1}p_0'^2u_1/\beta^2R, \quad (71)$$

where

$$\left.\begin{aligned}&X=\beta^2R^2-p_0^2-i\beta^2\nu_0R,\\&X'=\beta'^2R'^2-p_0'^2-i\beta'^2\nu_0'R'.\end{aligned}\right\} \quad (72)$$

Equations (70) and (71) specify purely longitudinal waves. Their equation of dispersion is

$$XX'-p_0^2p_0'^2=0,$$

i.e., the following quartic in L:

$$(R^2-i\nu_0R)(R'^2-i\nu_0'R')-\beta'^{-2}p_0'^2(R^2-i\nu_0R) -\beta^{-2}p_0^2(R'^2-i\nu_0'R')=0. \quad (73)$$

Thus there are four longitudinal wave modes.

For small drift velocities Eqs. (70), (71), and (73) agree with Eqs. (14), (15), and (28) of the earlier nonrelativistic theory.[2] Equation (73) also then corresponds to a differential equation given previously by Schumann.[7]

When $\nu_0=\nu_0'=0$, Eq. (73) is equivalent to Eq. (137) discussed below in Sec. VIII. From that discussion it follows that with negligible collisions the noise frequency-band for purely longitudinal waves is given by (142).

Similarly on taking in succession the values 2 and 3 for i in Eqs. (18) and (20) we obtain

$$Yu_2-iZ\beta\Omega u_3=\sigma p_0^2\beta Ru_2', \quad (74)$$

$$Yu_3+iZ\beta\Omega u_2=\sigma p_0^2\beta Ru_3', \quad (75)$$

$$Y'u_2'+iZ\beta'\Omega'u_3'=\sigma^{-1}p_0'^2\beta'R'u_2, \quad (76)$$

$$Y'u_3'-iZ\beta'\Omega'u_2'=\sigma^{-1}p_0'^2\beta'R'u_3. \quad (77)$$

Equations (74) to (77) specify purely transverse waves. For small drift velocities they agree with the equations for the same case derived from the earlier nonrelativistic theory.[2]

On eliminating u_2' and u_3' from them we obtain

$$au_2=ibu_3, \quad (78)$$

$$au_3=-ibu_2, \quad (79)$$

where

$$\left.\begin{aligned}&a=Z^{-1}(YY'-p_0^2p_0'^2\beta\beta'RR')-Z\beta\beta'\Omega\Omega',\\&b=\beta\Omega Y'-\beta'\Omega'Y.\end{aligned}\right\} \quad (80)$$

From (78) and (79) we obtain the equation of dispersion,

$$a^2-b^2=0.$$

i.e.,

$$a=k_nb,\quad (n=1,2),\quad \text{where}\quad k_n=(-1)^{n-1}. \quad (81)$$

[7] W. O. Schumann, Z. Physik 121, 7 (1942).

Then (78), (74) and (75) yield

$$u_2 = k_n i u_3, \quad u_2' = k_n i u_3', \tag{82}$$

which relations enable us to deduce from (15) that

$$h_2 = k_n i h_3. \tag{83}$$

These formulas show that the transverse waves consist of two sets of circularly polarized waves with opposite senses of polarization. Also Eq. (83) is the same as Eq. (19) in the earlier publication[3] which discusses the same special case but neglects the motions of the positive ions.

From (81) and (80) we obtain the following quartic in L for the equation of dispersion of each set of these circular waves:

$$Z(R-k_n\Omega-i\nu_0)(R'+k_n\Omega'-i\nu_0')+p_0'^2R'(R-k_n\Omega-i\nu_0) + p_0^2R(R'+k_n\Omega'-i\nu_0')=0. \tag{84}$$

Thus there are eight circular wave modes.

For small drift velocities (84) agrees with Eq. (31) given previously.[2] When $\nu_0=\nu_0'=0$, its four roots may be studied conveniently by means of the graphical methods published jointly with Roberts.[8] But when the drift velocities are not too large the following procedure also allows us to find approximately the bands of frequency within which two of the roots are fully complex numbers and so lead to growing waves.

For simplicity we shall here limit ourselves to conditions in which

$$\bar{N}=\bar{N}' \quad \text{and} \quad \nu_0=\nu_0'=0. \tag{85}$$

Under these conditions when we set $V=V'=0$ Eq. (84) reduces to the following quadratic:

$$(c^2L^2-\omega^2)(\phi^2-\theta^2)+p^2\omega^2=0, \tag{86}$$

where

$$\left.\begin{aligned}\theta=\tfrac{1}{2}(\Omega+\Omega'), \quad \phi=\omega-\tfrac{1}{2}k_n(\Omega-\Omega'),\\ p^2=p_0^2+p_0'^2.\end{aligned}\right\} \tag{87}$$

Each of the roots L_0 of (86) may be taken as a first approximation to a root of (84) when $\phi^2\neq\theta^2$. Then a second approximation is derived by means of Newton's method. In this way we obtain two roots L_1, L_2 where

$$L_1, L_2=\alpha\pm i\beta, \tag{88}$$

$$\left.\begin{aligned}\alpha&=\frac{k_n}{2c^2}\left[\frac{-p_0^2\Omega V}{(\omega-k_n\Omega)^2}+\frac{p_0'^2\Omega' V'}{(\omega+k_n\Omega')^2}\right],\\ \beta&=\frac{\omega}{c}\left(\frac{p^2}{\phi^2-\theta^2}-1\right)^{\frac{1}{2}}\\ &\text{and}\\ &\omega\neq k_n\Omega \quad \text{or} \quad -k_n\Omega'.\end{aligned}\right\} \tag{89}$$

The roots L_1, L_2 are complex when

$$|\theta|<|\phi|<(\theta^2+p^2)^{\frac{1}{2}}. \tag{90}$$

With ω taken always positive we shall now adopt the following labels for the two sets of waves:

$$\left.\begin{aligned}E_1 \text{ waves when } k_n(\Omega-\Omega')>0;\\ E_2 \text{ waves when } k_n(\Omega-\Omega')<0.\end{aligned}\right\} \tag{91}$$

Then from (90) and (87) it follows that L_1, L_2 are conjugate complex roots of (84) when the frequency lies within the bands indicated under (92):

$$\left.\begin{aligned}\text{With } E_1 \text{ waves when } |\Omega|<\omega<\omega_1, \text{ (band } B_1);\\ \text{With } E_2 \text{ waves when } |\Omega'|<\omega<w_2, \text{ (band } B_2);\end{aligned}\right\} \tag{92}$$

where

$$\omega_1, \omega_2=\tfrac{1}{2}[\{(\Omega+\Omega')^2+4(p_0^2+p_0'^2)\}^{\frac{1}{2}}\pm|\Omega-\Omega'|]. \tag{93}$$

We may therefore conclude that circular electromagnetic noise-waves of the types E_1 and E_2 may arise within the respective frequency bands B_1 and B_2.[9]

On comparing this result with the corresponding one in the earlier discussion[3] of circular waves, which neglected the motions of the positive ions, we find that it differs only in setting the lower edge of the band B_2 at $|\Omega'|$ instead of zero.

As a concrete example we consider a discharge in helium with $p_0/2\pi=100$ Mc/sec and $\Omega/2\pi=600$ Mc/sec. Then the band B_1 lies between 600 and 616.2 Mc/sec and the band B_2 lies between 16.3 Mc/sec and 70 kc/sec.

VII. WAVES IN INTERPENETRATING PARALLEL DOUBLE STREAMS OF ELECTRONS

We shall here study the situation considered in Sec. VI when the stream of positive ions is replaced by a second stream of electrons and collisions are neglected.

The drift velocities are now independent of each other. Also there is a net mean space charge which prevents these velocities from being strict constants, but for certain applications we may in a first approximation take them as such or alternatively assume that this space charge is neutralized by inert heavy positive ions.

On applying the rules of transformation, given at the end of Sec. III, to Eqs. (70) to (84) and using primed symbols for quantities relating to the second stream of electrons, we obtain the following results.

There are longitudinal waves, with the electron velocities specified by

$$Xu_1/\beta^2R=p_0'^2u_1'/\beta'^2R', \tag{94}$$

$$X'u_1'/\beta'^2R'=p_0^2u_1/\beta^2R, \tag{95}$$

where

$$\left.\begin{aligned}p_0^2=4\pi\bar{N}e^2/m_0, \quad p_0'^2=4\pi\bar{N}'e^2/m_0,\\ X=\beta^2R^2-p_0^2, \quad X'=\beta'^2R'^2-p_0'^2,\end{aligned}\right\} \tag{96}$$

and with the equation of dispersion,

$$R^2R'^2-\beta'^{-2}p_0'^2R^2-\beta^{-2}p_0^2R'^2=0. \tag{97}$$

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[8] V. A. Bailey and J. A. Roberts, Australian J. Sci. Res. **A2**, 307 (1949).

[9] It is noteworthy that B_1 and B_2 have the same widths.

For low velocities (97) approximates to the equations of dispersion given by Pierce[10] and Haeff.[11]

Equation (97) is formally equivalent to Eq. (137) which is discussed below in Sec. VIII. It follows from that discussion that (97) has a pair of complex roots when, and only when, (142) is true. This supplies a general criterion for amplification of longitudinal waves in a double stream of electrons. It appears to be much simpler than the criterion published by Nergaard.[12]

Besides the longitudinal waves there are also circularly polarized transverse waves with the electron velocities specified by

$$\left.\begin{aligned} Yu_2-iZ\beta\Omega u_3&=-p_0'^2\beta Ru_2',\\ Yu_3+iZ\beta\Omega u_2&=-p_0'^2\beta Ru_3',\\ Y'u_2'-iZ\beta'\Omega'u_3'&=-p_0^2\beta'R'u_2,\\ Y'u_3'+iZ\beta'\Omega'u_2'&=-p_0^2\beta'R'u_3,\end{aligned}\right\}\quad(98)$$

where

$$Y=\beta R(Z+p_0^2),\quad Y'=\beta'R'(Z+p_0'^2),\tag{99}$$

$$\beta\Omega=\beta'\Omega'=-eH/m_0c.\tag{100}$$

These waves divide into two sets with their equations of dispersion given by

$$Z(R-k_n\Omega)(R'-k_n\Omega')+p_0'^2R'(R-k_n\Omega)+p_0^2R(R'-k_n\Omega')=0,\tag{101}$$

where $n=1$, 2 and $k_n=(-1)^{n-1}$.

For small drift velocities (101) formally approximates to Eq. (31) given previously[2] when the collision frequencies are negligible.

A complete discussion of the roots of the quartic (101) will not be given here. It will suffice to discuss two of these roots, on the lines given near the end of Sec. VI, when the drift velocities V, V' are small enough to make

$$\Omega'\doteqdot\Omega\doteqdot\Omega_0=-eH/m_0c.\tag{102}$$

For $V=V'=0$ Eq. (101) reduces to the quadratic

$$(L^2c^2-\omega^2)(\omega-k_n\Omega_0)+\omega(p_0^2+p_0'^2)=0$$

which has the two roots L_{01}, L_{02}.

On taking L_{01} and L_{02} as first approximations to two of the roots L_1, L_2 of (101) and using Newton's method we obtain the following second approximations when $\omega\neq k_n\Omega_0$:

$$L_1,\,L_2=\alpha\pm i\beta,\tag{103}$$

where

$$\left.\begin{aligned}\alpha&=-\frac{k_n\Omega_0(p_0^2V+p_0'^2V')}{2c^2(\omega-k_n\Omega_0)^2},\\ \beta&=\frac{\omega}{c}\left\{\frac{p_0^2+p_0'^2}{\omega(\omega-k_n\Omega_0)}-1\right\}^{\frac{1}{2}}.\end{aligned}\right\}\quad(104)$$

The roots L_1, L_2 are complex when

$$0<\omega^2-k_n\Omega_0\omega<p_0^2+p_0'^2.$$

We may now always take ω as positive and label the two sets of waves as follows:

$$\left.\begin{aligned}&E_1\text{ waves when }k_n\Omega_0>0;\\ &E_2\text{ waves when }k_n\Omega_0<0.\end{aligned}\right\}\quad(105)$$

It then follows that circular electromagnetic waves can be amplified when the frequency lies within the bands indicated below:

$$\left.\begin{aligned}&\text{With }E_1\text{ waves when }|\Omega_0|<\omega<\omega_1,\quad(\text{band }B_1);\\ &\text{With }E_2\text{ waves when }\quad 0<\omega<\omega_2,\quad(\text{band }B_2);\end{aligned}\right\}\quad(106)$$

where

$$\omega_1,\,\omega_2=\tfrac{1}{2}([\Omega_0^2+4p_0^2+4p_0'^2]^{\frac{1}{2}}\pm|\Omega_0|).\tag{107}$$

We shall now show that for small enough drift velocities and frequencies not near the electron gyro-frequency, the following approximate equivalence holds true between the present circular waves and those considered under Sec. VI:

The growing waves in a double stream of electrons which correspond to a given magnetic field, a given total charge-density Q and a given total electron current-density J are approximately like the growing circular waves in a single stream of electrons with the same magnetic field, charge density Q and current density J when associated with an equal number of infinitely heavy positive ions.

For the formulas (104) can be expressed in the forms,

$$\left.\begin{aligned}\alpha&=-\frac{k_n\Omega_0 4\pi eJ}{2m_0c^2(\omega-k_n\Omega_0)^2},\\ \beta&=\frac{\omega}{c}\left\{\frac{4\pi eQ}{m_0\omega(\omega-k_n\Omega_0)}-1\right\}^{\frac{1}{2}}.\end{aligned}\right\}\quad(104.1)$$

Also with infinitely heavy ions the second term in the expression for α under (89) vanishes,

$$k_np_0^2\Omega V=k_n\Omega_0 4\pi eJ/m_0,$$

and in the expression for β we now have

$$p^2/(\phi^2-\theta^2)=4\pi eQ/m_0\omega(\omega-k_n\Omega_0).$$

Thus with infinitely heavy ions the formulas (89) also assume the forms (104.1).

Furthermore, the statements (91) and (92) now reduce to the statements (105) and (106), respectively, and lastly (93) now reduces to a result equivalent to (107).[13] This completes the proof of equivalence.

VIII. MAGNETIC FIELD ABSENT AND COLLISIONS NEGLIGIBLE

In this section we consider the drift and vibratory motions of both the electrons and the ions.

[10] J. R. Pierce, J. Appl. Phys. **19**, 231 (1948).
[11] A. V. Haeff, Phys. Rev. **74**, 1532 (1948).
[12] L. S. Nergaard, R. C. A. Review **9**, 585 (1948).

[13] For in (107) $p_0^2+p_0'^2$ is proportional to the total electron density.

Since $\mathbf{H}\delta=0$ and the effects of the collisions are neglected we here use Eqs. (18) and (20) with all the terms in ϕ_{ij}, ν_0 and ν_0' omitted. Also for convenience we now choose a frame of reference in which $\mathbf{V}'$ is parallel to $\mathbf{V}$.

For convenience we will introduce new symbols s, s', t, t', W, X, X' defined as follows:

$$\left.\begin{aligned} l_k u_k &= -\beta R s, \quad l_k u_k' = -\beta' R' s', \\ \bar{U}_j u_j' &= -t, \quad \bar{U}_j' u_j = -t'. \end{aligned}\right\} \quad (108)$$

$$W=\bar{U}_j\bar{U}_j'=-c^2\beta\beta'(1-VV'/c^2), \quad (109)$$

$$X=\beta^2R^2-p_0^2, \quad X'=\beta'^2R'^2-p_0'^2. \quad (110)$$

The physical meanings of s and s' are provided by Eqs. (3.2) and (4.2) which show that

$$s=n/\bar{N}, \quad s'=n'/\bar{N}'. \quad (111)$$

Then (18) becomes

$$PYu_i+(l_ic^2+\beta R\bar{U}_i)s = \sigma\{\beta Ru_i'+l_it-(l_iW-\beta R\bar{U}_i')s'\}. \quad (112)$$

On multiplying (112) by l_i, summing over i and dividing through by Z we obtain

$$t=Ws'-\sigma^{-1}c^2PXs. \quad (113)$$

By symmetry (or from (20)) we have

$$t'=Ws-\sigma c^2P'X's'. \quad (114)$$

Again on multiplying (112) by $\bar{U}_i'$ and summing over i we obtain

$$PYt'-(\beta RW+\beta'R'c^2)s = \sigma\{-\beta'R't+(\beta'R'W+\beta Rc^2)s'\}. \quad (115)$$

On substituting in (115) the expressions for t and t' under (113) and (114) we obtain

$$R(as+bs')=0 \quad (116)$$

where

$$\left.\begin{aligned} a&=P(WZ-c^2\beta R\beta'R'), \\ b&=-\sigma c^2P'(PZX'+\beta'^2R'^2). \end{aligned}\right\} \quad (117)$$

By symmetry we must also have

$$R'(a's'+b's)=0 \quad (118)$$

where

$$\begin{aligned} a'&=P'(WZ-c^2\beta R\beta'R'), \\ b'&=-\sigma^{-1}c^2P(P'ZX+\beta^2R^2). \end{aligned} \quad (119)$$

When

$$n\neq0, \quad n'\neq0 \quad (120)$$

then (116) and (118) yield two waves with dispersion equations,

$$R=0, \quad (121)$$

$$R'=0, \quad (122)$$

and a set of waves for which

$$as+bs'=0, \quad (123)$$

$$a's'+b's=0. \quad (124)$$

The last set have the following equation of dispersion:

$$aa'-bb'=0,$$

i.e., by (117) and (119)

$$XX'(PP'Z+P+P')+X+X' -W^2c^{-4}Z+2Wc^{-2}\beta R\beta'R'=0. \quad (125)$$

From (109) we have

$$W^2/c^4=1+\beta^2\beta'^2(V-V')^2/c^2$$

and so (125) reduces to

$$(XX'-p_0^2p_0'^2)(Z+p_0^2+p_0'^2)=p_0^2p_0'^2\beta^2\beta'^2Q, \quad (126)$$

where

$$Q=Z(V-V')^2/c^2-(R-R')^2+(V'R-VR')^2/c^2. \quad (127)$$

From (12) and (13) we obtain

$$\left.\begin{aligned} Z/c^2&=l_1^2+l_2^2+l_3^2+l_4^2, \\ R&=l_1V_1+l_2V_2+l_3V_3+l_4ic, \\ R'&=l_1V_1'+l_2V_2'+l_3V_3'+l_4ic. \end{aligned}\right\} \quad (128)$$

If θ is the angle between the direction of phase propagation and the common direction of the drift velocities and if

$$L^2=l_1^2+l_2^2+l_3^2, \quad (129)$$

then by (128)

$$\left.\begin{aligned} Z&=c^2L^2-\omega^2, \\ R&=-LV\cos\theta+\omega, \\ R'&=-LV'\cos\theta+\omega. \end{aligned}\right\} \quad (130)$$

On using (130) we find from (127) that

$$Q=L^2(V-V')^2\sin^2\theta.$$

Hence the equation of dispersion (126) reduces to the form,

$$(XX'-p_0^2p_0'^2)(\dot{Z}+p_0^2+p_0'^2) =p_0^2p_0'^2\beta^2\beta'^2(V_T-V_T')^2L^2, \quad (131)$$

where V_T, V_T' are the components of the drift velocities transverse to the direction of propagation. This is an irreducible sextic in L.

Another set of solutions of the simultaneous equations (116) and (118) is given by

$$s=0, \quad s'=0,$$

i.e.

$$n=0, \quad n'=0. \quad (132)$$

Here we have, by (113) and (114),

$$t=0, \quad t'=0,$$

and so by (112) and by symmetry we obtain

$$PYu_i=\sigma\beta Ru_i', \quad (133)$$

$$P'Y'u_i'=\sigma^{-1}\beta'R'u_i. \quad (134)$$

Hence the equation of dispersion for this set is

$$PYP'Y'-\beta R\beta' R'=0,$$

which is equivalent to another pair of equations like (121) and (122) and the equation,

$$Z+p_0^2+p_0'^2=0. \quad (135)$$

We thus find that there are twelve different wave modes; four corresponding to (121) and (122), two to (135), and six corresponding to (131). Of these only the last six include growing waves.

In the special case when propagation is along the direction of drift motions (131) reduces to the equations,

$$XX'-p_0^2p_0'^2=0, \quad (131.1)$$

$$Z+p_0^2+p_0'^2=0. \quad (131.2)$$

We shall now consider the growing waves more fully. These correspond to fully complex roots L of Eq. (131) and to solutions of Eqs. (123) and (124).

For propagation in the direction of drift motions the only fully complex roots are those which satisfy (131.1), i.e., (73). The corresponding waves are purely longitudinal and have no associated Poynting flux.

For propagation perpendicular to the drift motions we have from (130)

$$R=R'=\omega,$$

and then (131) is found to yield only real or pure imaginary roots, i.e., there are then no growing *waves.*

For waves propagated *obliquely* to the drift motions it will be shown that all the non-real roots of (131) are now *fully* complex and that the corresponding growing waves possess transverse magnetic vectors and therefore also Poynting fluxes.

For any specified numerical values of p_0, p_0', V, V' and θ the frequency-bands in which growing waves can occur can be determined immediately from the curve representing the equation (131) when the co-ordinates are taken as $x=\omega$, $y=L$. To draw this curve we may conveniently use the graphical method published previously[8] in which the given equation is expressed in the form,

$$f(\omega^2, \sigma)=0,$$

where $\sigma=L/\omega$, and values of ω^2 corresponding to selected real values of σ are determined. In the present instance this equation is a quadratic in ω^2 and so pairs of values of ω and L are easily determined.

For a general discussion it is, however, necessary to proceed by successive approximations such as the following:

When V_T and V_T' are small, first approximations to the roots of (131) are given by the roots L_1 of the quartic (131.1) and the roots L_2 of the quadratic (131.2).

When developed in full by means of (110) Eq. (131.1) becomes

$$R^2R'^2-\beta'^{-2}p_0'^2R^2-\beta^{-2}p_0^2R'^2=0 \quad (136)$$

i.e., by (130),

$$\frac{\beta^{-2}p_0^2}{(V_1L-\omega)^2}+\frac{\beta'^{-2}p_0'^2}{(V_1'L-\omega)^2}=1, \quad (137)$$

where

$$V_1=V\cos\theta, \quad V_1'=V'\cos\theta, \quad (138)$$

i.e., V_1, V_1' are the components of the drift velocities in the direction of propagation.

When we set

$$L=\tfrac{1}{2}\omega(V_1^{-1}-V_1'^{-1})z+\tfrac{1}{2}\omega(V_1^{-1}+V_1'^{-1}) \quad (139)$$

Eq. (137) assumes the form,

$$\frac{A^2}{(z-1)^2}+\frac{B^2}{(z+1)^2}=1, \quad (140)$$

where

$$\left.\begin{aligned} A&=|2p_0V_1'/\omega\beta(V_1-V_1')|,\\ B&=|2p_0'V_1/\omega\beta'(V_1'-V_1)|.\end{aligned}\right\} \quad (141)$$

Equation (140) is discussed in Appendix II. It is there shown that when A and B are positive numbers then (140) has two real and two complex roots or four real roots according as

$$A^{\frac{2}{3}}+B^{\frac{2}{3}}\gtrless 2^{\frac{2}{3}}.$$

We thus conclude that L has a pair of (conjugate) complex values when, and only when,

$$\omega^{\frac{2}{3}}<\left(\frac{\beta^{-1}p_0V_1'}{V_1-V_1'}\right)^{\frac{2}{3}}+\left(\frac{\beta'^{-1}p_0'V_1}{V_1-V_1'}\right)^{\frac{2}{3}}. \quad (142)$$

For $V_1=c/10$, $V_1'=-c/100$, $p_0'=p_0/10$, (142) yields $\omega<0.257p_0$.

This agrees well with the (ω, L) curve given for this case in Fig. 4 of the paper by Bailey and Roberts.[8]

As shown in Appendix II the two real roots z can be obtained by means of a simple geometrical procedure and the two complex roots can then easily be derived from the real ones. The complex roots L_1 are then given by (139).

The roots L_2 of Eq. (131.2) are complex when, and only when,

$$\omega<(p_0^2+p_0'^2)^{\frac{1}{2}} \quad (143)$$

and then they are given by

$$L_2=\pm ic^{-1}(p_0^2+p_0'^2-\omega^2)^{\frac{1}{2}}. \quad (144)$$

Taking L_1 and L_2 as first approximations to the roots of (131) we now proceed to determine the second approximations L'.

First we write (131) in the form,

$$f(L)\equiv F(L)G(L)-hL^2=0, \quad (145)$$

where

$$\left.\begin{aligned} F(L)&=R^2R'^2-\beta'^{-2}p_0'^2R^2-\beta^{-2}p_0^2R'^2,\\ G(L)&=L^2-L_2^2,\\ h&=p_0^2p_0'^2\beta^2\beta'^2c^{-2}(V-V')^2\sin^2\theta.\end{aligned}\right\} \quad (146)$$

Then

$$f(L_1)=-hL_1^2,$$
$$f(L_2)=-hL_2^2,$$
$$f'(L_1)=F'(L_1)G(L_1)-2hL_1,$$
$$f'(L_2)=F(L_2)G'(L_2)-2hL_2.$$

Therefore the second approximations are

$$\left.\begin{aligned} L_1'&=L_1\{1+hL_1/F'(L_1)(L_1^2-L_2^2)\},\\ L_2'&=L_2\{1+h/2F(L_2)\}.\end{aligned}\right\}\quad(147)$$

Since for oblique propagation $h\neq 0$ it follows that in the bands of frequencies specified by (142) and (143) L_1' and L_2', respectively, have two fully complex values.

The imaginary part of L_2' approximates to L_2 and the real part approximates to

$$-\tfrac{1}{2}hL_2\cdot \mathrm{Im}\{1/F(L_2)\},$$

which is of the second order of smallness in V and V'.

We thus see that wave noise can arise in the two (overlapping) bands through propagation obliquely to the drift motions, i.e., from small initial random perturbations in planes oblique to these motions.

We now proceed to determine the components f_{ij} of the field of these waves and the resulting Poynting flux.

From (112) we obtain

$$PY(l_ju_i-l_iu_j)+(l_j\bar{U}_i-l_i\bar{U}_j)\beta Rs =\sigma[\beta R(l_ju_i'-l_iu_j')+(l_j\bar{U}_i'-l_i\bar{U}_j')\beta Rs'],$$

i.e.,

$$(PZ+1)v_{ij}+V_{ij}s=\sigma v_{ij}'+\sigma V_{ij}'s',\quad(148)$$

where

$$\left.\begin{aligned} v_{ij}&=l_iu_j-l_ju_i,\\ v_{ij}'&=l_iu_j'-l_ju_i',\\ V_{ij}&=l_i\bar{U}_j-l_j\bar{U}_i,\\ V_{ij}'&=l_i\bar{U}_j'-l_j\bar{U}_i'.\end{aligned}\right\}\quad(149)$$

By symmetry we must also have

$$v_{ij}+V_{ij}s=\sigma(P'Z+1)v_{ij}'+\sigma V_{ij}'s'.\quad(150)$$

From (150) and (148) we obtain

$$Pv_{ij}=-\sigma P'v_{ij}'\quad(151)$$

and so

$$Gv_{ij}=-p_0^2V_{ij}s+\sigma p_0^2V_{ij}'s',\quad(152)$$
$$Gv_{ij}'=-p_0'^2V_{ij}'s'+\sigma^{-1}p_0'^2V_{ij}s,\quad(153)$$

where

$$G=Z+p_0^2+p_0'^2.\quad(154)$$

Also in the notation of (149) Eq. (15) now becomes [on using (108)]

$$-ikPZf_{ij}=v_{ij}+V_{ij}s-\sigma v_{ij}'-\sigma V_{ij}'s'.\quad(155)$$

On eliminating v_{ij} and v_{ij}' by means of (152) and (153) this yields

$$-ikPGf_{ij}=V_{ij}s-\sigma V_{ij}'s'.\quad(156)$$

Then eliminating s' by means of (123) yields

$$f_{ij}=A_{ij}\psi\quad(157)$$

where

$$A_{ij}=bV_{ij}+\sigma aV_{ij}',\quad(158)$$
$$\psi=si/kPbG.\quad(159)$$

We will now choose Ox in the direction of propagation and Oz perpendicular to the drift motions. Then

$$\left.\begin{aligned} &l_1=-L,\quad l_2=l_3=0,\quad l_4=\omega/ic,\\ &V_1'=qV_1,\quad V_2'=qV_2,\quad V_3=V_3'=0,\end{aligned}\right\}\quad(160)$$

where q is some constant,

$$\left.\begin{aligned} Z&=c^2L^2-\omega^2,\\ R&=\omega-V_1L,\\ R'&=\omega-V_1'L,\end{aligned}\right\}\quad(161)$$

$$V_{ij}=\begin{bmatrix} 0, & l_1\bar{U}_2, & 0, & l_1\bar{U}_4-l_4\bar{U}_1\\ -l_1\bar{U}_2, & 0, & 0, & -l_4\bar{U}_2\\ 0, & 0, & 0, & 0\\ -l_1\bar{U}_4+l_4\bar{U}_1, & l_4\bar{U}_2, & 0, & 0\end{bmatrix}$$

and V_{ij}' is expressed by a similar matrix.

Hence by (158)

$$A_{ii}=A_{i3}=A_{3j}=0,$$
$$A_{12}=-A_{21}=l_1(b\bar{U}_2+\sigma a\bar{U}_2'),$$
$$A_{14}=-A_{41}=l_1(b\bar{U}_4+\sigma a\bar{U}_4')-l_4(b\bar{U}_1+\sigma a\bar{U}_1'),$$
$$A_{24}=-A_{42}=-l_4(b\bar{U}_2+\sigma a\bar{U}_2').$$

It then follows from (157) that the components of the electromagnetic field are given by

$$\left.\begin{aligned} \mathbf{h}&=(0,0,A_{12})\psi,\\ \mathbf{e}&=(iA_{14},iA_{24},0)\psi.\end{aligned}\right\}\quad(162)$$

Thus the magnetic vector is transverse to the plane containing the directions of the propagation and drift motions and the electric vector lies in this plane. In general the electric vector has components along and transverse to the direction of propagation.

The mean Poynting vector $\langle\mathbf{P}\rangle$ also lies in the same plane and has the components,

$$\langle P_1\rangle=(c/8\pi)\,\mathrm{Re}(if_{24}f_{12}^*),\quad \langle P_2\rangle=(c/8\pi)\,\mathrm{Re}(-if_{14}f_{12}^*),$$

i.e.,

$$\langle P_1\rangle=(\omega/8\pi)\,\mathrm{Re}(L)\,|b\beta V+\sigma a\beta'V'|^2\sin^2\theta|\psi|^2,\quad(163)$$

$$\langle P_2\rangle=[(c^2/8\pi)|L|^2\,\mathrm{Re}\{(b\beta+\sigma a\beta')(b\beta V+\sigma a\beta'V')^*\}\sin\theta -(\omega/8\pi)\mathrm{Re}(L)|b\beta V+\sigma a\beta'V'|^2\sin\theta\cos\theta]|\psi|^2.\quad(164)$$

Since L is a root of the irreducible sextic (131) the quartic polynomials,

$$b\beta V+\sigma a\beta'V'\quad\text{and}\quad b\beta+\sigma a\beta',$$

cannot in general vanish. Hence for $\theta \neq 0$ both $\langle P_1 \rangle$ and $\langle P_2 \rangle$ have nonzero values.

We thus arrive at the conclusion that *growing Poynting fluxes* can arise from waves propagated obliquely to the drift motions.

These waves are plane polarized with the electric vector in the plane containing the directions of propagation and drift and with the magnetic vector perpendicular to this plane.

For a disturbance which arises initially at a plane (x= constant) fixed to a frame of reference which is at rest in the medium as a whole we find from (147) and (163) that the Poynting fluxes, of these oblique growing waves, along the direction of the electron drift component V_1 are always negative. But for a disturbance which arises initially at a plane fixed in a frame which moves against the electron drift at least as fast as the positive ions (i.e., a frame in which $V_1 > 0$, $V_1' \geqslant 0$) the Poynting flux for the growing L_1' wave-mode is positive. We may also describe such an initial disturbance as one which moves against both streams of charged particles.

We shall now derive the following interesting complement to the formula (142).

For assigned real values of the wave number $L/2\pi$ below a certain value the dispersion Eq. (137) yields a pair of conjugate complex values of the frequency ω. The physical interpretation of this result is that one of the corresponding waves grows in time.

By means of the substitution,

$$\omega = pz + q,$$

where

$$p = \tfrac{1}{2}L(V_1 - V_1'), \quad q = \tfrac{1}{2}L(V_1 + V_1'),$$

Eq. (137) reduces to (140) with

$$A = |2p_0/\beta L(V_1 - V_1')|, \quad B = |2p_0'/\beta' L(V_1 - V_1')|.$$

Hence ω has a pair of (conjugate) complex values when, and only when,

$$L^{\frac{2}{3}} < \left(\frac{\beta^{-1}p_0}{V_1 - V_1'}\right)^{\frac{2}{3}} + \left(\frac{\beta'^{-1}p_0'}{V_1 - V_1'}\right)^{\frac{2}{3}}. \tag{165}$$

For $V_1 = c/10$, $V_1' = -c/100$, $p_0' = p_0/10$, this yields $cL < 12.15 p_0$, in good agreement with the (ω, L) curve in Fig. 4 of the paper by Bailey and Roberts.[8]

For waves in interpenetrating double streams of electrons with no static magnetic field present the theory is formally similar to that developed in the present section with σ replaced by $-\sigma$. It follows that in such double streams growing Poynting fluxes can arise from waves propagated obliquely to the streams. For a disturbance which arises initially at a plane (x= constant) fixed to a frame of reference which moves against one stream at least as fast as the second stream moves, we find from (147) and (163) that for a growing L_1' wave mode the Poynting flux along the direction of the drift components V_1 and V_1' is positive. Also within a certain band wave numbers which grow in time can occur.

IX. SUMMARY AND GENERAL DISCUSSION

The results obtained may now be summarized as follows.

In Sec. II the general equations for the uniform medium and the equations for small perturbations are derived. By an appropriate choice of the system of co-ordinates these may be used to study plane or cylindrical perturbations.

Section III contains the general dispersion equation and other equations which govern plane perturbations. In general there are twelve wave modes.

Sections IV and V are concerned with the approximation in which the motions of the positive ions are negligible, i.e., for wave frequencies which are not too low. The resulting dispersion Eq. (4) agrees exactly with that previously obtained by means of an entirely different method.

In Sec. VI is discussed the special case of the theory of Sec. III in which the drift velocities of the electrons and positive ions and the static magnetic field are all parallel to the direction of propagation. It is found that the possible wave modes are four purely longitudinal and eight purely transverse and circularly polarized modes. Their dispersion and other equations agree substantially with those previously obtained by means of the nonrelativistic theory. The bands of frequency in which growing circularly polarized waves (e.g., circular noise-waves) can occur are determined approximately.

In Sec. VII the theory of Sec. III is used to study the analogous waves in interpenetrating parallel double streams of electrons with a parallel static magnetic field present. The results obtained are formally similar to those given under Sec. VI. It is also found that with small drift velocities and frequencies not near the electron gyro-frequency, the growing circularly polarized waves which correspond to a given total charge-density Q and a given total current-density J are approximately like the growing circular waves in a single stream of electrons with the same densities Q and J when associated with an equal number of infinitely heavy positive ions.

In Sec. VIII the theory of Sec. III is discussed in the special case in which there is no static magnetic field and the effects of collisions are negligible. It is found that there are twelve different wave modes. Of these only the six which correspond to the dispersion Eq. (131) include growing wave modes. For waves propagated *obliquely* to the drift motions there are two such growing modes. The corresponding frequency bands in which they occur are given approximately by (142) and (143). It is also found that these *growing* oblique wave-modes have associated Poynting fluxes. The components of these fluxes along the direction of the electron stream are negative when they arise initially from a disturbance at a plane (x= constant) at rest in the medium as a whole. But when the plane of this disturbance moves against the electron stream at least as fast as the positive ions, one of these fluxes is positive. Lastly it is shown that these growing wave modes are plane polarized with the electric vector in the plane containing the directions of propagation and drift. Similarly, in double streams of electrons growing Poynting fluxes can arise from waves propagated obliquely to the streams. When the initial disturbance occurs in a plane which moves against one stream at least as fast as the second stream moves, the flux in the direction of the drift components V_1 and V_1' is positive for the growing L_1' wave mode.

We may now consider briefly the source of energy of the growing waves. As was pointed out or implied in earlier publications[2,3] the momentum and energy in the growing waves are necessarily derived from the static current in the medium, i.e., from the momentum and energy corresponding to the drift motions of the electrons and ions. In order to see in more detail how the latter are in part transformed into momentum and energy of growing waves (of the charged particles and of the field) we may here use the well-known relativistic equations, derivable from the fundamental equations in Secs. I and II with $\nu_0=\nu_0'=0$, which express the principles of conservation of momentum and energy by the vanishing of the divergence of a tensor T_{ij}. This tensor is given by

$$T_{ij}=M_{ij}+E_{ij},$$

where M_{ij} is the sum of the momentum energy tensors of the two kinds of moving charged particles and E_{ij} is the momentum energy tensor of the electromagnetic field.

Each of these tensors consists of a part independent of the perturbations and a part depending on them. Hence we may set

$$T_{ij}=\bar{T}_{ij}+t_{ij},$$

where only t_{ij} depends on the perturbations.

On taking Ox along the direction of propagation and averaging over time the divergence equation yields

$$\partial(\bar{T}_{i1}+\bar{t}_{i1})/\partial x=0, \quad (i=1, 2, 3, 4). \qquad (166)$$

The first of Eqs. (166) is, in effect, the same as Eq. (72) in the publication on circular waves[3] when the collisions and gradients of electron partial pressure are neglected. It implies that a growing wave acts on the total momentum of the streams with a mean force which opposes this momentum. This provides a criterion for the direction in which such a wave does actually grow.

Consistently with this the fourth of Eqs. (166) shows that when a wave of a given frequency arises which grows in the direction of Ox, i.e., when the component along Ox of the total mean flux $-ic\bar{t}_{41}$ of wave-energy, of the electrons, ions and the field, increases in the direction Ox, then the component along Ox of the total flux $-ic\bar{T}_{41}$ of drift energy of the electrons and ions decreases in the direction Ox by the same amount per unit length.[14]

With very heavy ions their contributions to the momentum and energy exchanges may be neglected and we may then conclude that the wave will grow in the direction of the component along Ox of the drift motion of the electrons.

This last result entails a revision of that part of the recent discussion[3] of the growth of circularly polarized waves in a sunspot, which is concerned with E_1 waves. It is now clear that, like the E_2 waves, the E_1 waves also must grow in the direction of the electron drift motion.[15] But unlike the others the E_1 waves cannot escape into free space since their associated Poynting fluxes are directed inwards. As a result the Sec. 5 of that paper requires correcting by deleting all references to escaping theoretical E_1 waves and expunging their frequency bands from the Fig. 4.

The E_1 waves were there related to the extraordinary waves in the magneto-ionic theory. In order therefore to explain the cited report of Ryle and Vonberg that extraordinary solar waves can be occasionally observed it is necessary to study the growth and escape of waves which are propagated obliquely to the sunspot axis. A preliminary investigation shows that such extraordinary waves can arise from E_2 waves as a result of the change in their polarization in passing through a region in which $p_0(1-V_1)^{-1}$ is equal to ω, the wave frequency. It is hoped to publish this investigation of oblique propagation in due course.

We shall now briefly consider the application of the E.M.I. theory to the problem of the origin of cosmic noise and solar "isolated bursts" and "outbursts" of of noise, all of which published hitherto have been found to be randomly polarized.[16]

First it should be noted that no current method of observation can distinguish between noise which is a mixture of plane waves with their planes of polarization scattered completely at random and noise which is a mixture of elliptically polarized waves with their senses of rotation scattered completely at random.

The theory shows that elliptical waves can arise only from an ionized medium in which a magnetic field is present. Such a medium will be referred to as a magnetic source. So when elliptic (e.g., circular) waves are observed (such as solar noise from sunspots) they provide unimpeachable evidence that they come from or pass through a magnetic source. But randomly polarized waves cannot by themselves alone serve to distinguish

[14] Since $\bar{t}_{i1}$ is of the second order in the perturbations, it follows that $\bar{T}_{i1}$ and therefore $\bar{N}$, $\bar{U}$, etc., differ from constants only by quantities of the second order. Hence in Eqs. (1.0) to (9.1) of Sec. II, the quantities $\bar{N}$, $\bar{U}$, etc. may be taken as constants without introducing errors of the same order as the perturbations.

[15] As I have learned from Mr. J. A. Roberts, this fact was first pointed out by Dr. Twiss in a colloquium address given by him in August last at the University of Cambridge.

[16] The use by some authors of the term "unpolarized" for such noise is theoretically incorrect and very apt to be misleading in regard to the problem of the origin of this noise.

between magnetic and nonmagnetic sources of noise. These waves can be attributed to magnetic sources only through observations by means of highly directional aerial systems (e.g., radio interferometers) and observation of the Zeeman effect. It follows that the recorded observations of cosmic noise (and of some solar bursts and outbursts) do not by their random polarization alone indicate whether they come from magnetic or nonmagnetic sources.

As indicated by the results of Sec. VIII, which are summarized above, the relativistic theory of E.M.I. waves provides one explicit example and perhaps a second example of nonmagnetic sources of noise in which certain waves can grow and escape. These are, respectively:

(i) A stream of electrons moving relatively to the positive ions and an initial disturbance in a plane moving against this stream as fast as or faster than the ions;

(ii) Two streams of electrons and an initial disturbance in a plane moving against both streams. This example is as yet uncertain since the exchanges of momentum and energy between the growing waves and the streams have not yet been fully investigated. The situation is like that envisaged by Haeff[11] for solar noise but here the waves are propagated obliquely to the drifts.

From both sources the escaping waves would be plane polarized but if the streams in a source change their direction much, and at random within distances that cannot be resolved by extant receiving aerial systems, these waves would appear to be randomly polarized. In these nonmagnetic sources the electrons and ions need not possess appreciable temperatures. If they did have sufficient temperatures then, according to the earlier nonrelativistic theory additional modes of growing waves could arise; but whether these waves can escape requires further investigation.

In magnetic sources there can arise additional modes of growing waves of which at least one can escape. These are in general elliptically polarized. An example of the escaping mode is the E_2 wave referred to above. If the magnetic field in a source changes its direction much, and at random within distances that cannot be resolved by the receiving system these modes would be observed as randomly polarized waves.

It thus appears theoretically that magnetic sources are more effective than the others. This conclusion is amply confirmed by numerous laboratory experiments, in the University of Sydney and elsewhere, with discharge tubes or electron tubes subjected to magnetic fields,[17] and by the fact that strong solar noise is most frequently associated with sun-spot activity and usually comes from the neighborhood of sun spots (or from places where sunspots were observed one solar month earlier).

All these facts lend support to the hypothesis, suggested in the original publication[18] on E.M.I. waves, that a notable part of cosmic noise and strong solar noise commonly originates in magnetic sources. This is consistent with the view of Pawsey and his collaborators[19] that cosmic noise and strong solar noise may be due to similar processes, but it allows some room for nonmagnetic sources.

From this hypothesis it would follow that our Galaxy and the Great Nebula in Andromeda (M31)[20] probably contain a large number of strongly magnetized ionized regions. This is consistent with the known existence of magnetic stars and with the view, now becoming current, that turbulence in ionized media can set up local magnetic fields. Visible and invisible magnetic stars could be magnetic sources of cosmic noise, but it does not follow that known magnetic stars are sufficiently strong sources for observation of their noise by current methods. Also it is an interesting question whether most "point sources" (radio stars) are magnetic and other sources are nonmagnetic.

All these considerations suggest that it would be worth while for astronomers to examine carefully the neighborhood of all visible "point-sources" of cosmic noise for traces of the Zeeman effect and all point-sources (including solar ones) for traces of an excess of elliptically polarized noise. Even transient traces would be worth seeking.[21]

I wish to acknowledge with thanks the help with this work which I have received from Messrs. R. F. Mullaly, W. Moriarty, and J. W. Dungey through their checking of the calculations and criticisms. Discussions, by correspondence, with Mr. J. A. Roberts and discussions with Mr. Dungey have also been valuable in regard to the question of escape of E_1 and other waves.

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APPENDIX I

The relativistic equation of motion for a mean charged particle moving in a gas does not appear to have been published before. It will therefore be derived here.

If $\mathbf{V}$ is the velocity of the particle of mass m_0, $\mathbf{V}_g$ the velocity of the gas and $m_0\mathbf{L}$ the mean rate of loss of momentum of the particle through its collisions with gas molecules then from Maxwell's equations of transfer of momentum we have

$$\mathbf{L}=\nu(\mathbf{V}-\mathbf{V}_g), \tag{1a}$$

where ν is a positive constant which depends on the nature of the collisions and on the number-density of the molecules. On account of its dimensions ν may be called the collision frequency.

[17] V. A. Bailey and K. Landecker, Nature **166**, 259 (1950); P. C. Thonemann and R. B. King, Nature **158**, 414 (1946); J. D. Cobine and C. J. Gallagher, Phys. Rev. **70**, 112 (1946); W. T. Ruthberg, Phys. Rev. **70**, 112 (1946); J. Denisse and J. L. Steinberg, Compt. rend. **224**, 646 (1947); E. Aström, Trans. Roy. Inst. Techn. Stockholm **22**, 70 (1948); D. Bohm, Division 1 in the National Nuclear Energy Series (McGraw-Hill Book Company Inc., New York, 1949), Chapter 1, Vol. 5.

[18] V. A. Bailey, Nature **161**, 599 (1948).

[19] Pawsey, Payne-Scott, and McCready, Nature **157**, 158 (1946).

[20] R. H. Brown and C. Hazard, Nature **166**, 901 (1950).

[21] Since this paragraph was written, Dr. Pawsey has called my attention to the recent observation by Payne-Scott and Little of transient traces of circularly polarized solar noise immediately following "outbursts."

In the present theory the frequency of collisions between the particle and oppositely charged particles is taken as negligible by comparison with ν. Also all the velocity-distribution functions will be taken as constant.

The four-vectors of velocity of the particle and gas, in any inertial frame of reference S, will be denoted by U_i and U_{gi}, respectively.

We shall now search for a four-vector L_i which is such that the relativistic equation of motion of the particle is

$$U_j\partial U_i/\partial x_j+L_i=kF_{ij}U_j \tag{2a}$$

and also such that the nonrelativistic Eq. (1a) approximately specifies its space components L_1, L_2, L_3.

In (2a) F_{ij} is a skew-symmetrical tensor and for U_i we have

$$U_iU_i=-c^2. \tag{3a}$$

On multiplying (2a) by U_i and summing over i we obtain

$$U_iL_i=0. \tag{4a}$$

Also we necessarily require

$$L_iL_i=\phi^2, \tag{5a}$$

where ϕ is an invariant.

Without loss of generality we may now take the x_1 axis of our inertial frame of reference S parallel to the relative velocity $\mathbf{V}-\mathbf{V}_g$ of the particle and the gas. This will help us to determine ϕ in a simple manner.

The relations (3a), (4a) and (5a) are conveniently illustrated by the Fig. 1 in which the four-vectors U_i ,U_{gi} and L_i are represented by vectors OP, OG and OR in the x_1x_4 plane of the frame S.

The vectors U_i and U_{gi} have the same length ic and are inclined to the x_4 axis at the angles θ and θ_g, respectively. On account of the relations (4a) and (5a) the vector L_i is perpendicular to U_i and has the length ϕ. Since also $\mathbf{L}$ has the sign of $\mathbf{V}-\mathbf{V}_g$ we must have the vector L_i inclined at the angle $\theta+\pi/2$ to the x_4 axis.

The angle α between the vectors U_i and U_{gi} is clearly also an invariant with respect to any frame S. To determine if we take the frame S_g which is at rest in the gas. Then if V_0 is the velocity of the particle in this frame we must have

$$\tan\alpha=V_0/ic,\quad \cos\alpha=\beta_0,\quad \sin\alpha=\beta_0V_0/ic,$$

where

$$\beta_0=(1-V_0^2/c^2)^{-\frac{1}{2}}. \tag{6a}$$

If the line PQ is drawn perpendicular to OP then by (6a)

$$PQ=V_0,\quad OQ=\beta_0^{-1}ic. \tag{7a}$$

In the frame S_p which moves with the particle we have $V_g=-V_0$ and so the relation (1a) yields

$$L_1=\nu_p(-V_g)=\nu_pV_0.$$

Also in this frame L_i lies along the x_1 axis, i.e., $L_1=\phi$. Hence

$$\phi=\nu_pV_0. \tag{8a}$$

The line QP is, by (7a) and (8a), of length ϕ/ν_p and is parallel to L_i. Therefore QP represents the vector L_i/ν_p. Similarly OQ represents the vector $\beta_0^{-1}U_{gi}$.

We therefore have the vectorial relation

$$L_i=\nu_p(U_i-\beta_0^{-1}U_{gi}). \tag{9a}$$

Also since the number of collisions νdt made by the particle in the time dt must be an invariant and dt/β is also invariant we have

$$\nu_p=\beta\nu=\beta_0\nu_0,$$

where

$$\beta=(1-V^2/c^2)^{-\frac{1}{2}} \tag{10a}$$

and ν_0 is the collision frequency in the frame S_g.

Then (9a) may be written in the symmetrical form,

$$L_i=\nu_pU_i-\nu_0U_{gi}. \tag{11a}$$

This or (9a) may now be taken as the exact relativistic equivalent to the relation (1a). It will be noted that in the form (11a) the collision frequency ν associated with each velocity four-vector is that for the proper frame of the corresponding particle.

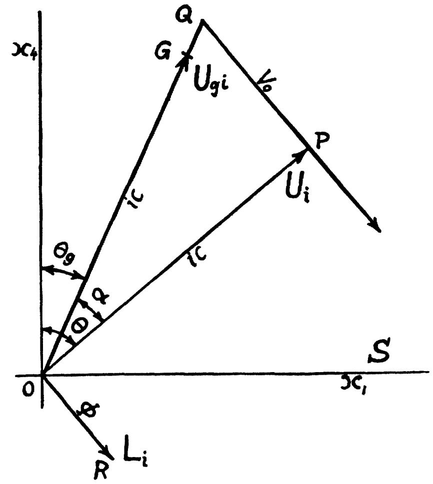

FIG. 1. Representation of the four-vectors U_i, U_{gi}, and L_i.

If we now rotate the frame S about its x_4 axis through an arbitrary angle, the relations (9a) and (11a) remain unaltered and U_i, U_{gi} have the general forms,

$$U_i=(U_1,\ U_2,\ U_3,\ U_4),\quad U_{gi}=(U_{g1},\ U_{g2},\ U_{g3},\ U_{g4}) \tag{12a}$$

with U_4 and U_{g4} the same as before rotation.

Thus

$$V_1^2+V_2^2+V_3^2=V^2,\quad V_{g1}^2+V_{g2}^2+V_{g3}^2=V_g^2.$$

Also, since α is unaltered, we have

$$OP\cdot OQ\cos\alpha=U_i(\beta_0^{-1}U_{gi}),$$

with summation over i, and so by (7a) and (6a)

$$U_iU_{gi}=-\beta_0c^2. \tag{13a}$$

By means of (13a) it is easily verified that the expression (9a) or (11a) for L_i satisfies the relations (4a) and (5a) as is required. It also clearly yields a space vector $\mathbf{L}(L_1, L_2, L_3)$ which agrees with that specified approximately by (1a).

On account of (10a) and (13a) Eq. (11a) may be written as

$$L_i=\nu_0W_i \tag{14a}$$

where

$$W_i=-c^{-2}U_{gj}U_jU_i-U_{gi}. \tag{15a}$$

In the frame S_g at rest in the gas we have therefore

$$W_i=(-i/c)(U_4U_1,\ U_4U_2,\ U_4U_3,\ c^2+U_4^2),$$

which is identical with the expression for W_i in Eq. (9) of Sec. II.

Equation (2a) then becomes the dynamical Eq. (5) adopted in Sec. III with S_g as the frame of reference.

In a general frame S the fundamental Eqs. (1) to (8) remain valid provided that in (5) and (6) W_i is expressed by (15a) given above. Then (9.1) is replaced by $w_i=-c^2U_{gj}(\bar{U}_iu_j+\bar{U}_ju_i)$, etc. The derived equations for the perturbations are then modified only through the terms involving ν_0 and ν_0'.

Hence, when these terms are neglected the perturbation equations obtained in Secs. II and III are valid in any frame of reference.

In the present discussion it is assumed for simplicity that the collision frequencies ν_0, ν_0' are independent of the drift velocities. This is correct only when the mean random motions of the particles are large compared with the drift motions. But with all applications in which the effects of the collisions are small this assumption will not lead to any serious error. When the mean random motions are not large compared with the drift motions the exact effects of collisions can be calculated only by means of more complicated methods which will not be considered here.

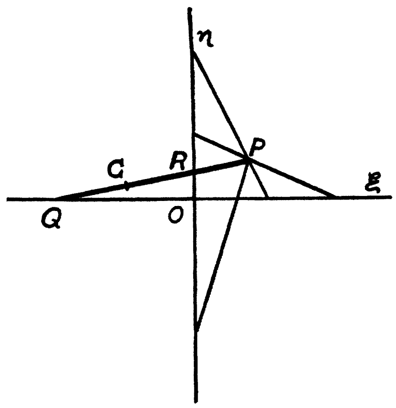

FIG. 2. Graph of Eq. (18a).

APPENDIX II

The quartic equation,

$$A^2/(z-1)^2+B^2/(z+1)^2=1, \tag{16a}$$

with A and B given positive numbers, is satisfied by the substitution,

$$z-1=A\sec\theta,\quad z+1=B\operatorname{cosec}\theta, \tag{17a}$$

where θ is a root of

$$A\sec\theta-B\operatorname{cosec}\theta+2=0. \tag{18a}$$

If A and B are taken as the coordinates of a point P in the (ξ, η) plane shown in Fig. 2, then for any given value of θ Eq. (18a) represents a straight line QRP inclined at the angle θ to the ξ-axis and with the following intercepts on the axes:

$$OQ=-2\cos\theta,\quad OR=2\sin\theta.$$

The intercept QR, of this line by the axes, is therefore of constant length 2.

If C is the midpoint of QR then, by (17a), the corresponding value of the root z is given by

$$z=CP. \tag{19a}$$

Also it is easily proved that the envelope of the line (18a) is the four-cusped hypocycloid[22]

$$\xi^{\frac{2}{3}}+\eta^{\frac{2}{3}}=2^{\frac{2}{3}}, \tag{20a}$$

which is a closed curve symmetrical about 0.

A real tangent to this curve from the point $P(A, B)$ necessarily has its angle of inclination θ equal to one of the real roots of (18a) and its intercept by the axes equal to 2.

Since there are two or four such real tangents according as $P(A, B)$ is outside or inside the curve, it follows that Eq. (16a) has two real and two complex roots or four real roots according as

$$A^{\frac{2}{3}}+B^{\frac{2}{3}}\gtrless 2^{\frac{2}{3}}. \tag{21a}$$

The real roots can be determined easily by drawing through the given point $P(A, B)$ all the lines which have their intercepts QR by the axes equal to 2.

When two complex roots z_3, z_4 occur they can be determined as follows by means of the known real roots z_1, z_2.

On expressing (16a) as a polynomial equation we see that

$$z_1+z_2+z_3+z_4=0,\quad z_1z_2z_3z_4=1-A^2-B^2.$$

It then follows that

$$z_3,\, z_4=x\pm iy, \tag{22a}$$

where

$$x=-\tfrac{1}{2}(z_1+z_2),\quad y=[(1-A^2-B^2)z_1^{-1}z_2^{-1}-x^2]^{\frac{1}{2}}. \tag{23a}$$

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[22] See for example J. Edwards, *Differential Calculus* (The Macmillan Company, New York, 1912), p. 301.