

to be negligible. Correcting for the 12.8-hour activity of Cu^{64} , and using the known total energy flux, and assuming that the absorption is strongly peaked about an average energy of 19.1 Mev,⁴ it is found that $\int \sigma_{\gamma n}(E)dE = (0.77 \pm 0.15) \times 10^{-24} \text{ cm}^2 \text{ Mev}$ for Cu^{63} . In our opinion the above error is a fair estimate of uncertainties introduced by such factors as counting statistics, calibration of the counter, sensitive volume and input resistance of the chambers, assumed thin target spectrum, assumption of a resonance narrow with respect to 50 Mev, extrapolation to zero chamber thickness, Walker correction, etc.

An error yet to be mentioned is the uncertainty in resonance energy, which is here assumed to be 19.1 Mev but has been reported by others as 17.6 Mev.^{5,6} The calculated cross section is almost exactly inversely proportional to the assumed resonance energy.

The quantum-mechanical sum rule^{7,8} predicts for

⁴ L. Marshall, Phys. Rev. **82**, 300(A) (1951); **83**, 345 (1951). This issue.

⁵ B. C. Diven and G. M. Almy, Phys. Rev. **80**, 408 (1950).

⁶ Johns, Katz, Douglas, and Haslam, Phys. Rev. **80**, 1062 (1950).

⁷ M. Goldhaber and E. Teller, Phys. Rev. **74**, 1046 (1948).

⁸ J. S. Levinger and H. A. Bethe, Phys. Rev. **78**, 115 (1950).

Cu^{63} a maximum integrated dipole cross section of 0.95 Mev barns.

Previous determinations of this cross section have yielded the values (in Mev barns): 1.5,⁹ 0.6,⁵ and 0.7.⁶

CALIBRATION OF VICTOREEN THIMBLE

The intensity of the beam did not vary greatly over the area accepted by the collimator, so that its intensity per square cm was roughly known. Hence, it was determined that a Victoreen thimble, enclosed within $\frac{1}{4}$ " Pb walls and inserted into the beam, measured one r per minute for every $2.5 \times 10^7 \text{ Mev cm}^{-2} \text{ sec}^{-1}$ of beam intensity.

As can be seen from Fig. 2, $\frac{1}{4}$ " Pb lies close to the shower maximum, so that the exact thickness of the wall is not critical. The Victoreen reading with $\frac{1}{8}$ " walls was only 5 percent lower.

We wish to acknowledge the help of D. W. Connor, who advised us in the design of the experiment, and of C. R. McKinney and his betatron crew, K. Benford, F. Sammons, and W. Humphrey.

⁹ J. L. Lawson and M. L. Perlman, Phys. Rev. **74**, 1190 (1948).

On the Stability of Magneto-Hydrostatic Fields

S. LUNDQUIST

Royal Institute of Technology, Stockholm, Sweden

(Received March 15, 1951).

The stability of static magnetic fields in an electrically conducting liquid is investigated. The result of the study is applied to the stability of twisted cylindrical magnetic fields. It is shown that instabilities may be caused by the twisting of a homogeneous field.

I. INTRODUCTION

IN recent years several theories of the earth's magnetic field have been developed by Elsasser and Bullard,¹ the essence of which is the induction due to liquid motion in a magnetic field. The interactions between motion and magnetic field which they consider are very complicated. Some similar but simpler processes by which mechanical energy is transferred into magnetic energy have been suggested by Alfvén.² One of the most attractive of his suggestions is founded on the hypothesis that a homogeneous magnetic field which is twisted by liquid motion becomes unstable and will form a loop. The magnetic energy of this loop is used to amplify the original field.

This paper attempts to present a mathematical treatment of the stability of magneto-hydrostatic fields, i.e., magnetic fields in equilibrium in a liquid at rest. The

theory is applied to Alfvén's problem. It is found that a magnetic field in a long cylinder becomes unstable when the increase of magnetic energy due to the twisting becomes of the same order of magnitude as the energy of the original field.

II. SOME PROPERTIES OF MAGNETO-HYDROSTATIC FIELDS

For hydrostatic equilibrium it is necessary that the force acting on the liquid be balanced by a hydrostatic pressure. In a medium with the permeability $\mu=1$, a current density \mathbf{i} and a magnetic field \mathbf{H} give a volume force $=\mathbf{i} \times \mathbf{H}$ emu. Since $\mathbf{i} = \text{curl} \mathbf{H} / 4\pi$, the condition for equilibrium is

$$\text{curl} \mathbf{H} \times \mathbf{H} = \nabla \phi, \quad (1)$$

where $\phi/4\pi$ is the hydrostatic pressure.

According to Eq. (1) the lines of force and current are situated in the surfaces $\phi = \text{const}$.

¹ W. M. Elsasser, Revs. Modern Phys. **22**, 1 (1950).

² H. Alfvén, Tellus **2**, 74 (1950).

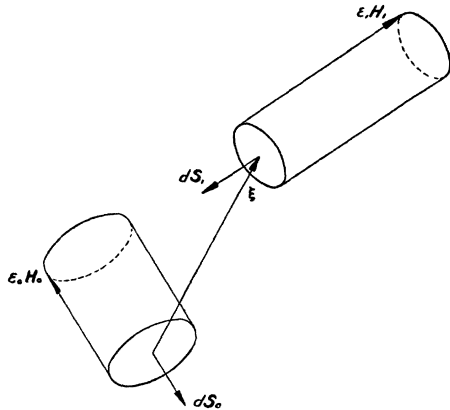


FIG. 1. Displacement of flux tube.

A special type of field are those which give no force at all on the medium. They are given by

$$\text{curl} \mathbf{H} \times \mathbf{H} = 0. \tag{2}$$

These force-free fields may exist in a compressible medium of uniform density. Hence, we may expect them to be a type of magnetic field that can exist in interstellar space, where a pressure gradient would give an increased density accompanied by increased recombination. Diffusion of the neutral atoms would then tend to decrease the pressure gradient. A solution of Eq. (2) for an infinite cylinder is a field with the axial component

$$H_z = AJ_0(\alpha r) \tag{3}$$

and the tangential component

$$H_\varphi = AJ_1(\alpha r). \tag{4}$$

A fuller discussion was given in a previous paper.³

III. MOTION OF THE LINES OF FORCE IN AN IDEAL LIQUID

A motion of a conducting liquid in a magnetic field is, in general, connected with induction effects which cause induced currents. The magnetic field of these

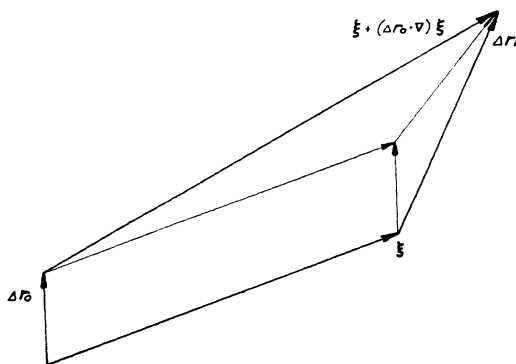


FIG. 2. Geometry of displacement.

³ S. Lundquist, Arkiv. fysik Bd. 2, 35 (1950).

currents deforms the original field. The purpose of this section is to derive an analytic expression for the change of the magnetic field connected with a finite displacement of the liquid. We consider an incompressible liquid, which is an ideal conductor with the permeability $\mu=1$. The liquid is moving with the velocity \mathbf{v} in a magnetic field \mathbf{H} . The current is given by

$$i = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{H}). \tag{5}$$

Since the conductivity $\sigma = \infty$, we must have

$$\mathbf{E} + \mathbf{v} \times \mathbf{H} = 0. \tag{6}$$

From Maxwell's equations we know

$$\text{curl} \mathbf{E} = -\partial \mathbf{H} / \partial t. \tag{7}$$

Combining Eqs. (6) and (7), we obtain

$$\partial \mathbf{H} / \partial t = \text{curl}(\mathbf{v} \times \mathbf{H}). \tag{8}$$

A discussion of this equation where \mathbf{H} is replaced by the vorticity vector, $\text{curl} \mathbf{v}$, will be found in any textbook on hydrodynamics.⁴

Equation (8) means that the flux of \mathbf{H} through any closed curve moving with the liquid is constant. Considering an isolated flux tube we see that this will move with the liquid. This is true even if the tube is not isolated and hence for each line of force, as the following discussion, originating from Helmholtz, will show.

The flux through any element of a surface wholly composed of lines of force is zero. This property is conserved when the surface moves with the liquid, according to Eq. (8). Hence, the surface will always be composed of lines of force.

Also, their intersection will be a line of force. Hence, we infer that the lines of force move with the liquid.

From this conclusion we may obtain the wanted expression for the change of \mathbf{H} .⁵ Consider a small part of a flux tube with the length $= \Delta \mathbf{r}_0 = \epsilon_0 \mathbf{H}_0$ and the cross section $d\mathbf{S}_0$. After displacement, it has the length $\Delta \mathbf{r}_1 = \epsilon_1 \mathbf{H}_1$ and the cross section $d\mathbf{S}_1$ (see Fig. 1). As the volume is the same, we have $\epsilon_0 \mathbf{H}_0 \cdot d\mathbf{S}_0 = \epsilon_1 \mathbf{H}_1 \cdot d\mathbf{S}_1$. The constancy of flux implies $\mathbf{H}_0 \cdot d\mathbf{S}_0 = \mathbf{H}_1 \cdot d\mathbf{S}_1$, and hence, $\epsilon_1 = \epsilon_0$.

This means that \mathbf{H}_1 is obtained from \mathbf{H}_0 in the same way as $\Delta \mathbf{r}_1$ from $\Delta \mathbf{r}_0$. Since

$$\Delta \mathbf{r}_1 = \Delta \mathbf{r}_0 + (\Delta \mathbf{r}_0 \cdot \nabla) \xi, \tag{9}$$

(see Fig. 2), we obtain

$$\mathbf{H}_1 = \mathbf{H}_0 + (\mathbf{H}_0 \cdot \nabla) \xi. \tag{10}$$

Equation (10) is *not* limited to small ξ but is valid for any finite value.

We complete this discussion by giving a direct analytic proof of Eq. (10). Using the vector identity

$$\text{curl}(\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla) \mathbf{a} - (\mathbf{a} \cdot \nabla) \mathbf{b} + \mathbf{a} \text{ div} \mathbf{b} - \mathbf{b} \text{ div} \mathbf{a} \tag{11}$$

⁴ H. Lamb, *Hydrodynamics* (Dover Publications, New York, 1945), p. 203.

⁵ This was kindly pointed out to me by Professor T. G. Cowling.

and the relations

$$\operatorname{div}\mathbf{H}=\mathbf{0}, \tag{12}$$

$$\operatorname{div}\mathbf{v}=\mathbf{0}, \tag{13}$$

we write Eq. (8)

$$(\partial\mathbf{H}/\partial t)+(\mathbf{v}\cdot\nabla)\mathbf{H}=d\mathbf{H}/dt=(\mathbf{H}\cdot\nabla)\mathbf{v}. \tag{14}$$

This equation may be integrated. We introduce the displacement $\xi(\mathbf{r}, t)$ as the vector which moves the point \mathbf{r} into its original position $\mathbf{r}_0=\mathbf{r}+\xi$ (see Fig. 3). The position vector of a point moving with the liquid is, accordingly, $\mathbf{r}=\mathbf{r}_0-\xi(\mathbf{r}, t)$, and hence,

$$\mathbf{v}=-d\xi/dt=-\partial\xi/\partial t-(\mathbf{v}\cdot\nabla)\xi \tag{15}$$

for any ξ , small or not.

We introduce Eq. (15) into Eq. (14) and write the equation in suffix notation (summation over equal suffices in the same term):

$$\begin{aligned} \frac{dH_\alpha}{dt} &= -H_\beta \frac{\partial}{\partial x_\beta} \left(\frac{\partial \xi_\alpha}{\partial t} + v_\gamma \frac{\partial \xi_\alpha}{\partial x_\gamma} \right) \\ &= - \left(\frac{\partial}{\partial t} + v_\gamma \frac{\partial}{\partial x_\gamma} \right) \left(H_\beta \frac{\partial \xi_\alpha}{\partial x_\beta} \right) \\ &\quad + \left(\frac{\partial H_\beta}{\partial t} + v_\gamma \frac{\partial H_\beta}{\partial x_\gamma} - H_\gamma \frac{\partial v_\beta}{\partial x_\gamma} \right) \frac{\partial \xi_\alpha}{\partial x_\beta}. \end{aligned} \tag{16}$$

The last term vanishes according to Eq. (13), and hence,

$$d\mathbf{H}/dt = - (d/dt)[(\mathbf{H}\cdot\nabla)\xi]. \tag{17}$$

Integrating and putting $\xi=0$ when $\mathbf{H}=\mathbf{H}_1$, we obtain Eq. (10):

$$\mathbf{H}_1 = \mathbf{H}_0 + (\mathbf{H}_0 \cdot \nabla)\xi$$

as before.

IV. THE CHANGE OF MAGNETIC ENERGY AT AN ARBITRARY DISPLACEMENT

We consider a magnetic field \mathbf{H} given in a volume V . At an arbitrary deformation given by the displacement vector ξ , the field changes to \mathbf{H}_1 in the region V_1 . The change of magnetic energy connected with this displacement is

$$\Delta W = \frac{1}{2} \int_{V_1} H_1^2 d\tau_1 - \frac{1}{2} \int_V H^2 d\tau, \tag{18}$$

where $d\tau$ means an element of the liquid in V , and $d\tau_1$ the same element in V_1 .

From Eq. (11) we have $d\tau_1 = d\tau$. From Eq. (10) we obtain \mathbf{H}_1 as a function of the original coordinates of $d\tau_1$. Hence, the first integral in Eq. (18) may be transformed into an integral over the original volume, giving

$$\Delta W = \int_V \mathbf{H} \cdot (\mathbf{H} \cdot \nabla)\xi d\tau + \frac{1}{2} \int_V [(\mathbf{H} \cdot \nabla)\xi]^2 d\tau. \tag{19}$$

The field \mathbf{H} is an equilibrium field, and hence, from Eq. (1),

$$(\operatorname{curl}\mathbf{H}) \times \mathbf{H} = (\mathbf{H} \cdot \nabla)\mathbf{H} - \nabla(\frac{1}{2}H^2) = \nabla\phi, \tag{20}$$

or

$$(\mathbf{H} \cdot \nabla)\mathbf{H} = \nabla\psi, \tag{21}$$

with

$$\psi = \phi + \frac{1}{2}H^2 + \text{const.} \tag{22}$$

Transforming ΔW_1 by two partial integrations we find

$$\begin{aligned} \Delta W_1 &= \int H_\alpha H_\beta (\partial \xi_\alpha / \partial x_\beta) d\tau = \int \mathbf{H} \cdot \xi \mathbf{H} \cdot d\mathbf{S} \\ &\quad - \int \psi \xi \cdot d\mathbf{S} + \int \psi \operatorname{div} \xi d\tau, \end{aligned} \tag{23}$$

where $d\mathbf{S}$ denotes an element of the surface of V , and the integrals are taken over V .

In most cases the surface integrals of Eq. (23) vanish, due to the fact that ψ , $\mathbf{H} \cdot d\mathbf{S}$ or $\mathbf{H} \cdot \xi$ is zero. If so, we

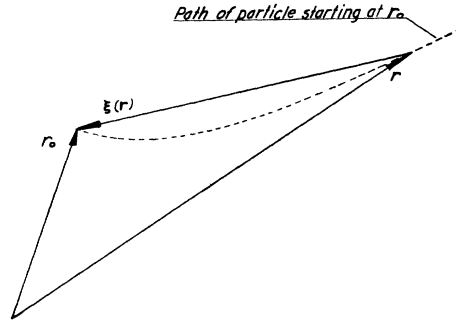


FIG. 3. Definition of displacement ξ .

are left with the expression

$$\Delta W = \int \psi \operatorname{div} \xi d\tau + \frac{1}{2} \int [(\mathbf{H} \cdot \nabla)\xi]^2 d\tau. \tag{24}$$

Putting $\psi=0$ at infinity, we see that $\psi = p + \frac{1}{2}H^2 - p_\infty$ is usually positive. The second integral is always positive. Since a negative ΔW means instability, we infer that the deformation ξ of an unstable field is such that $\operatorname{div} \xi$ is negative. The first integral of Eq. (24) is apparently of the first order in ξ , but actually both integrals are of the second order. The exact expression for $\operatorname{div} \xi$ may be calculated from the incompressibility condition that the functional determinant for the transformation from \mathbf{r} to $\mathbf{r} + \xi$, $\partial(\mathbf{r} + \xi)/\partial(\mathbf{r})$, be equal to one. However, since we only need $\operatorname{div} \xi$ to second order in ξ in order to treat initial stability, we may obtain it more easily from Eqs. (13) and (15).

Regarding ξ as a small quantity of first order we obtain from Eq. (15)

$$\mathbf{v} = -(\partial \xi / \partial t) + (\partial \xi / \partial t \cdot \nabla)\xi, \tag{25}$$

and hence, from Eq. (13)

$$\operatorname{div} \mathbf{v} = -\frac{\partial \xi_\alpha}{\partial x_\alpha} + \frac{\partial}{\partial x_\alpha} \left(\frac{\partial \xi_\alpha}{\partial x_\beta} \right) = 0$$

$$= -\frac{\partial}{\partial t} \left(-\frac{\partial \xi_\alpha}{\partial x_\alpha} + \frac{1}{2} \frac{\partial \xi_\alpha}{\partial x_\beta} \frac{\partial \xi_\beta}{\partial x_\alpha} \right) \quad (26)$$

or, finally,

$$\operatorname{div} \xi = \frac{\partial \xi_\alpha}{\partial x_\alpha} = \frac{1}{2} \frac{\partial \xi_\alpha}{\partial x_\beta} \frac{\partial \xi_\beta}{\partial x_\alpha} \quad (27)$$

correct to second order. Hence, we write

$$\Delta W = \frac{1}{2} \int \left(\psi \frac{\partial \xi_\alpha}{\partial x_\beta} \frac{\partial \xi_\beta}{\partial x_\alpha} + H_\alpha H_\beta \frac{\partial \xi_\gamma}{\partial x_\alpha} \frac{\partial \xi_\gamma}{\partial x_\beta} \right) d\tau, \quad (28)$$

which is the required expression for the change of magnetic energy.

We see that fields with $\psi = \text{const}$ are stable. Since

$$\nabla \psi = (\mathbf{H} \cdot \nabla) \mathbf{H} = H(\partial H / \partial s) \mathbf{t} + (H^2 / R) \mathbf{n}, \quad (29)$$

where \mathbf{t} and \mathbf{n} denote unit tangent and principal normal, and R the radius of principal curvature of a line of force, we see that $\psi = 0$ means that the lines of force are straight lines.

We split the deformation tensor $\partial \xi_\alpha / \partial x_\beta$ into a symmetric part $S_{\alpha\beta} = \frac{1}{2}(\partial \xi_\alpha / \partial x_\beta + \partial \xi_\beta / \partial x_\alpha)$ and a skew part $A_{\alpha\beta} = \frac{1}{2}(\partial \xi_\alpha / \partial x_\beta - \partial \xi_\beta / \partial x_\alpha)$. In this way Eq. (27) becomes

$$\operatorname{div} \xi = \frac{1}{2}(S_{\alpha\beta} S_{\alpha\beta} - A_{\alpha\beta} A_{\alpha\beta}), \quad (30)$$

the right-hand side of the equation being the difference between two sums of squares.

Since

$$A_{\alpha\beta} A_{\alpha\beta} = \frac{1}{2}(\operatorname{curl} \xi)^2, \quad (31)$$

a negative value of $\operatorname{div} \xi$ or a decrease of energy is due to the rotation of the elements of the liquid.

V. STABILITY OF A TWISTED CYLINDER FIELD

In this section we investigate the stability of the magnetic field in a long cylinder of radius R and length L . In cartesian coordinates the magnetic field is supposed to be

$$\mathbf{H} = [-H_\varphi(r)y/r, H_\varphi(r)x/r, H_z(r)], \quad (32)$$

where $r^2 = x^2 + y^2$, $\cos \varphi = x/r$. The lines of force are helical. The magnetic force $\mathbf{F}_m = (\operatorname{curl} \mathbf{H}) \times \mathbf{H}$, is radial, and

$$F_m = -\frac{H_\varphi}{r} \frac{\partial}{\partial r} (rH_\varphi) - H_z \frac{\partial H_z}{\partial r} = -\frac{\partial \phi}{\partial r}. \quad (33)$$

Hence,

$$\frac{\partial \psi}{\partial r} = \frac{\partial}{\partial r} \left(\phi + \frac{H_\varphi^2 + H_z^2}{2} \right) = -\frac{H_\varphi^2}{r}. \quad (34)$$

In order to simplify the calculation we put $\xi = \xi' + \xi''$, with $\operatorname{div} \xi' = 0$, ξ'' of the order ξ^2 , and $\operatorname{div} \xi'' = \frac{1}{2}(\partial \xi_\alpha /$

$\partial x_\beta)(\partial \xi_\beta / \partial x_\alpha) = \frac{1}{2}(\partial \xi_\alpha' / \partial x_\beta)(\partial \xi_\beta' / \partial x_\alpha)$ to second order. Further, $[(\mathbf{H} \cdot \nabla) \xi]^2 = [(\mathbf{H} \cdot \nabla) \xi']^2$ to the same order. Accordingly, Eq. (28) together with

$$\operatorname{div} \xi = 0 \quad (35)$$

gives ΔW correct to the second order.

We study a simple displacement of the form

$$\xi = [c \cos bx \sin az, 0, c' \sin bx \cos az]. \quad (36)$$

It is easily seen that the surface integrals of Eq. (23) vanish, provided a is chosen so as to give a whole number of full arcs in the z direction.

From Eq. (35) we obtain

$$bc + ac' = 0. \quad (37)$$

The deformation tensor is

$$\frac{\partial \xi_\alpha}{\partial x_\beta} = \begin{pmatrix} -bc \sin bx \sin az, & 0, & ac \cos bx \cos az \\ 0, & 0, & 0 \\ bc' \cos bx \cos az, & 0, & -ac' \sin bx \sin az \end{pmatrix} \quad (38)$$

and

$$\frac{1}{2} \frac{\partial \xi_\alpha}{\partial x_\beta} \frac{\partial \xi_\beta}{\partial x_\alpha} = (bc)^2 \{ \sin^2 bx \sin^2 az - \cos^2 bx \cos^2 az \}. \quad (39)$$

Assuming $bR \ll 1$, i.e., the field long compared with the radius, we obtain

$$\frac{1}{2} \frac{\partial \xi_\alpha}{\partial x_\beta} \frac{\partial \xi_\beta}{\partial x_\alpha} \approx - (bc)^2 \cos^2 az \quad (40)$$

in which we have used Eq. (37).

Further,

$$(\mathbf{H} \cdot \nabla) \xi \approx [H_z ac \cos az, 0, -H_\varphi (y/r) bc' \cos az] \quad (41)$$

and

$$[(\mathbf{H} \cdot \nabla) \xi]^2 \approx H_z^2 (ac)^2 \cos^2 az + H_\varphi^2 (bc')^2 \sin^2 \varphi \cos^2 az. \quad (42)$$

Integrating over φ and z (whole number of periods), we obtain

$$\Delta W \sim (ac)^2 \int_0^R \frac{1}{2} H_z^2 r dr + (bc')^2 \int_0^R \frac{1}{4} H_\varphi^2 r dr - (bc)^2 \int_0^R \psi r dr. \quad (43)$$

Observing that $acbc' = -(bc)^2$ and putting

$$\int_0^R \frac{1}{2} H_z^2 r dr = \alpha, \quad \int_0^R \psi r dr = \beta, \quad \int_0^R \frac{1}{4} H_\varphi^2 r dr = \gamma,$$

the condition for a negative ΔW is

$$\beta^2 > 4\alpha\gamma. \quad (44)$$

Since

$$\beta = \int_0^R \psi r dr = - \int_0^R \frac{1}{2} r^2 (\partial \psi / \partial r) dr = - \frac{1}{2} \int_0^R H_\varphi^2 r dr = 2\gamma, \quad (45)$$

the condition becomes

$$\int_0^R H_\varphi^2 r dr > 2 \int_0^R H_z^2 r dr \quad (46)$$

or

$$\langle H_\varphi^2 \rangle_{Av} > 2 \langle H_z^2 \rangle_{Av}, \quad (47)$$

the symbol $\langle \rangle_{Av}$ standing for a mean value over the cross section. The result may also be stated as follows: the field becomes unstable when the magnetic energy due to the twisting exceeds double the energy of the non-twisted field.

For the uniformly twisted field considered by Alfvén, $H_\varphi = kH_z r/R$ and $\langle H_\varphi^2 \rangle_{Av} = \langle H_z^2 \rangle_{Av} k^2/4$, and hence, the field is unstable if $k > 2$. This is of the same order of magnitude as Alfvén found by considering the initial and final states at the forming of a loop.

For the force-free field given by Eqs. (3) and (4), we find

$$\langle H_\varphi^2 \rangle_{Av} = \langle H_z^2 \rangle_{Av}. \quad (48)$$

This field is accordingly stable for this displacement.

The geometry of the disturbance is seen in Fig. 4. The rotation of the cross sections is obvious. Here we have neglected the influence of the field outside the cylinder. It is possible to take account of this by considering a field and a displacement, radially distributed according to a gaussian function. The result is that a harder twisting is required if R and L are of the same order. The main displacement will occur in the central part of the field.

This result makes a little doubtful the assumption by Bullard¹ of the existence of a strong tangential magnetic field inside the earth.

VI. THE MOST CRITICAL DISPLACEMENTS

The simple disturbance considered in Sec. IV is certainly not the most critical one. The coefficient 2 in inequality (47) should probably be less. A better value might be obtained by considering a more general displacement containing a larger number of adjustable coefficients. This is rather laborious, however. By means of a variational principle the problem of stability may be given a different form.

We try to find a ξ that makes

$$\int \psi \frac{\partial \xi_\alpha}{\partial x_\beta} \frac{\partial \xi_\beta}{\partial x_\alpha} d\tau = \min \quad (49)$$

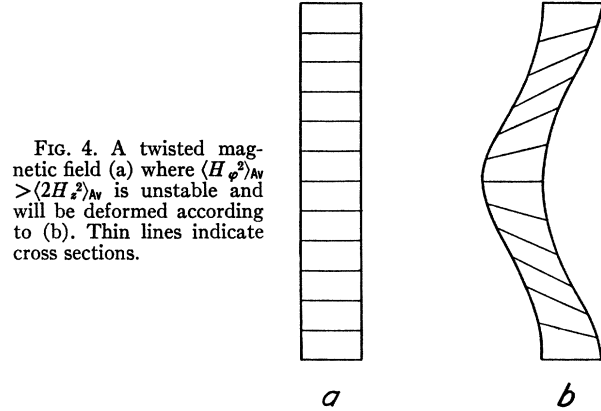


FIG. 4. A twisted magnetic field (a) where $\langle H_\varphi^2 \rangle_{Av} > 2 \langle H_z^2 \rangle_{Av}$ is unstable and will be deformed according to (b). Thin lines indicate cross sections.

when

$$\int H_\alpha H_\beta \frac{\partial \xi_\gamma}{\partial x_\alpha} \frac{\partial \xi_\gamma}{\partial x_\beta} d\tau = \text{const} = \Delta W_0. \quad (50)$$

This is equivalent to making

$$I(\xi) = \int \left\{ \psi \frac{\partial \xi_\alpha}{\partial x_\beta} \frac{\partial \xi_\beta}{\partial x_\alpha} + \lambda H_\alpha H_\beta \frac{\partial \xi_\gamma}{\partial x_\alpha} \frac{\partial \xi_\gamma}{\partial x_\beta} \right\} d\tau = \min. \quad (51)$$

The variational equations for this problem are

$$\frac{\partial}{\partial x_\gamma} \left(H_\alpha H_\beta \frac{\partial \xi_\beta}{\partial x_\alpha} \right) + \frac{1}{\lambda} \frac{\partial \psi}{\partial x_\alpha} \frac{\partial \xi_\alpha}{\partial x_\beta} = 0 \quad (\beta = 1, 2, 3) \quad (52)$$

in which Eq. (35) has been used. Multiplying Eq. (52) by ξ_β and integrating over the volume, we find

$$0 = \int \left\{ \psi \frac{\partial \xi_\alpha}{\partial x_\beta} \frac{\partial \xi_\beta}{\partial x_\alpha} + \lambda H_\alpha H_\beta \frac{\partial \xi_\gamma}{\partial x_\alpha} \frac{\partial \xi_\gamma}{\partial x_\beta} \right\} d\tau, \quad (53)$$

and the total change of energy is then

$$\Delta W = (1 - \lambda) \Delta W_0. \quad (54)$$

Hence, if Eq. (52) has a solution with an eigenvalue $\kappa = 1/\lambda$, where $0 < \kappa < 1$, the field is unstable. This question is rather complicated, however, owing to the weak boundary conditions for ξ . The method would probably be to put Eq. (52) in the form of an integral equation.

The facts of main physical interest seem to be given by inequality (47), and there is therefore no reason to use more elaborate methods at present.

Grants from Naturvetenskapliga Forskningsrådet and Stiftelsen Lars Hiertas minne made possible this investigation at the University of Leeds.

The author wishes to thank Professor T. G. Cowling for his great interest in this work and for his invaluable help.