# Nonlinear Spinor Fields

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A nonlinear spinor field, suggested by the symmetric coupling between nucleons, muons, and leptons, has been investigated in the classical approximation. Solutions of the field equations having simple angular and temporal dependence were obtained, subject to the boundary conditions that the fields be regular and that all observable integrals be finite. These b.c. lead to a nonlinear eigenvalue problem, whose solutions may be systematically discussed in the phase plane. Numerical solutions were obtained with a differential analyzer. If charge and mass of the particle-like solutions are defined in terms of $\int s_4 d\mathbf{x}$ and $\int T_{44} d\mathbf{x}$, then the number of masses corresponding to the same charge turns out to be small in all cases investigated. For certain lagrangians the nonlinearity leads to solutions having positive energy *only*. The mass ratio between the lightest stable particle and the heaviest unstable particle can be taken of the order of $10^{-3}$, if the nonlinear coupling constant is properly chosen. Although our specific model is too simple to meet certain obvious requirements, a theory of this general type has some interesting features.

## INTRODUCTION

THIS paper contains a classical investigation of certain properties of a unitary field theory which seem to correspond to some of the needs of elementary particle physics. Such a theory has these principal features: the equations of motion are derivable from a variational principle whose lagrangian density is invariant under a particular group; the equations of motion are nonlinear; the physically admissible solutions of the field equations are everywhere finite and quadratically integrable, so that the classical infinities never appear; the particles, instead of having an independent existence as field singularities, appear only as intense localized regions of field. Extensive investigations[1] have been made along these lines at a very fundamental level; but these efforts have attempted to extract the specifically nuclear fields, as well as the electromagnetic and gravitational fields, so far probably with little hope of success, from an underlying total field. A program, similar in spirit but technically much less formidable, may be based on the invariance of the lagrangian density under the group, not of general relativity, but only of special relativity.

The Maxwell, Dirac, and Yukawa fields, including their usually assumed interactions, are the simplest possibilities permitted by the Lorentz group. The Dirac-Maxwell field has been able to account for most of the facts about electronic systems, and in conjunction with the Yukawa field, has been able to describe some of the properties of nuclei and mesons. Although a composition of the lagrangian densities of these three fields and their Lorentz invariant interactions, which is simply additive and which excludes the gravitational field as well, may not seem natural from a standpoint more general than that of special relativity, still the total field resulting from this synthesis does account for much of what is now known about elementary particles; and since the differential equations form a nonlinear system, one has the possibility of a unitary theory.[2] Therefore, one possible procedure is to look for localized solutions of this total field, which is exactly the one currently used. The partial success of the field equations as currently interpreted does not, of course, guarantee even the same limited success of the same equations interpreted according to a unitary theory. Nevertheless, the possibility seems well worth investigating because the usual procedure does not treat the nonlinear terms correctly and of course also leads to infinities.

Here, however, we follow the different procedure of studying a simpler nonlinear field, first, to avoid the mathematical complexity still associated with the three simultaneously interacting fields, and, second, to explore the possibility that a simpler lagrangian in the richer nonlinear theory can accomplish as much as a more complicated lagrangian in the linear theory.

The simple case considered here is that of a single nonlinear spinor field which is not coupled to any other field. Although the nonlinearities of the usual theories result from just the interaction of different fields, the mathematical situation there is quite similar to what we treat here. (We intend to present at a later date corresponding results for the interaction of different fields.) The field described here, however, since it is spinor, may be of some interest in itself, because the symmetric coupling[3] between leptons, muons, and nucleons, as well as the possibility of building bosons out of fermions, seems to indicate a fundamental role for spinors in ele-

[1] See, for example, A. Einstein and E. G. Strauss, Ann. Math. **47**, 731 (1946); E. Schrödinger, Proc. Roy. Irish Acad. **49**, 275 (1944).

[2] R. Finkelstein, Phys. Rev. **75**, 1079 (1948).

[3] J. Tiomno and J. A. Wheeler, Revs. Modern Phys. **21**, 153 (1949); O. Klein, Nature **161**, 897 (1948); Lee, Rosenbluth, and Yang, Phys. Rev. **75**, 905 (1948); M. Ruderman and R. Finkelstein, Phys. Rev. **76**, 1458 (1949).

mentary particle theory. In addition, spinor theories of the pion[4] and photon (neutrino theory of light) already have been proposed; and, although unsuccessful, they do not discourage further study.

The special theory to be given here conforms to the following general pattern. The invariance of the lagrangian density to gauge transformations and to the inhomogeneous Lorentz group leads, in view of the equations of motion, to the conservation laws:

$$\partial_\mu s_\mu=0,\quad \partial_\mu\theta_{\mu\alpha}=0,\quad \partial_\mu M_{\mu\alpha\beta}=0,$$

where $s_\mu$, $\theta_{\mu\alpha}$, and $M_{\mu\alpha\beta}$ are well-known expressions which may be written down as soon as the lagrangian is specified. From these equations of continuity it follows that $Q$, $G_\alpha$, and $M_{\alpha\beta}$, defined by

$$iQ=\int s_4 d\mathbf{x},\quad iG_\alpha=\int\theta_{4\alpha}d\mathbf{x},\qquad iM_{\alpha\beta}=c^{-1}\int M_{4\alpha\beta}d\mathbf{x}, \tag{A}$$

are, respectively, scalar, vector, and 6-vector, not approximately but exactly, and that they are time independent. A field confined to a small region will then carry a definite charge $Q$, energy-momentum $G_\alpha$, and angular-momentum $M_{\alpha\beta}$. This localization of charge, energy-momentum, and spin may in general be interpreted as a single particle, or as a cluster of particles. If the theory is quantized, the charge, energy, and (total angular momentum)$^2$ become operators; but they mutually commute—again because of gauge and Lorentz invariance—so that the usual classification of elementary particles according to mass, charge, and spin is not invalidated.

## THE NONLINEAR SPINOR FIELD

The Dirac lagrangian may be written in terms of the two invariants $I_0$ and $I_1$:

$$\mathfrak{L}=\int Ld^4\mathbf{x},\quad L=\mu I_0+I_1, \tag{1}$$

where

$$I_0=i\psi^\dagger\psi,\quad I_1=\tfrac{1}{2}i[\psi^\dagger\gamma_\mu\partial_\mu\psi-(\partial_\mu\psi^\dagger)\gamma_\mu\psi],$$

and $\mu$ is a constant of dimensions $[L]^{-1}$. Since the theory to be given is classical, $\mu$ is a fundamental constant not necessarily related to the Compton wavelength appearing in the usual formulation. A generalization of Eq. (1) may be written as

$$L=\mu I_0+I_1+gW(I_0, I_1, J), \tag{2}$$

where $J$ indicates other possible invariants of the spinor field, $g$ is a constant, and $W$ is a simple function which nevertheless makes the theory nonlinear. We assume that $J$ contains no field derivatives. The charge, rest mass, and spin, generally defined by Eq. (A), now become

$$Q=\int L_1\psi^*\psi d\mathbf{x}, \tag{B1}$$

$$Mc^2=-\int T_{44}d\mathbf{x}=-\int\tfrac{1}{2}L_1[\psi^*\partial_4\psi-(\partial_4\psi^*)\psi]d\mathbf{x}+\int Ld\mathbf{x}, \tag{B2}$$

$$S_z=c^{-1}\int L_1\psi^*(-i(\partial/\partial\varphi)+\tfrac{1}{2}\sigma_z)\psi d\mathbf{x}, \tag{B3}$$

where $\partial L/\partial I_1$. We suppose that these integrals are all computed in the proper system ($\mathbf{G}=0$) and that the $z$-axis is taken parallel to the spin.

Now $L_1=1+g\partial W/\partial I_1$. Hence, if $I_1$ does not appear in $W$, it follows from Eq. (B1) that the charge density will be definite as in the usual Dirac theory. Different possibilities, such as $W=I_0I_1$ and $W=I_1^2$ have also been studied in detail. For these the charge density is not definite so that neutral particles could conceivably result from compensation; however it was found,[5] for the particular cases studied, that a node in the charge density precluded the existence of localized solutions of the field equations.[6] We therefore limit ourselves to the case

$$W=W(I_0, J), \tag{3a}$$

for which

$$L_1=1. \tag{3b}$$

The equations of motion now become

$$\delta\mathfrak{L}/\delta\psi^\dagger=0,\ \text{or}\quad \gamma_\alpha\partial_\alpha\psi+\mu\psi-ig\partial W/\partial\psi^\dagger=0; \tag{4a}$$

$$\delta\mathfrak{L}/\delta\psi=0,\ \text{or}\ (\partial_\alpha\psi^\dagger)\gamma_\alpha-\mu\psi^\dagger+ig\partial W/\partial\psi=0. \tag{4b}$$

In addition, there is the useful invariant equation,

$$\psi^\dagger(\delta\mathfrak{L}/\delta\psi^\dagger)-(\delta\mathfrak{L}/\delta\psi)\psi=0,$$
$$I_1+\mu I_0+g/2[(\partial W/\partial\psi)\psi+\psi^\dagger\partial W/\partial\psi^\dagger]=0. \tag{4c}$$

## SPECIALIZATION OF LAGRANGIAN

The very simplest possibility for $W$ is

$$W=I_0^2. \tag{5a}$$

Another interesting form is

$$W=\sum_1^5(\psi^\dagger\gamma_\mu\psi)^2. \tag{5b}$$

Equation (5b) is closely related to one variant (Møller-

[4] E. Fermi and C. N. Yang, Phys. Rev. **76**, 1739 (1949); G. Wentzel, Phys. Rev. **79**, 710 (1950).

[5] M. Ruderman, thesis, California Institute of Technology, 1951.

[6] In addition, these theories have the property that charge and spin are proportional, so that neutral particles have no spin. This feature is, of course, entirely wrong; but no effort to rectify it will be made here.

Rosenfeld) of meson theory, in which the coupling is

$$\sum_1^5 (\psi^\dagger\gamma_\mu\psi)\chi_\mu,$$

where the five-dimensional vector $\chi_\mu$ consists of the vector meson field $\chi_\mu$ ($\mu=1\cdots4$) and the pseudosclar meson field $\chi_5$. This form was singled out, first, because it is (like the M.R. invariant) formally simple, and easy to handle; and, second, because it involves a pseudoscalar coupling, which not only behaves quite differently from other couplings, but also seems to be required—in some measure at least—by experiment. It was then decided to choose a $W$ which varies continuously between Eqs. (5a) and (5b); therefore the following form was chosen:

$$W_\lambda=\left[\frac{(6-\lambda)}{8}\right]I_0^2+\left[\frac{(2+\lambda)}{8}\right]\sum_1^5(\psi^\dagger\gamma_\mu\psi)^2, \quad (6)$$

and the results studied as a function of $\lambda$. (Because of the quadratic identities existing between the different Dirac tensors, $W_\lambda$ can also be expressed as a linear combination of invariants different from $I_0$ and $\sum_1^5(\psi^+\gamma_\mu\psi)^2$.)

### SEPARATION OF THE FIELD EQUATIONS

The field equations are a set of partial differential equations. The time separation may be made by the substitution,

$$\psi=e^{i\zeta(t)}\chi(x,y,z). \quad (7)$$

The angular separation is, in general, not possible; but we shall assume the following form for $\psi$:

$$\psi_\pm=\tfrac{1}{2}e^{i\zeta}\{(F+iG)\Omega_\pm+(F-iG)\beta\Omega_\pm\}, \quad (8)$$

where $F$ and $G$ are functions of $r$ only, and $\Omega_\pm$ are functions of angle only, and

$$\Omega_+=\begin{bmatrix} a & Y_{j-\frac12,m-\frac12} \\ b & Y_{j-\frac12,m+\frac12} \\ c & Y_{j+\frac12,m-\frac12} \\ d & Y_{j+\frac12,m+\frac12}\end{bmatrix}, \quad \begin{matrix} a=(j+m/2j)^{\frac12}, \quad a^2+b^2=1, \\ c=(j-m+1/2j+2)^{\frac12}, \\ c^2+d^2=1, \\ \Omega_-=\gamma_5\Omega_+. \end{matrix} \quad (8a)$$

These angular dependent spinors, $\Omega_\pm$, are common eigenfunctions of the commuting operators $\alpha_r$ and $k$:

$$\alpha_r\Omega_\pm=\Omega_\pm, \quad (9a)$$

$$k\Omega_\pm=\pm(j+\tfrac12)\Omega_\pm, \quad (9b)$$

where

$$k=\beta[-i\sigma(\mathbf{r}\times\nabla)+1],$$

$$\alpha_r=\left(\frac{\mathbf{r}\alpha}{r}\right)=\begin{pmatrix}0 & a\\ a & 0\end{pmatrix}; \quad a=\begin{pmatrix}\cos\theta & \sin\theta e^{-i\varphi}\\ \sin\theta e^{i\varphi} & -\cos\theta\end{pmatrix}. \quad (10)$$

There is the identity

$$\gamma_\lambda\partial_\lambda=-i\beta\alpha_r[(d/dr)+1/r]-i\alpha_r(k/r)-i\beta\partial/\partial t \quad (11)$$

by means of which one can easily show that the assumption (8) "separates" the linear parts of Eq. (4). If $k=\pm1$, the nonlinear terms also separate; but if $|k|>1$, then the differential equations indicate that $F$ and $G$ depend on angle, which is contrary to our initial assumption. We therefore consider only the solutions for which $k=\pm1$; these are the solutions having the simplest angular dependence. Then we have

$$\psi_+=\frac{e^{i\zeta(t)}}{(4\pi)^{\frac12}}\begin{bmatrix}F\\0\\iG\cos\theta\\iG\sin\theta e^{i\varphi}\end{bmatrix}, \quad (12)$$

$$I_0=(G^2-F^2)/4\pi, \quad (13a)$$

$$I_1=[GF'-FG'-2FG/r-\omega(F^2+G^2)]/4\pi, \quad (13b)$$

$$I_1^2=(G^4+F^4-2F^2G^2)/16\pi^2, \quad (14a)$$

$$\sum_1^5(\psi^+\gamma_\mu\psi)^2=(G^4+F^4+6F^2G^2)/16\pi^2, \quad (14b)$$

$$W_\lambda=(F^4+G^4+\lambda F^2G^2)/16\pi^2. \quad (15)$$

In $I_1$ we have written$'$ for $d/dr$ and $\omega$ for $d\zeta/dt$. By Eqs. (4a), (11), and (15) the equations of motion become

$$F'+(\mu-\omega)G+\gamma(2G^2+\lambda GF^2)=0, \quad (16a)$$

$$G'+(2/r)G+(\mu+\omega)F-\gamma(2F^2+\lambda FG^2)=0. \quad (16b)$$

Since $F$ and $G$ are time-independent, it is necessary to assume that $\omega$ also does not depend on $t$. The invariant equation is

$$\mu I_0+I_1+8\pi\gamma W_\lambda=0, \quad (16c)$$

where

$$\gamma=g/4\pi. \quad (16d)$$

### BOUNDARY CONDITIONS

We impose the b.c. that the functions $F$ and $G$ be everywhere finite and regular and that all the observable integrals also be finite. Since Eq. (16a) contains the term $2G/r$, the finiteness condition can be met at the at the origin only if $G$ vanishes there. (It follows from Eq. (16b) that $F'$ also vanishes there.) Equations (16) are two differential equations of the first order, so that there are only two boundary conditions; these may be taken to be the values of $F$ and $G$ at the origin. Since $G$ necessarily vanishes there, the solutions bounded at the origin depend on the initial value of $F$ only. It turns out, however, that these solutions usually do not vanish at infinity, and that the only solutions which satisfy the b.c. everywhere correspond to a discrete set of initial values for $F$.

## THE NONLINEAR SCALAR FIELD

In order to motivate the procedure for solving the eigenvalue problem corresponding to Eq. (16), let us first consider the nonlinear scalar field described by the lagrangian:

$$-L=(\partial_\mu\psi)(\partial_\mu\psi^*)+\mu^2\psi\psi^*-(\gamma/2)(\psi\psi^*)^2. \quad (17)$$

The quartic coupling has been chosen in order to simulate the spinor case. The equations of motion are

$$\Box\psi=\mu^2\psi-\gamma(\psi^*\psi)\psi. \quad (18)$$

We try to find spherically symmetric solutions having harmonic time-dependence

$$\psi(\mathbf{r},t)=e^{i\omega t}y(r), \quad (19)$$

where $y$, as indicated, is a function of $r$ only, and satisfies the total differential equation,

$$y''+(2/r)y'+(\omega^2-\mu^2)y+\gamma y^3=0. \quad (20)$$

We assign $y'=0$ at the origin and look for quadratically integrable functions; it will turn out that the solutions corresponding to a given $\omega$ form a discrete set. They exist only when $\gamma>0$. Note first that the equation has the three trivial special solutions,

$$y=\text{constant}=A;\quad A=0,\ \pm[(\mu^2-\omega^2)/\gamma]^{\frac{1}{2}}. \quad (21)$$

Let $A_\pm=\pm[(\mu^2-\omega^2)/\gamma]^{\frac{1}{2}}$. Linearize in the neighborhood of the special solutions by writing

$$y=A+u \quad (22)$$

and regarding $u$ as small. The linear equation satisfied by $u$ is

$$u''+(2/r)u'=(\mu^2-\omega^2)u,\qquad \text{near } A=0, \quad (23a)$$

$$=-(\mu^2-\omega^2)u,\qquad \text{near } A=A_\pm. \quad (23b)$$

The solutions are the spherical bessel functions. We want the eigensolutions to approach the axis exponentially in order to have quadratic integrability, and therefore take

$$\mu^2-\omega^2>0. \quad (24)$$

But then other solutions getting into the neighborhood of the special solutions $A_\pm$, will approach these lines as $r\to\infty$ according to $[\sin(\mu^2-\omega^2)^{\frac{1}{2}}r]/r$. The different behaviors near $A_\pm$ and the axis correspond to the different natures of these special solutions, as we shall now see more clearly by going to the phase-plane: the $y$, $y'$-plane.

If one lets $y$ and $y'$ correspond to position and velocity of a representative point, then the differential equation describes a nonconservative, one-dimensional motion, since $r$ ("time") appears explicitly. The energy for the corresponding conservative motion (defined by Eq. (20) after the $1/r$ term has been deleted) is

$$K=\tfrac{1}{2}(y')^2+V(y), \quad (25)$$

where

$$V(y)=(\gamma/4)y^4-[(\mu^2-\omega^2)/2]y^2. \quad (25a)$$

The equilibrium points of this motion, defined by $\partial V/\partial y=0$, correspond to the special solutions $y=A_\pm$ and $y=0$; and it is clear that $A_\pm$ are stable equilibria, while the other is not. Figure 1 summarizes the situation in the phase plane. The point representative of the conservative motion moves on the curves of constant $K$. The curve $K=0$ is a figure-eight through the origin; the curves $K>0$ enclose both $A_\pm$ and 0, but the curves $K<0$ enclose only one equilibrium point; the origin is a saddle point. The nonconservative motion, which corresponds to our actual problem, may now be described. For it we may calculate $K$ from the exact equations of motion, and we find

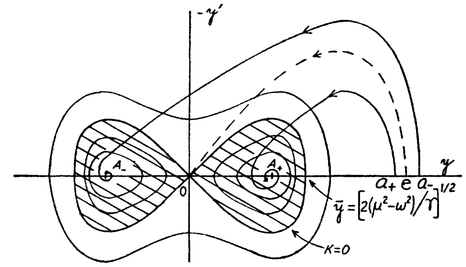


FIG. 1. Phase-plane for scalar equation.

$$dK/dr=-2(y')^2/r. \quad (26)$$

Hence $K'$ is never positive, and the representative point of the actual motion will always move inward across the lines of constant $K$. Such a trajectory must terminate at either $A_\pm$ or the origin, no matter where it starts. In particular, if we denote the two parts of the area enclosed by the curve $K=0$ by $\mathcal{A}_+$ and $\mathcal{A}_-$, then it is clear that any trajectory getting into the region $\mathcal{A}_+$ must terminate on $A_+$; any curve entering $\mathcal{A}_-$ must end at $A_-$.

The dashed trajectory $e0$, shown in Fig. 1 is an eigensolution. An eigensolution may be located by starting on the $y$ axis and continuously increasing the initial ordinate $\bar{y}$. At first when $0<\bar{y}<[2(\mu^2-\omega^2)/\gamma]^{\frac{1}{2}}$, all trajectories are certain to terminate at $A_+$. For $\bar{y}$ slightly greater than this critical value, the situation is still unchanged and such a trajectory, starting at $a_+$, has been shown in Figs. 1 and 2. If the initial ordinate is increased to $a_-$, a trajectory is found which terminates, in the other lobe of the figure-eight, on the special solution $A_-$; this is also shown in both Figs. 1 and 2 and is seen to node in the $y-r$ plot. By narrowing the interval $a_+a_-$ the eigensolutions may be determined with arbitrary accuracy.[7]

## SOLUTION OF THE SPINOR EIGENVALUE PROBLEM

There are three special solutions of the set (16):

$$G=\text{constant}=0,\quad F_0=\pm[(\mu+\omega)/4\gamma]^{\frac{1}{2}},\quad F=\text{constant}=F_0,\quad \gamma>0. \quad (27)$$

[7] We are indebted to Professor F. Bohnenblust for a rigorous proof of this together with a demonstration of the analyticity of $y$ in a neighborhood of $r=0$. We wish to thank him for an extremely helpful discussion.

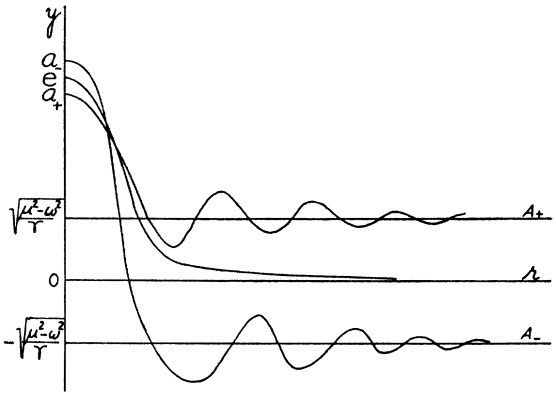


FIG. 2. Radial solution for scalar case.

Linearizing in the vicinity of the special solutions by putting

$$G=g,\ F=F_0+f, \tag{28}$$

where $f$ and $g$ are small, we obtain

$$f''+(2/r)f'-(\mu^2-\omega^2)f=0 \tag{29a}$$

near $F_0=0$, and

$$f''+(2/r)f'+2[(\mu^2-\omega^2)+(\mu+\omega)^2(\lambda/2)]f=0 \tag{29b}$$

near

$$F_0=\pm[(\mu+\omega)/4\gamma]^{\frac{1}{2}}.$$

According to Eq. (29a) we require $\omega<\mu$ in order to guarantee exponential solutions. According to Eq. (29b), $\omega$ is also limited by the condition,

$$\omega(2-\lambda)<\mu(2+\lambda), \tag{30}$$

to force oscillatory behavior near the other two constant solutions; it turns out that there are eigensolutions if (30) is satisfied. These conditions again become clearer in the phase plane to which we now turn. One sees from the asymptotic form of the differential equations that $G$ behaves like the derivative of $F$; in the special solutions where $F$ is a constant, $G$ plays the same role, since in these cases it vanishes. The $FG$ plane is therefore chosen as the phase plane.

It is possible, and it is now convenient, to obtain the differential equations from a one-dimensional variational equation, namely,

$$\delta\mathfrak{L}=0, \quad \text{where} \quad \mathfrak{L}=\int Lr^2dr \tag{31}$$

and

$$L=\mu I_0+I_1+gW_\lambda.$$

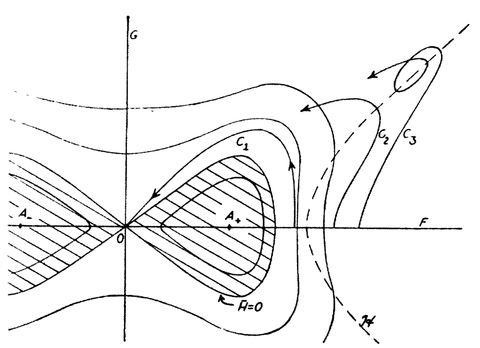


FIG. 3. Phase-plane for spinor equations.

The corresponding Euler-Lagrange equations are

$$\left(\frac{d}{dr}+\frac{2}{r}\right)\frac{\partial L}{\partial F'}-\frac{\partial L}{\partial F}=0, \tag{32a}$$

$$\left(\frac{d}{dr}+\frac{2}{r}\right)\frac{\partial L}{\partial G'}-\frac{\partial L}{\partial G}=0. \tag{32b}$$

It can be verified, by use of Eqs. (13) and (15), that the set (32) is equivalent to the set (16).

These differential equations may be regarded as describing the nonconservation motion of a point having position $F$ and momentum $G$. It is again convenient to study the conservative motion described by the same equations without the $1/r$ term. To do this a pseudohamiltonian, $\bar{H}$, is constructed as follows:

$$-\bar{H}\equiv(\partial L/\partial F')F'+(\partial L/\partial G')G'-\tilde{L}, \tag{33}$$

where $\tilde{L}$ is obtained from the lagrangian density by dropping the term which contains $r$ explicitly. Thus, we have

$$4\pi L=GF'-FG'-2FG/r-\omega(F^2+G^2)+\mu(G^2-F^2)+\gamma(F^4+G^4+\lambda F^2G^2), \tag{34a}$$

$$4\pi\tilde{L}=4\pi L+2FG/r. \tag{34b}$$

Explicit calculation of $\bar{H}$ according to Eqs. (33) and (34) leads to

$$-4\pi\bar{H}=(\mu+\omega)F^2-(\mu-\omega)G^2-\gamma(F^4+G^4+\lambda F^2G^2). \tag{35}$$

The representative point in the $FG$ plane moves on curves of constant $\bar{H}$, when its motion is governed by the set (16) simplified by omission of the term $2G/r$. On the other hand, in the actual (nonconservative) motion, $\bar{H}$ will vary according to the exact Eqs. (16) in the following way:

$$\begin{aligned}-4\pi\bar{H}'&=\left(\frac{d}{dr}\frac{\partial L}{\partial F'}\right)F'+\left(\frac{d}{dr}\frac{\partial L}{\partial G'}\right)G'+\frac{\partial L}{\partial F'}F''+\frac{\partial L}{\partial G'}G''-\tilde{L}'\\&=-\frac{2}{r}\left(\frac{\partial L}{\partial F'}F'+\frac{\partial L}{\partial G'}G'\right)+\frac{\partial L}{\partial F}F'+\frac{\partial L}{\partial G}G'\\&\quad+\frac{\partial L}{\partial F'}F''+\frac{\partial L}{\partial G'}G''-\left[L'+\frac{d}{dr}\left(\frac{3}{r}FG\right)\right]\\&=-\frac{2}{r}\left(\frac{\partial L}{\partial F'}F'+\frac{\partial L}{\partial G'}G'\right)-\frac{2}{r}\frac{d}{dr}(FG),\\4\pi\bar{H}'&=+(4/r)GF',\end{aligned} \tag{36}$$

and by Eq. (16),

$$4\pi\bar{H}' = -(4/r)\{(\mu-\omega)G^2+\gamma G^2(2G^2+\lambda F^2)\}. \quad (37)$$

It will now be shown that the exact Eq. (37), describing the variation of expression (35), is sufficient for a discussion of the solutions.

In the $F^2G^2$-plane the level lines of $\bar{H}$ are confined to the first quadrant and are conic sections. If $|\lambda|>2$, they are hyperbolic; if $|\lambda|<2$, they are elliptic; and if $|\lambda|=2$, they are parabolic. The corresponding curves in the $FG$ plane are topologically the same but must be completed by reflection in both axes. If $\lambda\geq 0$, then $\bar{H}'$ is negative definite, and the situation is qualitatively the same as with the scalar field already described: the representative point must ultimately arrive at $A_+$, $A_-$, or, if it is an eigensolution, at the origin. If $\lambda<0$, then there are always two regions in the $FG$ plane, separated by a hyperbola, in each of which $\bar{H}'$ is definite. The equation of this hyperbola, $\mathcal{H}$, is $2G^2+\lambda F^2+(\mu-\omega)/\gamma=0$. In Fig. 3, $\mathcal{H}$ is shown for $\lambda=1$. An eigenfunction may lie entirely between the two branches of $\mathcal{H}$, as $C_1$ does, or it may resemble either $C_2$ or $C_3$. The actual numerical solutions of the differential equations, obtained with a differential analyzer, displayed all the features to be expected from this analysis. Figure 4 illustrates the appearance of typical radial solutions.

Suppose $\mu$ fixed; then the solutions form a two-parameter ($\omega$ and $\lambda$) family. However, $|\omega|<\mu$ and Eq. (30) must be satisfied. The necessity of Eq. (30) follows from fact that all curves in the phase plane leading from $(F, 0)$ to $(0, 0)$ violate the equations of motion when Eq. (30) is not satisfied. The condition (30) confines solutions to the shaded region in Fig. 5. For the range of initial values investigated with the differential analyzer, no eigensolutions could be found outside of the interval $-2\leq\lambda\leq 0$. The solutions obtained are indicated in Fig. 5 except for some members of the family at $\lambda=-2$. From the point of view of the results the most interesting value of $\lambda$ is $\lambda=-2$, since a family of polynodal solutions was obtained there. The physical discussion will be based on this family.

## DISCUSSION

The lagrangian employed in this paper has been introduced mainly because it is simple; but even if physical justification should be found for it, one still cannot discuss these results without reference to the quantum theory. On the other hand, it seems clear that the quantization ought not be attempted according to the canonical formalism, since the singular commutators thereby introduced lead to new divergences and ambiguities. Furthermore, a spinor field quantized in the usual way (by anticommutators) has no classical limit. The case considered here is therefore the "abnormal" one;[8] but it cannot be excluded by arguments based on the negative energy states, because, as we shall see, the nonlinear lagrangian may be so chosen that there are no localized solutions having negative energy. Finally, one may remark that the usual quantum theory of fields employs a formalism, which, although applicable, appears unnatural for describing the localized solutions characteristic of unitary theories.

[8] W. Pauli, Phys. Rev. 58, 716 (1940).

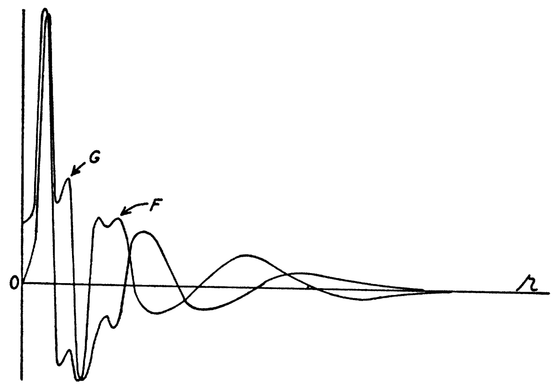


FIG. 4. Radial solution for spinor case.

Lacking an adequate formalism, one may nevertheless quantize in a very crude way by postulating that the coupling constant, $\epsilon$, with the electromagnetic field has its usual value, namely, $e/\hbar c$, and by requiring that the total charge be an integral multiple of $e$. Then the lowest charge state is described by

$$e=(e/\hbar c)\int\psi^*\psi d\mathbf{x} \quad (38a)$$

or

$$\int\psi^*\psi d\mathbf{x}=\hbar c. \quad (38b)$$

The $z$-component of the angular momentum is now

$$c^{-1}\int\pi^*\left(-i\frac{\partial}{\partial\varphi}+\tfrac{1}{2}\sigma_z\right)\psi d\mathbf{x}=(2c)^{-1}\int\psi^*\psi d\mathbf{x}=\hbar/2 \quad (39)$$

for a field having the angular dependence (12). (The $x$ and $y$ components will vanish. If the quantization is done correctly however, the components of angular momentum are, of course, subject to the usual commutation rules.)

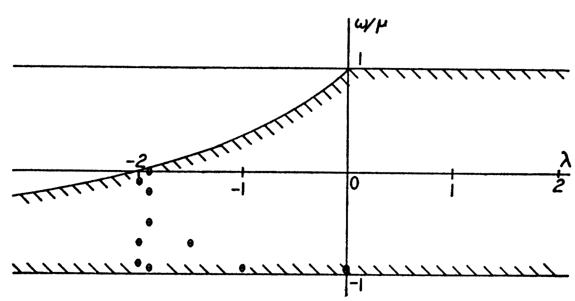


FIG. 5. Each point corresponds to an eigensolution obtained on the differential analyzer; $\omega$ is frequency and $\lambda$ specifies coupling. No eigensolutions can exist outside the shaded region if $\lambda<-2$.

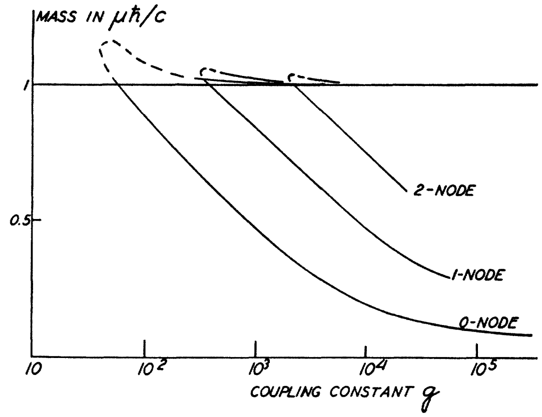


FIG. 6. Spinor mass spectrum for $\lambda=-2$. The solutions corresponding to $M<1$ were integrated by the analyzer. For the $M>1$ solutions an asymptotic form was used. The mass spectrum in the intermediate region (dotted curve) is slightly uncertain.

Now assume that the fundamental length $\mu^{-1}$ is assigned and that $\lambda=-2$. Then consider three eigenfunctions having zero, one, and two nodes. Each is characterized by a value of $\omega$ which may be varied within limits without spoiling the eigensolution, although the initial value of $F$ (and therefore the complete solution) must be varied simultaneously. Each of these solutions may now be completely fixed by choosing $\omega$ so that Eq. (38) is satisfied. Then Eq. (39) is also satisfied and each of the three field structures in question carries the same charge, $e$, and the same spin, $\hbar/2$. Since the charge quantization fixes the solution uniquely, it follows that the masses of these three particles are also fixed (by the equation)

$$Mc^2=-\hbar\omega+g\int W_{-2}dx.$$

The results are shown in Fig. 6 where the mass is given as a function of $g$, the coupling constant. The mass spectrum terminates and the number of particles carrying the charge $e$ is small and depends on the value chosen for $g$. There are no solutions corresponding to $\omega>0$. As a result all masses are positive; the nonlinear term in this case excludes the negative energy solutions of the Dirac equation, as mentioned before. The existence of a discrete spectrum is a joint consequence of the classical eigenvalue problem and the charge quantization condition (38).

For a particular $g$ those solutions having a mass greater than $\mu\hbar/c$ are unstable against expanding to infinity, while the amplitude approaches zero; the nonlinear term then becomes negligible and the $F$ and $G$ functions form the usual $S$-wave solutions of the Dirac equation. Those solutions which correspond to masses less than $\mu\hbar/c$ *are stable*: when perturbed they will not spread to infinity because of the (rigorous) conservation of charge and total energy.[9]

The mass ratio between the lightest stable particle and the heaviest unstable one can be made of the order of $10^{-3}$ if $g$ is chosen sufficiently large, but such a $g$ ought to be in agreement with the rates of the various instabilities and with the strength of nuclear forces. Although the nonlinearity leads to ordinary and tensor forces having different ranges, the theory is too simple to provide neutrons and, therefore, a description of nuclear forces. In addition, although the "nucleons" (the two-nodal solutions) quite properly bind to form heavier nuclei, the nodeless solutions, which ought to be identified with electrons, also bind. It is entirely possible that difficulties of this nature may be avoided with other nonlinear invariants, but even in that case a satisfactory appraisal of the whole approach waits upon a natural quantization procedure.

[9] All the particlelike solutions of the nonlinear scalar equation (18), with $\gamma>0$, when normalized to unit charge, have masses greater than $\mu\hbar/c$ and are unstable. Unlike the spinor case, there are an infinite number of masses corresponding to a given $g$.