# On the Interpretation of Resonance Levels and Their Widths in Terms of the Scattering Length and the Effective Range

T. TEICHMANN
*Palmer Physical Laboratory, Princeton University, Princeton, New Jersey*
(Received January 17, 1951)

The so-called effective range formula for nuclear scattering is deduced from the nuclear dispersion formalism for a fairly general class of potentials. It is shown that only in the case of zero angular momentum scattering is the angular momentum uniquely determined from the low energy variation of the cross section. The relation between the scattering length and effective range on the one hand and the parameters of nuclear dispersion theory (resonance levels and reduced widths) is briefly discussed.

## I. INTRODUCTION

LOW energy nuclear collisions may be described by two types of parameters. In the case of $n$–$p$ scattering at zero angular momentum, for instance, it is usual to consider $\tan\delta=Q$ ($\delta$=phase shift), and to express $k\cot\delta=k/Q$ in the form

$$k\cot\delta=-1/a+\tfrac{1}{2}r_0k^2+pr_0^3k^4+\cdots, \qquad (1.1)$$

where $a$ is the scattering length, $r_0$ the effective range, etc.[1] The cross section is then given by

$$\sigma=(4\pi/k^2)[Q^2/(1+Q^2)]. \qquad (1.2)$$

Similar formulas may also be obtained for $p$–$p$ scattering (see Bethe, reference 1) and for scattering at higher angular momenta. In each case the quantity $k\cot\delta$ (or something closely related) is expanded in a Taylor series in $k^2$ (i.e., substantially in the energy), and the cross section is characterized by various *length* parameters, $a$, $r_0$, $p$, $\cdots$etc., which are in some way characteristic of the scattering potential.

On the other hand, when rather larger energy ranges are considered, or when the compound nucleus (formed by the two colliding nuclei) is rather complicated, it is also common to use certain *energy* parameters characteristic of the compound system, *viz.*, the resonance levels $E_\lambda$, and the reduced widths $\gamma_\lambda^2$. To this end one introduces the so-called derivative matrix[2] $R$ (which is, of course, a scalar in the case of pure scattering), given by

$$R=\sum_\lambda[\gamma_\lambda^2/(E_\lambda-E)] \qquad (1.3)$$

in terms of which[3]

$$Q=-(F-RF')/(G-RG'). \qquad (1.4)$$

Here $F/r_c$, $G/r_c$ are the regular and irregular solutions[4] of the two-particle colliding system, evaluated at $r=r_c$, $r$ being the relative separation of the two particles. $r_c$ is the "radius" of the compound system. For $r>r_c$, the system may be treated as a two-particle collision with the well-known centrifugal and coulomb interaction between the particles, for which the corresponding functions $F$, $G$ have been calculated by Yost, Wheeler, and Breit.[4] (However, the following treatment does not depend on the particular forms of $F$ and $G$, and can be presumed to include a wide class of external potentials, for which $F$, $G$ have the forms given in (2.5) and (2.7). See also (2.8).) For $r<r_c$, the system is supposed merged into a compound nucleus characterized by $R$.[5]

From (1.1) and (1.4) it is clear that it must be possible to derive the right-hand side of (1.1) from the known properties of $R$, $F$, and $G$. This should render it possible to derive at once the generalization of (1.1) for all angular momenta, for a coulomb potential, and generally for a large class of external potentials (see the remark above). The result should be expected to illuminate the nature both of the energy parameters $E_\lambda$, $\gamma_\lambda^2$ and of the length parameters $a$, $r_0$, $\cdots$ etc. (Such a connection is also mentioned on related grounds by Blatt and Jackson.[1])

Such a relation is to be expected, not only for the reason given above, but also from the nature of the derivations of the two quantities $k\cot\delta$ and $R$, as given by Bethe[1] and Wigner,[6] respectively. Both these quantities are simple fractional linear forms in the logarithmic derivations of incoming and outgoing waves of unit amplitude (with energy dependent coefficients), and in both cases a functional equation of the type

$$f(E_2)-f(E_1)=(E_2-E_1)g(E_2,E_1) \qquad (1.5)$$

is obtained. In the derivation of (1.1) given by Bethe, $f=k\cot\delta$, and a solution for $f$ is obtained as a power series in $E$ (i.e., in $k^2$). In the dispersion-theoretic treatment of Wigner, $f=R$, and solution of the functional equation (1.5) (taking into account the particular form of $g(E_2,E_1)$) yields the expression (1.3) for $R$. It is thus natural to expect a fairly intimate connection

[1] See H. A. Bethe, Phys. Rev. **76**, 38 (1949), for a full and neat derivation of this result, which was originally surmised by Landau and Smorodinsky, J. Phys. Acad. Sci. U.S.S.R. **8**, 154 (1944), and first proved by variational methods by J. Schwinger [see J. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949)].

[2] E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947), $R$ is simply the inverse logarithmic derivative of $r\times$ the internal wave function ($r<r_c$) evaluated at $r=r_c$.

[3] See E. P. Wigner, "On the statistical distribution of the widths and spacings of nuclear resonance levels" (unpublished).

[4] Yost, Wheeler, and Breit, Phys. Rev. **49**, 174 (1937).

[5] The validity of such a model, and the changes implied by a more rigid adherence to reality have been discussed elsewhere. See reference 4 and also E. P. Wigner, Phys. Rev. **73**, 1002 (1948); T. Teichmann, Ph.D. thesis, Princeton (1949).

[6] E. P. Wigner, Phys. Rev. **70**, 15 (1946).

between $k\cot\delta$ and $R$, apart from the purely formal one mentioned earlier.

## II. DERIVATION: GENERAL METHOD

The method of deriving the "effective range" formula (1.1), or its analogs, from the expression for $Q$ is very simple, though some of the details may be a bit messy in the general case (arbitrary external potential) if the functions $F$ and $G$ are rather complicated.

To begin with, one writes

$$\cot\delta=1/Q=-(G-RG')/(F-RF')$$
$$=-\frac{G}{F}+\frac{k}{FF'}-\frac{k/F'^2}{(F/F')-R}. \quad (2.1)$$

This follows from the usual normalization adopted for $F$ and $G$

$$F'G-FG'=k, \quad (2.2)$$

the dash denoting differentiation with respect to $r$, not with respect to $kr$. The form (2.1) is used in order to avoid mixing the irregular function $G$ (with its often displeasing properties for small values of the argument), and expressions involving $R$. One also has

$$R=\sum_\lambda \frac{\gamma_\lambda^2}{E_\lambda-E}=\sum_{j=0}^{\infty} R_j k^{2j}, \quad (2.3)$$

where

$$\left.\begin{aligned} R_0&=\textstyle\sum_\lambda(\gamma_\lambda^2/E_\lambda),\\ R_j&=(\hbar^2/2m)^j(\textstyle\sum_\lambda \gamma_\lambda^2/E_\lambda^{j+1}).\end{aligned}\right\} \quad (2.4)$$

Since $F$ and $G$ are solutions of a second-order differential equation of the type

$$u''(r)+[k^2-W(r)]u(r)=0 \quad (2.5)$$

and satisfy the linear independence (wronskian) relation (2.2), they are expressible respectively in the forms

$$F=C(kr)^\mu\Phi, \quad \mu>0, \quad (2.6)$$

$$G=[(2\mu+1)C]^{-1}(kr)^{-\mu+1}\Psi+A(\log r+\bar g)F. \quad (2.7)$$

Here $C=C_\mu(k)$, $A=A_\mu(k)$, and $\bar g=\bar g_\mu(k)$ depends at most on $\mu$ and $k$, but not on $r$, while $\Phi$, $\Psi$ are power series in $kr$, beginning with the term 1, whose coefficients may also involve $k$. A particular example is given by the coulomb wave functions (see reference 4).

It should be noted that the assumption of the forms (2.6) and (2.7) for $F$ and $G$, respectively, implies a certain restriction on the potential $W(r)$. Roughly speaking, this means that $W(r)$ must satisfy the condition[7]

$$r^2W(r)\longrightarrow\text{constant}<\infty, \text{ as } r\longrightarrow 0. \quad (2.8)$$

Of course, all the usual potentials, such as centrifugal, coulomb, exponential, Yukawa, and gaussian satisfy this requirement.

$\Phi$ and $\Psi$ may both be expressed in the form[8]

$$\Phi(r)=\phi_0(r)+k^2\phi_1(r)+k^4\phi_2(r)+\cdots, \quad (2.9)$$

where the $\phi_n(r)$, $n=0, 1, 2, \cdots$ are independent of $k$, and satisfy

$$\phi_n(0)=\delta_{n0}.$$

Clearly $(r/\mu)\Phi'(r)+\Phi(r)=\Phi(r)$ has a similar form, and hence so have the functions $M=\Psi/\Phi$, $N=1/\Phi\bar\Phi$, $\Lambda=1/\bar\Phi^2$, and $\Upsilon=\Phi/\bar\Phi$. Thus

$$\frac{G'}{F'}=\frac{G}{F}-\frac{k}{FF'}=A(\log r+\bar g)+C^{-2}(kr)^{-2\mu+1}\left[\frac{M}{2\mu+1}-\frac{N}{\mu}\right]$$

and hence

$$\cot\delta=-A(\log r_c+\bar g)$$
$$-C^{-2}(kr)^{-2\mu+1}\left[\frac{M}{2\mu+1}-\frac{N}{\mu}+\frac{r_c\Lambda/\mu^2}{(r_c\Upsilon/\mu)-R}\right].$$

Writing

$$\bar g=\tfrac{1}{2}g+\tfrac{1}{2}g',$$

where $\frac{1}{2}g'$ is the $k$-independent part of $\bar g$, one readily finds that

$$\frac{1}{A}\cot\delta+\tfrac{1}{2}g+\frac{(kr)^{-2\mu+1}}{AC^2}\left\{\frac{M}{2\mu+1}-\frac{N}{\mu}+\frac{r_c\Lambda/\mu^2}{(r_c\Upsilon/\mu)-R}\right\}$$
$$=-(\log r_c+\tfrac{1}{2}g') \quad (2.10)$$

is independent of $k$, and hence of energy. If $A=0$, i.e., if both solutions of the differential equation (2.5) are expressible as a power series (about $kr=0$) times some power of $kr$, one finds instead

$$\cot\delta=-\frac{(kr_c)^{-2\mu+1}}{C^2}\left\{\frac{M}{2\mu+1}-\frac{N}{\mu}+\frac{(r_c\Lambda/\mu^2)}{(r_c\Upsilon/\mu)-R}\right\}. \quad (2.11)$$

Equations (2.3), (2.9) and the subsequent remarks ensure that the expression in the brackets in both (2.10) and (2.11) may be expanded in a power series in $k^2$, which will in general be of the form

$$-\alpha+\tfrac{1}{2}r_0k^2+pr_0^3k^4+\cdots, \quad (2.12)$$

where the coefficients $\alpha$, $r_0$, $p\cdots$ depend on $r_c$ and the various $R_j$, but not on $k$. (2.10) and (2.11) thus yield respectively, the formulas

$$k^{2\mu-1}C^2[\cot\delta+\tfrac{1}{2}Ag-A\beta]$$
$$=-r_c^{1-2\mu}[-\alpha+\tfrac{1}{2}r_0k^2+pr_0^3k^4\cdots] \quad (2.13)$$

and

$$k^{2\mu-1}C^2\cot\delta=-r_c^{1-2\mu}[-\alpha+\tfrac{1}{2}r_0k^2+pr_0^3k^4\cdots]. \quad (2.14)$$

[7] See P. M. Morse and M. Feshbach, *Methods of Theoretical Physics* (Technology Press, Cambridge, Massachusetts, 1946), for full mathematical details.

[8] Such an expansion has been given explicitly (for $\Phi$) in the case of a coulomb and centrifugal potential by J. G. Beckerley, Phys. Rev. **67**, 11 (1945).

The coefficients $C$, $A$, $g$, $\beta$, $\alpha$, $r_0$, $p$, $\cdots$, etc., may be evaluated as described above, but it is not of very great interest to do this in general. The particular cases usually treated are discussed below. Before leaving the general discussion it is perhaps not unimportant to note that for certain particular internal potentials ($r<r_c$), $R_0$ may have a value such that the term independent of $k$ in $(r_c\Upsilon/\mu)-R$ vanishes, and in this case the expression (2.12) must be augmented by a term in $1/k^2$. (See Eq. (3.8).)

### III. NO EXTERNAL INTERACTION

Consider first the case where there is no external interaction between the two colliding particles, so that the potential $W(r)$ has the form

$$W(r)=l(l+1)/r^2 \tag{3.1}$$

if the system has $l$ units of angular momentum. The case $l=0$ contains nearly all the essential details, and is treated at some length here, and the corresponding results and changes for $l>0$ will merely be stated at the end of this section.

For $l=0$

$$F(r_c)=\sin kr_c, \quad G(r_c)=\cos kr_c \tag{3.2}$$

and thus

$$\begin{aligned} k\cot\delta &= k\tan kr_c-(1+\tan^2 kr_c)/(k^{-1}\tan kr_c-R) \\ &= -a^{-1}+\tfrac{1}{2}r_0k^2+pr_0{}^3k^4+\cdots. \end{aligned} \tag{3.3}$$

One then finds directly that

$$a=r_c-R_0, \tag{3.4}$$

$$\tfrac{1}{2}r_0=r_c-r_c^2/a+(\tfrac{1}{3}r_c^3-R_1)/a^2, \text{ etc.} \tag{3.5}$$

Equation (3.4) shows that if the scattering length $a$ is greater than the radius $r_c$ of the compound system, then the latter must have at least one strongly bound state $E_1$, such that

$$\Big|\sum_{E_\lambda<0}(\gamma_\lambda^2/E_\lambda)\Big|>\sum_{E_\lambda>0}(\gamma_\lambda^2/E_\lambda).$$

Since $\delta$ (or the collision matrix $U$) is invariant to any changes made in the assumed radius $r_c$ of the compound system, provided only that the total potential is not changed, Eqs. (3.4), (3.5), etc., are to be regarded as expressing necessary relations which must hold between the parameter $r_c$ (which may be varied) and the various $R_n$. It has been pointed out elsewhere (Teichmann, reference 5) that the behavior of the "external" functions $F$ and $G$, and that of the derivative function $R$ just so compensate as to render $\delta$ (or $U$) independent of the choice of $r_c$ (or even of the boundary conditions on the internal system). The present treatment is interesting in that it shows much more plainly (though only in the case of pure scattering) the effect on $R$ of specific choices of $r_c$. (For instance if $r_c=a$ (provided $a>0$), then $R_0=0$.)

Equation (3.5) shows that $r_0$ is not necessarily positive, and that $r_0\gtrless 0$ only if

$$R_1\leq\frac{(r_c-R_0)^3}{3}+\frac{R_0{}^3}{3}. \tag{3.6}$$

This of course implies a certain restriction on the potentials. Examples of potentials for which $r_0<0$ have been given previously;[9] but even in the case of the square well $r_0$ may be less than zero for certain choices of $V_0r_c^2$ ($V_0$ = depth of well).

Since in this case

$$R_1=(k^2+\kappa^2)^{-\frac{1}{2}}\tan[(k^2+\kappa^2)^{\frac{1}{2}}r_c], \quad \kappa^2=2mV_0/\hbar^2,$$

one can readily calculate that $r_0<0$ if

$$\begin{aligned} (\lambda+\tfrac{3}{2})\pi-\frac{1}{(\lambda+\tfrac{3}{2})\pi(1-1/\sqrt{3})}\lesssim\kappa r_c\lesssim(\lambda+\tfrac{3}{2})\pi \\ -\frac{1}{(\lambda+\tfrac{3}{2})\pi(1+1/\sqrt{3})}, \end{aligned} \tag{3.7}$$

$\lambda=0, 1, 2, \cdots$. Qualitatively this means that $r_0$ will be less than zero[10] if there is at least one bound state and the first virtual state is just a little above zero energy. A little consideration shows that this result is also to be expected from the forms (2.4) of $R_0$ and $R_1$ combined with Eq. (3.6).

If $R_0$ should be fortuitously be equal to $r_0$, then the right-hand side of (3.3) must be replaced instead by the expression

$$-1/(\beta k^2)-\alpha_0+\alpha_1k^2, \tag{3.8}$$

where

$$\beta=\tfrac{1}{3}r_c^2-R_1. \tag{3.9}$$

Since the cross section is given by

$$\sigma=(4\pi/k^2)\sin^2\delta=4\pi/(k^2+k^2\cot^2\delta), \tag{3.10}$$

this means that while in the usual case (3.3) the cross section tends to the constant value

$$\sigma\to 4\pi a^2 \tag{3.11}$$

for zero energy, it may, under suitable conditions (*viz.*, for potentials for which $R_0=r_c$) go to zero as $k^4$ at low energies; i.e.,

$$\sigma\simeq 4\pi\beta^2k^4 \tag{3.12}$$

even though the scattering is at zero angular momentum (see the remarks below on scattering at $l>0$). In the case of the square well such a condition obtains

[9] See V. Bargmann, Revs. Modern Phys. **21**, 488 (1949). The author is indebted to Dr. Bargmann for this remark.

[10] Functions of the forms of $R$, whose imaginary part is positive in the upper half-plane and negative in the lower half-plane, and whose only singularities are poles, are known as $R$ functions, and have been considered in some detail by E. P. Wigner, Ann. Math. **53**, 36 (1951). He showed, for example, that a fractional linear transformation $(aR+b)/(cR+d)$, with positive determinant $ad-bc>0$ is also an $R$-function. The above simple example shows that $Q$ is *not* in general an $R$-function, though it may be approximated by one in any energy range in which the variation of $F$ and $G$ may be neglected.

when

$$\tan\kappa r_c=\kappa r_c, \tag{3.13}$$

i.e., for

$$\kappa r_c\simeq(\lambda+\tfrac{3}{2})\pi-[(\lambda+\tfrac{3}{2})\pi]^{-1}$$

which lies between the limits given in (3.7) above.

Equations (3.8) and (3.12) give the correct energy dependence of the scattering when the scattering length vanishes. Under more extreme conditions ($R_0=r_c$, $R_1=\frac{1}{3}r_c^3$) one can easily see that $k\cot\delta\sim k^{-4}$, so that the cross section tends to zero as $k^8$ (i.e., $E^4$), and so on. Finally, if there is no internal interaction at all, $R=F/F^1$ at all energies, and the cross section vanishes identically at all energies.

In the general case, $l>0$,

$$\begin{aligned}F_l(r)&=(\pi kr/2)^{\frac{1}{2}}J_{l+\frac{1}{2}}(kr),\\ G_l(r)&=(-1)^l(\pi kr/2)^{\frac{1}{2}}J_{-l-\frac{1}{2}}(kr).\end{aligned} \tag{3.14}$$

Referring to (2.6) and (2.7) one has $A=0$, $\mu=1$, and

$$C=C_l=1/(2l+1)!!=1/(2l+1)(2l-1)\cdots5\cdot3\cdot1. \tag{3.15}$$

Thus

$$k^{2l}(k\cot\delta_l)=-\alpha_l+\tfrac{1}{2}r_{0l}k^2+\cdots, \tag{3.16}$$

where

$$\begin{aligned}\alpha_l&=1/a_l\\ &=\frac{1}{2l+1}\left(\frac{r_c+lR_0^{(l)}}{r_c-(l+1)R_0^{(l)}}\right)\frac{r_c^{-2l-1}}{C_l^2},\end{aligned} \tag{3.17}$$

etc. The superscript $l$ on $R_0$, denotes that the $R$ function must be calculated for the compound system with angular momentum $l$; in general the $R^{(l)}$ will be different for different $l$'s. Hence, when there is no external interaction to affect the centrifugal field $l(l+1)/r^2$, the cross section will generally vary as $k^{4l}$ (i.e., as $E^{2l}$) for small values of energy, though in special cases (see remarks above in the case $l=0$) it may vary as[11] $k^{4l+4}$. For instance, in the case of a square well of radius $r_0$ and depth $V_0=\hbar^2\kappa^2/2m$ the condition for this exceptional case is

$$J_{l+\frac{1}{2}}(\kappa r_c)/J_{l-\frac{1}{2}}(\kappa r_c)=\kappa r_c/(1-l). \tag{3.18}$$

(See also (3.13) above.)

Consideration of the significance of the various quantities involved in (2.10), say, shows that the above phenomenon may generally be expected to occur when the logarithmic derivative of the zero energy internal wave function (i.e., the wave function of the finite isolated compound system) taken for $r=r_c$, is equal to the logarithmic derivative of the zero energy regular external wave function evaluated at the same radius, i.e., when

$$R\Big|_{\substack{k=0\\ r=r_c}}=\frac{F}{F'}\Big|_{\substack{k=0\\ r=r_c}} \tag{3.19}$$

It is thus clear that the energy variation of the cross section at low energies does *not* uniquely determine the angular momentum $l$ except in the case $l=0$. For instance, a low energy variation $\sigma\sim E^2$ may be due *eilher* to $P$ scattering, or to a peculiarity of the potential leading to condition (3.19), etc.

## IV. EXTERNAL COULOMB FIELD

With the preparation of Sec. II, and the formulas of Yost, Wheeler, and Breit, it is no great hardship to compute the expression for $k\cot\delta$ in the case of a coulomb force between the two colliding particles.

Using the notation of II and III, and the results of YWB, one has

$$C=C_l'=C_l\left[(1+\eta^2)\left(1+\frac{\eta^2}{4}\right)\cdots\left(1+\frac{\eta^2}{l^2}\right)\right]^{\frac{1}{2}}\times\left(\frac{2\pi\eta}{(\exp2\pi\eta)-1}\right)^{\frac{1}{2}}, \tag{4.1}$$

$C_l$ being given as $1/(2l+1)!!$ by (3.15) and

$$A=(e^{2\pi\eta}-1)/\pi, \tag{4.2}$$

$$g_l(k)=2\log k+2\sum_1^l\frac{s}{s^2+\eta^2}+2\eta^2\sum_1^\infty\frac{1}{s(s^2+\eta^2)}, \tag{4.3}$$

$$g_l'=2\left(\gamma-\sum_1^{2l+1}\frac{1}{s}\right)+2\log2,$$

$$\gamma=0.57\cdots\text{ (Euler's constant)}, \tag{4.4}$$

where

$$\eta=1/r_bk \tag{4.5}$$

with

$$r_b=\hbar^2/mZ_1Z_2e^2 \tag{4.6}$$

(the Bohr radius of the colliding system). Then one obtains

$$\prod_{s=1}^{l}\left(k^2+\frac{1}{s^2r_b^2}\right)\left[\frac{2\pi\cot\delta_l}{e^{2\pi\eta}-1}+g_l(k)\right]=r_b(-\alpha_l'+\tfrac{1}{2}r_{0l}'k^2+\cdots). \tag{4.7}$$

The quantity $[2\pi\cot\delta_l/(e^{2\pi\eta}-1)+g_l(k)]$ will be denoted by $f_l(k)$ in accordance with the notation of Breit, Condon, and Present.[12]

The coefficient

$$\prod_1^l\left(k^2+\frac{1}{s^2r_b^2}\right)\frac{2\pi}{e^{2\pi\eta}-1}$$

of $\cot\delta_l$ tends to $k^{2l}\cdot k$ for large $k$, but for small $k$ it tends to zero as $\exp(-2\pi/r_bk)$, so that $\cot\delta_l$ tends to infinity much more rapidly than $k^{-2l-1}$ (which occurs for uncharged particle ($n$–$p$) scattering), and the form

[11] See also N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, London, 1950).

[12] Breit, Condon, and Present, Phys. Rev. **50**, 825 (1936).

of the cross section as a function of $k$ (or energy) is substantially independent of $l$ for small energies. As had already been pointed out by Bethe[1] for the case $l=0$, this shows that the coulomb effects swamp both the specifically nuclear, and the centrifugal effects at low energies.

The quantities $\alpha'$, $r_0'$, $\cdots$, etc., are functions of the various $R_n$ and the functions $\phi_0(r_c)$, etc., described in Sec. II. Since $r_c/r_b$ is small ($\sim 1/20$), reference to YWB or Beckerley shows that it is a good approximation to use the expression (3.4) (3.5) (3.17) (*mutatis mutandis*) to determine $\alpha_l'$, $\gamma_{0l}'$, $\cdots$, etc. Thus one finds

$$\alpha_l' \simeq \alpha_l + \frac{1}{r_b}(r_b{}^l l!)^2[2\log r_c + g_l'], \tag{4.8}$$

etc., $\alpha_l$ being given in form by (3.17). It should be noted that $R$ will not in general be the same as in the case of uncharged particle scattering. Only in the case of identical compound systems may the same $R$ be used in (3.17) and (4.8).

## V. RELATION TO THE THEORY OF NUCLEAR REACTIONS

If the collision is one in which the compound system can disintegrate in several ways, then considerations of the above type become much more complicated and less perspicuous, both for "proper" reactions (in which the outgoing components are different from the incoming ones), and for scattering. Consequently, only some general comments are made here about the analogy.

In general, the cross section for a reaction[13] "$st$" is given by

$$\sigma_{st} = (\pi/k_s{}^2)\,|i(1-U)_{st}|^2, \tag{5.1}$$

where $U$ is the $n$-dimensional unitory symmetric collision matrix. (The subscripts $s$ and $t$, ranging from 1 to $n$, denote the $n$ possible initial or final states of the system.) $U$ itself is given by

$$U = \omega \frac{1-iC+iBRB}{1+iC-iBRB}\omega, \tag{5.2}$$

where $\omega$, $C$, and $B$ are diagonal matrices, given in terms of the regular and irregular functions $F$, $G$ of the outgoing or incoming pairs of components ($F$ is a diagonal matrix with elements $F_s$, $s=1, \cdots n$) by

$$\begin{aligned} \omega &= F'/(F'^2+G'^2)^{\frac{1}{2}} + iG'/(F'^2+G'^2)^{\frac{1}{2}}, \\ C &= (FF'+GG')/k, \\ B &= (F'^2+G'^2)^{\frac{1}{2}}/\sqrt{k}. \end{aligned} \tag{5.3}$$

(All these quantities are supposed evaluated at the boundary of the compound systems, i.e., for $r_s = r_{sc}$, etc.) $R$ is the symmetric derivative matrix with elements

$$R_{st} = \sum_\lambda [\gamma_{\lambda s}\gamma_{\lambda t}/(E_\lambda - E)]. \tag{5.4}$$

In the case of pure scattering considered in the earlier part of this paper $U$ is simply a scalar, and it is most convenient to express $\sin^2\delta = \frac{1}{4}|i(1-U)|^2$ in terms of $\cot\delta = (1+U)/i(1-U)$, *viz.*, $\sin^2\delta = 1/(1+\cot^2\delta)$. This simple relation is no longer true in general if $U$ is a matrix and it is necessary to consider $|i(1-U)|^2$ directly.

In order to see the significance of the quantities one need merely note that, in the case of pure scattering, formulas of the type

$$f(k)\cot\delta = -1/a + \tfrac{1}{2}r_0k^2 + \cdots \tag{5.5}$$

are obtained in most cases, and that the corresponding cross sections are

$$\sigma = 4\pi(f(k)/k)^2 a^2[1 + k^2a^2[r_0/a - f^2(k)/k^2]\cdots]. \tag{5.6}$$

The factor $(f(k)/k)^2$ gives the energy dependence of the cross section at small energies (e.g., $\sim 1$ in the case of no external interaction, $\sim k^{4l}$ for scattering at angular momentum $l$, etc.).

This energy dependence has already been investigated very fully in the general case (many reactions) by Wigner,[14] and, in fact, the value of the first coefficient $a^2$ is given implicitly in this treatment. Because $U$ is a rather complicated matrix quotient and

$$(1-U) = 1 + \omega^2 - 2\omega(1+iC-iBRB)^{-1}\omega, \tag{5.7}$$

it is rather difficult to see what the coefficients really look like in the general analog of (5.5), even in the case of proper reactions, ($s \neq t$) when one need only consider $[(1+iC-iBRB)^{-1}]_{st}$. Reference to (2.6), (2.7), and (5.3) shows that the matrices in this last expression do not always have the simple form of Taylor series in $k^2$, beginning with 1 (they may, for instance, have an extra factor $k$), and it is only after taking the absolute value of the matrix element squared that the desired form is obtained.

It is not proposed to carry this general discussion any further here. Even with simple examples (such as are treated in a different connection in reference 13) it is rather difficult to obtain perspicuous results beyond those given by Wigner. It is hoped to give a fuller discussion at some later opportunity.

## ACKNOWLEDGMENTS

The author is indebted to Professor V. Bargmann for some helpful discussions and to Professor E. P. Wigner for prepublication copies of his papers (references 3 and 10). This work was assisted in part by the AEC and by the Higgins Fund.

[13] See E. P. Wigner and L. Eisenbud (reference 2) and T. Teichmann, Phys. Rev. **77**, 506 (1950) for further details of this formalism.

[14] E. P. Wigner, Phys. Rev. **73**, 1002 (1948).