PHYSICAL REVIEW VOLUME 83, NUMBER 2 JULY 15, 1951

Radio Wave Generation by Multistream Charge Interaction

J. FEINSTEIN AND H. K. SEN*
Central Radio Propagation Laboratory, National Bureau of Standards, Washington, D. C.

(Received January 8, 1951)

The excitation of plasma oscillations in traveling beams is reviewed; analytic and graphical methods are employed to ascertain the ranges of parameters over which wave growth may exist and to determine contours of constant amplification factor. The effect of thermal velocities is taken into account by utilizing approximations to a maxwellian distribution. It is found that the possibility of growth exists even for beam injection velocities much smaller than the mean thermal motion in the region; narrow band widths are encountered under these circumstances. Possible applications to solar phenomena are indicated.

An investigation of the effect of the initiating disturbance upon the frequency band generated leads to the conclusion that where the possibility of growth in time exists, a spatial description of initial conditions gives rise to a broader band than does a temporal prescription. This is illustrated in specific cases.

Previously suggested mechanisms for the conversion of these longitudinal oscillations into transverse electromagnetic energy are considered quantitatively and found to be inadequate. A more promising mechanism is suggested.

I. INTRODUCTION

THE possibility of accounting for much of the abnormal radio noise received from the sun on the basis of conversion of the kinetic energy of ejected prominence material into electromagnetic radiation has led several workers[1] to consider the excitation of plasma oscillations by moving charged particles. The results obtained thus far have been mainly for the case of one or two uniform homogeneous beams, and even here some errors appear to have been committed;[1a] while these results serve to indicate the nature of the processes, their application to the solar atmosphere, in which thermal velocities are of the order of most injected beam velocities, or greater than them, has been questioned.[2] It is the purpose of this paper to re-examine and extend the two-beam case and to discuss the modifications introduced by appropriate models of thermal motion. The mechanisms available for the conversion of these longitudinal type oscillations into radiation fields are also investigated.

For the sake of completeness, we summarize below a simple derivation of the three-dimensional dispersion equation for a composite discrete and continuous electron velocity spectrum. We shall assume the absence of any static fields, and neglect the motion of the positive ions.

II. THE DISPERSION EQUATION

Let there be r interpenetrating streams of electrons of densities $N_{01}, \cdots, N_{0q}, \cdots, N_{0r}$ per unit volume, and traveling with uniform velocities $\mathbf{U}_{01}, \cdots, \mathbf{U}_{0q}, \cdots, \mathbf{U}_{0r}$; let there also be N_0 electrons with a continuous velocity distribution dN_0 in the range $d\mathbf{U}_0$ at $\mathbf{U}_0$. The first-order perturbation of the steady-state quantities are denoted by lower case letters.

The oscillating current density is, then, for a discrete stream

$$\mathbf{i}_q = e(n_q\mathbf{U}_{0q} + N_{0q}\mathbf{u}_q) \quad (1)$$

and for the continuous stream

$$d\mathbf{i} = e(\mathbf{u}dN_0 + \mathbf{U}_0 dn) \quad (1a)$$

to first order, where e is the electronic charge. We derive the fields from scalar and vector potentials through the familiar relations:

$$\mathbf{E} = -\nabla\psi - \partial\mathbf{A}/\partial t; \quad \mathbf{B} = \nabla\times\mathbf{A}, \quad (2)$$

where the potentials satisfy the equations:

$$\begin{aligned}\nabla^2\psi - (1/c^2)\partial^2\psi/\partial t^2 &= -\rho/\epsilon_0;\\ \nabla^2\mathbf{A} - (1/c^2)\partial^2\mathbf{A}/\partial t^2 &= -\mu_0\mathbf{i}.\end{aligned} \quad (3)$$

We seek solutions in which the space-time variation of the first-order perturbations is of the form $\exp(-\mathbf{\Gamma}\cdot\mathbf{r} + j\omega t)$. Then the equation of motion,[3]

$$(\partial\mu_q/\partial t) + (\mathbf{U}_{0q}\cdot\nabla)\mathbf{u}_q + \nu\mathbf{u}_q = (e/m)[\mathbf{E} + \mathbf{v}\times\mathbf{B}], \quad (4)$$

* Harvard College Observatory, on leave during summer of 1950 to work at the Central Radio Propagation Laboratory, National Bureau of Standards.

[1] J. R. Pierce, J. Appl. Phys. **19**, 231 (1948); Proc. Inst. Radio Engrs. **37**, 980 (1949). A. V. Haeff, Proc. Inst. Radio Engrs. **37**, 4 (1949); Phys. Rev. **75**, 1546 (1949). V. A. Bailey, J. Roy. Soc. N.S.W. **82**, 107 (1948); Australian J. Sci. Res. **1**, 351 (1948); Phys. Rev. **75**, 1104 (1949); **78**, 428 (1950). D. Bohm and E. P. Gross, Phys. Rev. **75**, 1851 (1949). J. A. Roberts, Phys. Rev. **76**, 340 (1949).

[1a] See Sec. III.

[2] M. Ryle, Proc. Phys. Soc. (London) **A62**, 483 (1949).

[3] The third term on the left-hand side of Eq. (4) refers to the loss of directed momentum as a result of collisions with heavy particles. Electron-electron collisions are already taken into account on a hydrodynamic basis by the scalar potential. If $\mathbf{U}_q$ is considered the average velocity, then ν may be shown to be exactly the number of collisions made by an electron per second, provided the momentum after a collision bears no relation to that prior to the collision. [E. V. Appleton and F. W. Chapman, Proc. Phys. Soc. (London) **44**, 246 (1932).] Since the density fluctuations arise from corresponding velocity variations, it suffices to take account of collisions at this point alone, at least when the collision time is much longer than the oscillation period. This formulation may be employed in general only for intrabeam collisions where the zero component of velocity is the same for both particles involved, since only the oscillatory component of momentum is presumed destroyed at each collision; for interbeam collisions, an electron will generally be thrown completely out of the region of velocity space occupied by its former beam, so that the zeroth-order quantities would be affected. The perturbation methods of the small signal

assumes the form,

$$(\nu+j\omega-\mathbf{\Gamma}\cdot\mathbf{U}_{0q})\mathbf{u}_q=(e/m)[\mathbf{\Gamma}\psi-j\omega\mathbf{A}+\mathbf{U}_{0q}\times(\mathbf{\Gamma}\times\mathbf{A})]. \quad (5)$$

Finally, we must satisfy the continuity condition

$$(\partial\rho/\partial t)+\nabla\cdot\mathbf{i}=0, \quad (6)$$

which becomes with the aid of Eq. (1)

$$(j\omega-\mathbf{\Gamma}\cdot\mathbf{U}_{0q})n_q=N_{0q}\mathbf{\Gamma}\cdot\mathbf{u}_q \quad (6a)$$

for a discrete beam, and

$$(j\omega-\mathbf{\Gamma}\cdot\mathbf{U}_0)dn=dN_0\mathbf{\Gamma}\cdot\mathbf{u} \quad (6b)$$

for the continuous stream. If one makes use of Eqs. (1), (3), and (6), Eq. (5) takes the form,

$$(\mathbf{\Gamma}\cdot\mathbf{U}_{0q}-j\omega-\nu)\mathbf{u}_q+\frac{1}{|\mathbf{\Gamma}|^2+\omega^2/c^2}\left\{\mathbf{\Gamma}\mathbf{\Gamma}\cdot\left[\sum_{s=1}^{r}\frac{\omega_s^2\mathbf{u}_s}{j\omega-\mathbf{\Gamma}\cdot\mathbf{U}_{0s}}+\int\frac{d(\omega_0^2)\mathbf{u}}{j\omega-\mathbf{\Gamma}\cdot\mathbf{U}_0}\right]-\frac{1}{c^2}(j\omega+\mathbf{U}_{0q}\cdot\mathbf{\Gamma}-\mathbf{\Gamma}\mathbf{U}_{0q}\cdot)\times\left[\sum_{s=1}^{r}\omega_s^2\left(1+\frac{\mathbf{U}_{0s}\mathbf{\Gamma}\cdot}{j\omega-\mathbf{\Gamma}\cdot\mathbf{U}_{0s}}\right)\mathbf{u}_s+\int d(\omega_0^2)\left(1+\frac{\mathbf{U}_0\mathbf{\Gamma}\cdot}{j\omega-\mathbf{\Gamma}\cdot\mathbf{U}_{0s}}\right)\mathbf{u}\right]\right\}=0, \quad (7)$$

($q=1\cdots r$, as well as all points in the continuous distribution), where $\omega_s^2=(N_{0s}e^2/\epsilon_0 m)$, $\omega_0^2=(N_0e^2/\epsilon_0 m)$ and the scalar products are to be taken with the first vector following the dot.

In general, the dispersion relation for the system of r discrete beams may be obtained only by employing the determinantal condition for a nontrivial solution of the expanded $3r$ simultaneous homogeneous equations equivalent to Eq. (7). Introduction of the continuous distribution would lead to an infinite determinant.

On physical grounds it is evident that stream-wave interaction leading to the possibility of wave growth requires a near match between beam and wave velocities. Since the particle speeds contemplated are much less than the speed of light, it is assumed that $\Gamma\simeq\omega/U_{0q}\gg\omega/c$. With this assumption, the magnetic force terms may be neglected; taking the dot product of the remainder of Eq. (7) with $\mathbf{\Gamma}$, summing over q and integrating over the continuous distribution, one obtains for the dispersion equation,

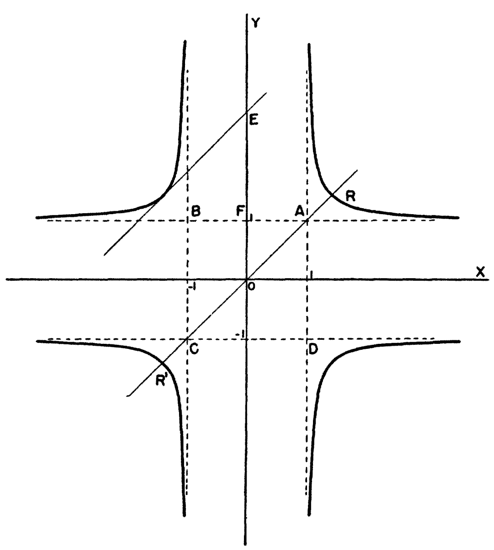

FIG. 1. Graphical determination of real propagation constants, and of regions in which wave amplification can occur, for the two-beam case.

$$\sum_1^r\frac{\omega_q^2}{(\omega+j\mathbf{\Gamma}\cdot\mathbf{U}_{0q})(\omega+j\mathbf{\Gamma}\cdot\mathbf{U}_{0q}-j\nu)}+\int\frac{d(\omega_0^2)}{(\omega+j\mathbf{\Gamma}\cdot\mathbf{U}_0)(\omega+j\mathbf{\Gamma}\cdot\mathbf{U}_0-j\nu)}=1, \quad (8)$$

which differs slightly in form from the Eq. (11) derived by Bohm and Gross[1] probably because of the different methods of averaging employed in the two cases. The physical result, however, can be shown to be the same, *viz.*, that electronic collisions damp the wave by a factor $1/e$ in the time $1/\nu$. These collisions, together with second-order saturation effects, set a limit to the amplitude of any growing wave which may arise. In addition, the maintenance of the oscillations in the presence of collisional damping is dependent upon a supply of directed, or mechanical-type, energy. This requires a departure from thermal equilibrium in the velocity distribution of the plasmas.

III. THE TWO-STREAM CASE

We consider first the interaction of two discrete electron beams, moving with velocities U_{01} and U_{02} along the Z axis. We will assume the initial disturbance to occur in this same direction; if collisions are neglected, Eq. (8) reduces to

$$[\omega_1^2/(\omega+j\Gamma U_{01})^2]+\omega_2^2/(\omega+j\Gamma U_{02})^2=1, \quad (9)$$

where Γ is now the component of the propagation vector in the z-direction. This process has been considered by Haeff;[1] but his results are vitiated by the omission of certain terms in his Eq. (19), which are of the same order as the terms retained.[4]

Before discussing the general solution of Eq. (9), we will note several special cases of interest. For an electron stream of velocity v_{01}, and plasma frequency ω_1 moving

theory are then no longer applicable. An exception exists for a thermal distribution of initial beam velocities, where detailed balancing may be relied upon to preserve the static momentum distribution.

[4] $-\gamma\delta$, and $+\gamma\delta$, respectively, from the right-hand side of the first and second equations.

into a static electron atmosphere of plasma frequency ω_2, we obtain

$$\Gamma = j(\omega/v_{01})[1 \pm \omega_1/(\omega^2-\omega_2^2)^{\frac{1}{2}}]. \quad (10)$$

Equation (10) has been considered by Pierce[1] for electrons moving through an ion atmosphere. Wave amplification is possible only when $\omega<\omega_2$. The singularity at $\omega=\omega_2$ may be removed by introducing collisional damping.

For $\omega_1/U_{01}=\omega_2/U_{02}$, the quartic, Eq. (9), is reducible to a biquadratic. The resulting algebraic solution affords an insight into the physical aspects of the process. With the substitutions,

$$\omega/\omega_1=\alpha, \quad \omega/\omega_2=\beta, \quad j\frac{\Gamma U_{01}}{\omega_1}=j\frac{\Gamma U_{02}}{\omega_2}=\gamma, \quad (11)$$

Eq. (9) reduces to

$$[1/(\gamma+\alpha)^2]+1/(\gamma+\beta)^2=1, \quad (12)$$

whose solution is

$$\gamma=-\tfrac{1}{2}(\alpha+\beta)\pm\{[\tfrac{1}{2}(\alpha-\beta)]^2+1\pm[(\alpha-\beta)^2+1]^{\frac{1}{2}}\}^{\frac{1}{2}}. \quad (13)$$

For wave growth one must consequently have

$$\omega<2\sqrt{2}\omega_1\omega_2/|\omega_1-\omega_2|. \quad (14)$$

The maximum value of the imaginary part of γ in Eq. (13) is $\frac{1}{2}$, at $(\alpha-\beta)=\sqrt{3}$.

To obtain the band width over which wave amplification is possible in the general case, we utilize a graphical solution of Eq. (9). With the substitutions

$$x=\alpha+j\Gamma U_{01}/\omega_1, \quad y=\beta+j\Gamma U_{02}/\omega_2 \quad (15)$$

there results

$$(1/x^2)+1/y^2=1. \quad (16)$$

Eliminating Γ in Eq. (15) yields the linear relation

$$y=(\beta r/\alpha)x+\beta(1-r), \quad (17)$$

where $r=U_{02}/U_{01}$.

The real roots, corresponding to purely imaginary Γ, are given by the points of intersections of Eq. (16) with Eq. (17). A graph of the quartic is shown in Fig. 1. Any straight line intersecting the rectangle ABCD formed by the asymptotes will cut the quartic is only two real points. The limit of zero amplification is reached at the points of tangency. Figure 2 indicates the ranges of δ/v over which the propagation constant contains a real part, as a function of β, for representative values of α/β. The band width reaches a minimum for either of the beams static ($\delta/v=\pm1$) and increases monotonically as matching of the two-beam velocities improves. For $r=-1$, ($\delta/v=\pm\infty$), the band width becomes zero. These results are what we should expect on physical grounds, as dispersion in the velocity militates against organized motion. For the case $\alpha/\beta=1$, it is interesting to note that as r increases from 0 to 1, the band width,

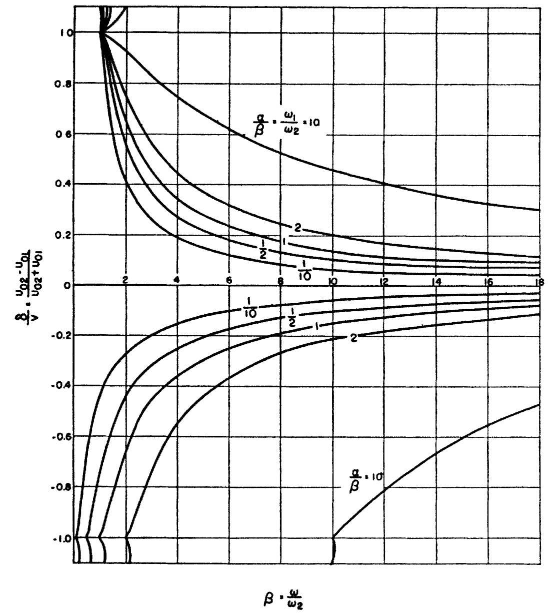

FIG. 2. Ranges of amplification for the two-beam case; temporal specification of initiating disturbance. Wave growth exists in the region to the left of a given curve.

in units of $\beta(1-r)$, increases from 1 to $2\sqrt{2}$ ($0F$ and $0E$ respectively, on Fig. 1).[5]

Figure 3 gives contours of constant amplification factor for the case $\beta/\alpha=1$. The region to the left of the

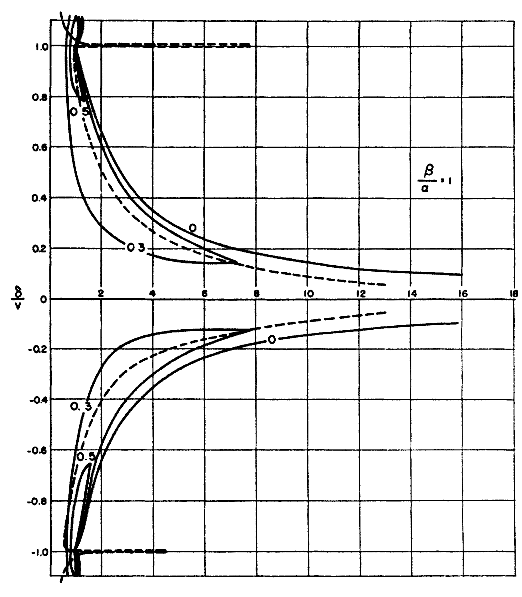

FIG. 3. Contours of constant amplification for the two-beam case, with $\beta/\alpha=1$. In the region to the left of the dotted lines the phase velocity is positive. Wave growth $=\exp[(\omega_1/v)\sqrt{2}\times$contour value]/unit length.

[5] Haeff's statement (reference 1) that there is amplification in this case when the inhomogeneity factor $(\delta/v)(\omega/\omega_1)$ lies between 0 and $\sqrt{2}$ is not correct. The maximum allowable value for this factor is itself a function of the beam velocities, increasing from 1 to $\sqrt{2}$ as r increases from 0 to 1.

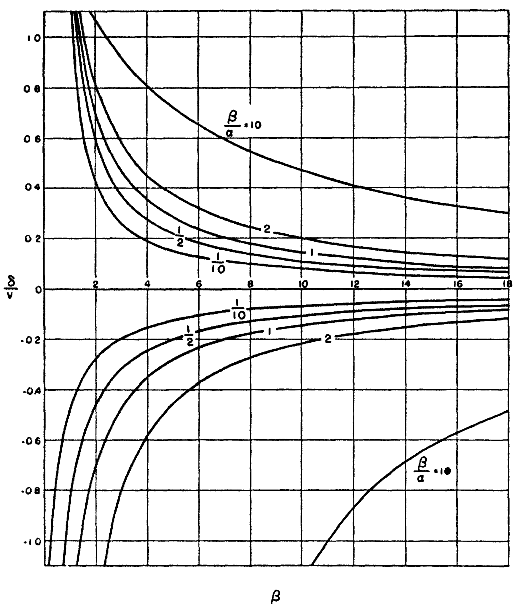

FIG. 4. Ranges of amplification for the two-beam case; spatial specification of initiating disturbance.

dotted line corresponds to positive phase velocities of the waves; to the right, these velocities are negative. In general, low amplication is associated with large band width, i.e., with closely matched beam velocities. Putting $1-r=\epsilon$, and solving Eqs. (16) and (17), we obtain the real part of Γ to be of order ϵ. For the limiting case of $r=1$ (homogeneous beam), the straight line (17) passes through the origin, a singular point of the quartic which corresponds to a double real root. The points of intersection, R and R', Fig. 1, give the well-known solutions of wave propagation along a uniform beam.

We have thus far considered the wave frequency ω to be real and ascertained the behavior of the wave number, $l=\Gamma/j$. This procedure may be reversed to find the ω corresponding to real l. A discussion of the physical significance of these two viewpoints will be given in Sec. V. For the case $\omega_1=\omega_2$, Eq. (9) becomes:

$$[1/(\alpha-lU_{01}/\omega_1)^2]+1/(\alpha-lU_{02}/\omega_1)^2=1. \quad (18)$$

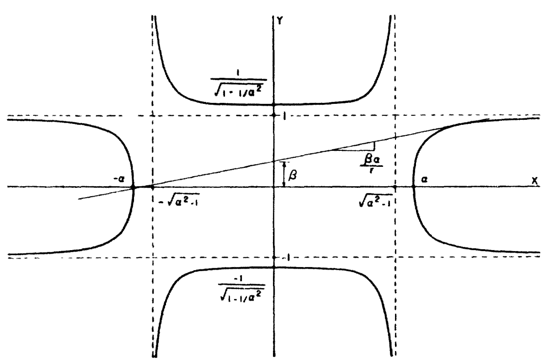

FIG. 5. Graphical determination of zones of amplification in the presence of a uniform, sharp cut-off velocity distribution.

This is of the form (12); it has the solution,

$$\alpha=(l/\omega_1)v\pm\{(l\delta/\omega)^2+1\pm[4(l\delta/\omega_1)^2+1]^{\frac{1}{2}}\}^{\frac{1}{2}}, \quad (19)$$

where all permutations of the algebraic signs are to be taken. The condition for wave growth (in time) is then

$$l\delta/\omega_1<\sqrt{2}; \quad (20)$$

and, since the real part of the frequency is, from Eq. (19), $\omega'=lv$, condition (20) may be written

$$(\delta/v)(\omega'/\omega_1)<\sqrt{2}. \quad (21)$$

This result illustrates a characteristic difference between the solutions based upon the two assumptions. As the limit on the right of Eq. (21) is no longer a function of δ/v, this latter case gives rise to the greater band width. Figure 4 indicates the ranges of amplification in the general case, and should be compared with its analog, Fig. 2.

IV. AMPLIFICATION IN THE PRESENCE OF THERMAL VELOCITIES

We take as model a homogeneous beam of electron density ρ_{0s}, injected with uniform velocity v_s into a plasma of density

$$\rho_0(U_0)=\text{constant}, \quad |U_0|<v_0,$$
$$=0, \quad |U_0|>v_0.$$

Equation 8 then yields

$$1=[\omega_s^2/(\omega+j\Gamma U_s)^2]+\omega_0^2/(\omega^2+\Gamma^2v_0^2), \quad (22)$$

where

$$w_0^2=\frac{e}{m}\int_{-v_0}^{v_0}\rho_0(U_0)dU_0. \quad (23)$$

It should be noted that the integration performed to obtain Eq. (22) is not valid if the singularity at $U_0=-\omega/j\Gamma$ lies on the path of integration. This represents a breakdown of the linear approximation, as pointed out by Bohm and Gross.[1] Since the equation is valid for complex values of Γ, however, it may be employed to investigate the characteristics of all ranges of amplification.

To reduce Eq. (22) to a form amenable to graphical solution we make the following substitutions:

$$l=j\Gamma, \quad \alpha=\omega/\omega_0, \quad \beta=\omega/\omega_s, \quad r=v_s/v_0,$$
$$y=(u+lv_s)/\omega, \quad x=lv_0/\omega_0. \quad (24)$$

Then the purely imaginary solutions of Γ correspond to the real points of intersection of the curve,

$$y^2=(x^2-\alpha^2)/(x^2-\alpha^2+1), \quad (25)$$

with the straight line

$$y=(\beta r/\alpha)x+\beta. \quad (26)$$

These solutions possess validity, of course, only if the singularity discussed above is not contained within the integration limits of U_0. For the determination of complex roots, these functions may nevertheless be considered as the analytic continuation of the expression which describes the physical situation only for complex Γ.

Figure 5 illustrates the salient features of the construction. In Fig. 6, the shaded regions indicate the ranges over which at least one pair of complex roots exists, i.e., over which amplification of an initial disturbance would take place. At large values of α, corresponding to considerable mismatch between the frequency of the disturbance and the equivalent static plasma density, the amplification region is bifurcated into two quite narrow bands of the parameters r and β. As α approaches unity, these bands broaden out, uniting into a single region at $\alpha=1$ and finally extending over the entire $r-\beta$ plane at $\alpha=0$. It is seen that amplification is possible when the thermal velocities are greater than the velocity of the homogeneous beam, i.e., for $r<1$.

Contours of constant growth factor for the $\alpha=2$ case are plotted in Fig. 7. The solid line contours belong to solutions in which the phase velocity of the wave is in the direction of the beam velocity v_s, while the dotted contours indicate phase velocities opposite to the beam motion. The magnitude of the wave growth is sufficiently large to produce a saturated equilibrium within a fraction of a wavelength. The amount of energy converted into the oscillatory state can be estimated, in the quasistatic approximation, by following the (hyperbolic) path corresponding to proportionate decreases in v_s and ω_s, in the $r-\beta$ plane, until the zero amplification contour is crossed. The validity of such quasi-static methods is questionable in view of the rapid rates of growth involved. It is interesting to note, nevertheless, that only a small amount of energy conversion would be obtained under highly mismatched conditions (as $\alpha\sim 10$).

In Fig. 8, the arrangement of parameters is more directly applicable to the solar problem. A situation is visualized wherein prominence material (ω_s, v_s) erupts into the corona (ω_0, v_0). The range of α over which the propagation constant contains a real component yields the approximate band width of the burst. In view of the high temperatures prevailing in the corona, "r" values of less than unity are to be expected. In the vicinity of $\alpha=10$, for $r=0.1$ the band width is about 5 percent (between zero amplification points; if reckoned between half-power points it would of course be considerably less, the exact amount being dependent upon the form of the initiating disturbance). Narrow band widths of this order have been obtained experimentally for many of the smaller bursts.

As the homogeneous beam velocity, v_s, increases, the zone of amplification broadens and shifts to somewhat lower frequencies. For $r>1$, a truncation appears, the branches broadening and shifting to lower frequencies as r increases. At $r=5$, the region of splitting has shifted off the plot. These curves indicate the possibility of bursts at separated frequencies with no activity at intermediate frequencies.

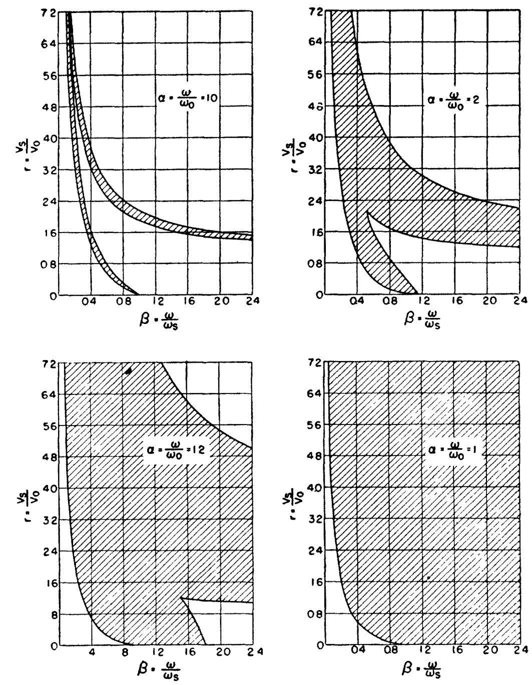

FIG. 6. The shaded regions indicate the ranges over which wave growth exists, for the distribution of Fig. 5.

To ascertain the dependence of the foregoing results

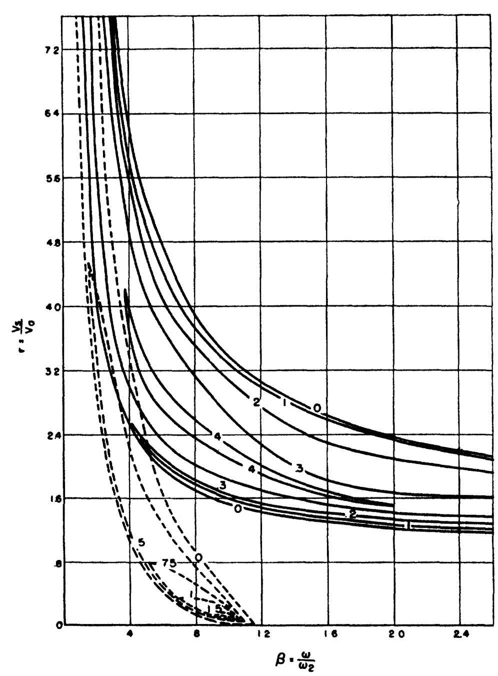

FIG. 7. Contours of constant amplification factor for $\alpha=2$; wave growth $=\exp[(\omega^0/v^0)\times$contour value$]$/unit length. Wave phase velocity is opposite in direction to injected beam motion along the dotted contours. Group velocity is in the same direction as phase velocity for $r>1$, but is oppositely directed for $r<1$.

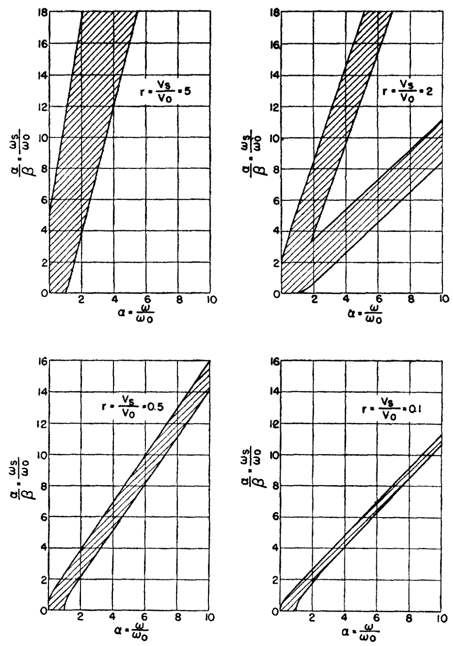

FIG. 8. Zones of amplification for several values of the ratio of injection to thermal velocities.

on the particular model assumed, it is desirable to employ an alternative functional representation of a thermal velocity distribution. To this end we expand the denominator of the integrand in Eq. (8) and take

$$f_0(U_0)=N_0(m/2\pi kT)^{\frac{1}{2}}\exp[-(m/2kT)U_0^2];\quad (27)$$

then we have

$$\int_{-\infty}^{\infty}\frac{\exp[-(m/2kT)U_0^2]}{(1+j\Gamma U_0/\omega)^2}dU_0$$
$$=\int_{-\infty}^{\infty}[1-(2j\Gamma U_0/\omega)+3(j\Gamma/\omega)^2U_0^2-\cdots]$$
$$\times\exp\left[-\left(\frac{m}{2kT}\right)\right]dU_0,\quad \left|\frac{j\Gamma U_0}{\omega}\right|<1.\quad (28)$$

If the mean square speed for a one-dimensional distribution, kT/m, is denoted by V_0^2, the equation for the propagation constant becomes

$$1=[\omega_s^2/(\omega+j\Gamma v_s)^2]+(\omega_0/\omega)^2[1+3(j\Gamma/\omega)^2v_0^2],\quad (29)$$

to second order in Γ. The nature of its roots may be investigated by graphical means similar to those employed previously. The quartic obtained possesses only two branches, which lie along the y axis. As a result, only an upper limit exists for "r" at any α and β. These upper limit curves lie fairly close to those of Fig. 6, for corresponding values of the remaining parameters. The addition of further terms to the expansion in Eq. (28) does not materially affect the situation, since the algebraic sign of all such contributions is the same as that of the Γ^2-term and so merely shift the asymptotes of the two branches without creating any additions.

An explanation for the lack of a lower "r" limit is to be found in the divergence of the series expansion at large U_0, which renders this type of approximation invalid in this region. It is the branches along the x axis in Fig. 5 which give rise to the lower "r" limit, and the roots obtained by intersections with these branches correspond to $\Gamma v_0/\omega>1$. Consequently, agreement would not be expected in this region. The y-branch roots, on the other hand, yield $\Gamma v_0/\omega<1$, which validates the upper "r" limit results obtained from our expansion.

A rather crude but nonetheless a useful criterion for the validity of the results obtained with a given approximation to a thermal distribution arises from the fact that the true distribution, in the absence of any directed beams, cannot alone produce growing oscillations, since the resultant conversion of thermal energy into directed wave energy would violate the second law of thermodynamics. Consequently, if for any set of parameter values a given distribution standing alone in the dispersion equation gives rise to growing waves, then one cannot expect the results in this range of the parameters to be a reliable index of the behavior of a true thermal distribution.

The uniform distribution, sharp cut-off model previously considered, for example, violates this criterion for $\alpha<1$, so that the results in this region are suspect. A distribution of the form,

$$f_0(U_0)=(N_0/\pi)\cdot V_0/(V_0^2+U_0^2),$$

which appears to give a good description of a thermal distribution, yields spurious results for $\alpha>1$. Physically, the slow rate of decrease of the number of high energy particles in this distribution is responsible for producing internal amplification in this region. For $\alpha<1$ the results obtained are qualitatively similar to those for the uniform distribution; and, in fact, as α approaches zero, the two dispersion relations become identical.

V. DESCRIPTION OF INITIATING DISTURBANCE

Boundary conditions, sufficient to determine the initial amplitudes of the modes of oscillation, may be prescribed either temporally or spatially; the type of description which is appropriate depends, of course, upon the physical situation envisaged. In an electron tube device, for example, a known time signal is generally impressed at a fixed space position; consequently, a temporal description is appropriate. For a medium such as the solar atmosphere, where the initiating disturbances take the form of random inhomogeneities of density and current, only a spatial description at a fixed instant would be physically permissible.

Mathematically, for the exponential space-time dependence employed here, these two modes of description appear to rest on an equal footing: spatially imposed boundary conditions result in the possibility of temporal amplification, while temporally imposed boundary conditions result in the possibility of spatial amplification. That this equivalence does not hold physically is evidenced by the fact that a spatial description gives rise to wave growth over a wider band of frequencies than does a temporal specification. Fundamentally, this appears to result from the prescription for all time of the form of the motion at the initiating point, employed in the temporal description. The analogous prescription of the motion over all space at a fixed instant of time utilized in the spatial method of specifying the initiating disturbance is less restrictive, since wave growth can occur at all points once the initial instant has passed. A quantitative comparison of the frequency bands in question was given in Sec. III for the two-beam case.

The determination of the initial mode amplitudes requires for the spatial case,[6] $\rho_q(z,0)$ and $\partial\rho_q(z,0)\partial t$, for each of the r interacting beams (or equivalent information). Then the coefficients of the oscillating components of charge for each beam, and in each mode follow from the $2r$ equations:

$$\sum_{n=1}^{2r} Aq_n(k)=\int_{-\infty}^{\infty}\rho_q(z,0)e^{-ikz}dz,$$
$$\sum_{n=1}^{2r} i\omega_n(k)Aq_n(k)=\int_{-\infty}^{\infty}\frac{\partial\rho_q(z,0)}{\partial t}e^{-ikz}dz,\quad q=1\cdots r, \tag{30}$$

together with the r relations between the components of oscillating charge density in each beam which are characteristic of each mode.

The reality of the wave growth associated with the complex propagation constants obtained here follows from their conjugate nature. It may be contrasted, for example, with the solution to the wave equation for a passive, lossy medium where one obtains $\Gamma=\pm jkK^{\frac{1}{2}}$, with complex dielectric constant K. Either choise of sign, in this case, corresponds to a diminution in wave amplitude along the propagation direction, since the relative sign of the real and imaginary parts of Γ is fixed. For complex conjugate values, on the other hand, one can always select a wave whose amplitude increases in any given direction as a result of the independence of the algebraic signs governing the real and imaginary portions of Γ.

VI. MECHANISMS OF CONVERSION INTO RADIATION FIELD ENERGY

The foregoing modes of oscillation are all of the longitudinal type, that is the electric field generated is irrotational. To flow through space devoid of matter, it is necessary that the waves in question possess a Poynting flux. This requires that the electric field contain a component transverse to the propagation direction. It is the purpose of this section to examine several mechanisms which have been proposed to effect this conversion.

The presence of a transverse static magnetic field suffices to couple the motion of the oscillating charge into a direction mutually perpendicular to the field and to the propagation vector, giving rise to a transverse component of electric field. The interaction of the static magnetic field, and of the newly created oscillating magnetic field with the forced vibrations in the transverse direction reacts back upon the original motion, altering its propagation constant. Taking account of these effects, we may write Eq. (8) in the determinantal form:

$$\begin{vmatrix} 1-\sum_q\frac{\omega_q^2}{(j\Gamma v_q+\omega)^2}\left(1+\frac{\omega_H^2}{[j\Gamma U_q+\omega]^2-\omega_H^2}\right) & \frac{j\omega_H}{\omega}\sum_q\frac{\omega_q^2}{[j\Gamma U_q+\omega]^2-\omega_H^2} \\ -j\frac{\omega_H}{\omega}\frac{\omega_q^2}{(j\Gamma v_q+\omega)^2-\omega_H^2} & 1+\frac{c^2\Gamma^2}{\omega^2}-\sum_q\frac{\omega_q^2}{\omega^2}\cdot\frac{1}{1-\omega_H^2/(j\Gamma v_q+\omega)^2}\end{vmatrix}=0, \tag{31}$$

where

$$\omega_H=eB_0/m, \tag{32}$$

B_0= static magnetic field flux density and the summations are carried out over all the beams present. The familiar splitting of each of the roots in the presence of the field appears here. The modes leading to growth arise from the upper diagonal term; since

$$|\Gamma|\sim\omega/v_q\gg\omega/c, \tag{33}$$

the off-diagonal cross product affects these mode propagation constants only to order $(v_q/c)^2$. Consequently, the magnitude of the splitting depends primarily on the ratio $[\omega_H/(j\Gamma v_q+\omega)]^2$. The energy division may be deduced from the ratio of the field intensities represented by the second row of Eq. (31), utilizing Eq. (33):

$$\frac{E_{\text{transverse}}}{E_{\text{longitudinal}}}\simeq\frac{\omega_q^2}{c^2\Gamma^2}\cdot\frac{\omega\omega_H}{[j\Gamma v_q+\omega]^2-\omega_H^2}\sim\left(\frac{v_q}{c}\right)^2\left(\frac{\omega_H}{\omega}\right)\cdots. \tag{34}$$

The presence of a component of mass velocity transverse to the direction of wave propagation gives rise to a corresponding motion on the part of the bunched charge which serves to generate a transverse electric field. A reaction back upon the original longitudinal motion occurs, in this case, only through the coupling

[6] The temporal case has been treated by V. A. Bailey, Phys. Rev. 78, 428 (1950).

produced by the newly created oscillating magnetic field. The determinantal equation assumed the form:

$$\begin{vmatrix} 1-\sum_q \dfrac{\omega_q^2}{(j\Gamma v_{qz}+\omega)^2} & +j\dfrac{\Gamma}{\omega}\sum_q \dfrac{\omega_q^2 v_{qy}}{(j\Gamma v_{qz}+\omega)^2} \\ +j\dfrac{\Gamma}{\omega}\sum_q \dfrac{\omega_q^2 v_{qy}}{(j\Gamma v_{qz}+\omega)^2} & 1+\dfrac{c^2\Gamma^2}{\omega^2}-\sum_q\left(\dfrac{\omega_q}{\omega}\right)^2\left[1-\dfrac{\Gamma^2 v_{qy}^2}{(j\Gamma v_{qz}+\omega)^2}\right] \end{vmatrix}=0, \tag{35}$$

where v_{qy} is the transverse component of mass motion of the qth beam.

The original number of longitudinal modes is preserved. The first-order change in the propagation constants of these modes may be shown to be $\simeq\frac{1}{2}(v_{qy}/c)^2\omega_q/v_{qz}$; the ratio of field intensities,

$$E_{\text{trans}}/E_{\text{long}}\simeq v_{qy}v_{qz}/c^2.$$

In the absence of a transverse velocity, the lower diagonal term gives rise to the modes characteristic of propagation of transverse waves as found in ionospheric theory. For $\omega_q>\omega$, a pair of positive and negative real values appear for Γ; the usual interpretation associates the propagation directions present when $\omega_q<\omega$ (Γ imaginary) with these real values in such a fashion as to correspond to attenuated fields for both roots. Roberts[1] has shown that the presence of a finite v_{qy} gives rise to a small imaginary part in the previously purely real pair of Γ's. He has interpreted this as making possible growing transverse waves.[7]

The irrotational nature of the longitudinal plasma oscillations herein considered arises from the perfect balancing of the rate of change of electric flux density, $\partial\mathbf{d}/\partial t$, against the oscillating current i. Martyn[8] has suggested that the introduction of collisional damping destroys this balance, giving rise to a radiation field. This argument is fallacious because it does not take account of the change in the characteristic plasma frequency brought about by the introduction of the damping term. If this is considered, the balance is restored. In general it follows from Poisson's equation (2) and the continuity equation (4) that

$$\nabla\cdot(\mathbf{i}+\dot{\mathbf{d}})=0. \tag{36}$$

Consequently if it is possible to write:

$$\nabla\cdot(\mathbf{i}+\dot{\mathbf{d}})=a(\mathbf{i}+\dot{\mathbf{d}})\cdots, \tag{37}$$

where "a" is an algebraic quantity, then no electromagnetic field exists. This will always be the case if $\mathbf{i}$ and $\mathbf{d}$ have components only along the direction in which space variations are present, provided this space dependence is the same for both.

To complete the discussion, it is instructive to consider the means utilized for the conversion of the energy of longitudinal space charge waves into transverse electromagnetic oscillations in many types of electron tubes. The cavity entered by the bunched beam is designed to produce a large electric field in the region traversed by the beam for a relatively low energy storage, at the frequency of the space charge wave. The wavelengths of the two oscillations will generally be quite different under these conditions, but this does not affect the energy transfer because the interaction is confined to a region which is small compared to either wavelength. It is this independence of the wavelengths of the two modes which makes it possible for each to satisfy its own dispersion relation. When the region of interaction extends over many wavelengths, on the other hand, a match is required both in frequency and in wavelength in order for a net interchange of energy to occur. For such a double matching to be in agreement with the dispersion relations for the two modes is a very special circumstance, and so there will normally be no excitation of the transverse mode. An exception arises if a region exists in which the wave characteristics vary considerably within a wavelength, or a period. Physically, the electron tube interaction model, or its temporal equivalent, then becomes applicable, since the rapidly varying conditions permit a net energy delivery with only one of the parameters of the two modes matched. Such a situation can arise if steep gradients are present in the medium characteristics, as might occur near the edges of prominence eruptions, or if the rate of growth of the longitudinal oscillations is such as to cause a considerable departure from uniformity in the amplitude of these oscillations within a wavelength spatially, or a period temporally. Since the usual calculated growths are very rapid, this last state of affairs may well provide the answer. A quantitative investigation of energy transfer under these conditions requires a nonlinear theory and so will be left to a future paper.

In conclusion, one of us (H.K.S.) wishes to express his gratitude for the opportunity to work at the National Bureau of Standards during the summer of 1950. We are indebted to Mr. Robert Lawrence for much of the graphical computation work.

[7] From a physical viewpoint an inconsistency appears to exist if one allows v_{qy} to approach zero, since one continues to obtain a growing field on this last interpretation, rather than the decaying field characteristic of normal ionospheric propagation. The resolution of this dilemma probably requires a better type of approximation than is provided by the currently employed geometric optical methods, since the wavelength becomes infinite at the limit in question.

[8] D. F. Martyn, Nature 159, 26 (1947).