Fredholm Structures in Positron Theory*

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THE integral equation for the matter field in positron theory—the theory of a quantized Dirac field interacting with a given external electromagnetic field—

$$\psi(x)=\psi_0(x)+e\int K^F(xx')\psi(x')dx' \quad (1)$$

was investigated from the point of view of the Fredholm method.[1] The symbol K^F is defined as

$$K_{\alpha\beta}{}^F(xx')\equiv iS_{\alpha\sigma}{}^F(xx')\gamma_{\sigma\beta}{}^\mu A^\mu(x'), \quad (2)$$

where $S^F(x)$ is the well-known Feynman propagation function. It was found that the Fredholm determinant $D_0(e)$ derived from (2) is equal to the vacuum expectation value of the Heisenberg S-operator. A set of operators $\Delta_0(e)$, $\Delta_1(e)$, $\cdots\Delta_n(e)\cdots$ [where $\Delta_0(e)=D_0(e)$], whose matrix elements $\langle x_1x_2\cdots x_n|\Delta_n(e)|y_1y_2\cdots y_n\rangle$ are determinantal representations of the nth particle parts of the S matrix, was also constructed. The set of operators $\{\Delta_n(e)\}$ is isomorphic to a set of operators $\{D_n(e)\}$ whose matrix elements $\langle x_1x_2\cdots x_n|D_n(e)|y_1y_2\cdots y_n\rangle$ are the nth Fredholm minors constructed on K^F. The set $\{\Delta_n(e)\}$ has many but not all of the functional properties of $\{D_n(e)\}$. The situation is somewhat similar to that obtaining between an ordinary exponential function and the chronologically ordered exponential of Dyson.

The properties of $\{D_n(e)\}$ have been extensively investigated in classical analysis. One can therefore utilize the isomorphism between $\{\Delta_n(e)\}$ and $\{D_n(e)\}$ to construct various representations of $\langle S\rangle_n$. The determinantal representation of $\{\Delta_n(e)\}$ are interesting in so far as they clearly exhibit the operation of the exclusion principle for real and virtual particles.

We shall illustrate the method by considering the somewhat trivial case of $\Delta_0(e)$ where the isomorphism degenerates into an equality $\Delta_0(e)=D_0(e)$.

We write

$$H\equiv ieT(\psi_\alpha(x)\phi_\alpha(x)). \quad (3)$$

Here T is the ordering operator of Wick[2]

$$T(\psi_\alpha(x)\phi_\alpha(x'))=\theta(x^0-x'^0)\psi_\alpha(x)\phi_\alpha(x')-\theta(x'^0-x^0)\phi_\alpha(x')\psi_\alpha(x); \quad (4)$$

$\theta(x)$ is the usual step function: $\theta(|x|)=1$, $\theta(-|x|)=0$ for $x\neq 0$; $\theta(0)=\frac{1}{2}$. The letter $\phi_\alpha(x)$ stands for $\bar{\psi}_\beta(x)\gamma_{\beta\alpha}{}^\mu A^\mu(x)$. We also have

$$\langle T(\psi_\alpha(x)\phi_\beta(x'))\rangle_0=-iK_{\alpha\beta}{}^F(xx'). \quad (5)$$

$\langle S\rangle_0$ may then be written as

$$\langle S\rangle_0=\sum_{n=0}^{\infty}[n!]^{-1}(-ie)^n\langle T[\psi(X_1)\phi(X_1)\cdots\psi(X_n)\phi(X_n)]\rangle_0, \quad (6)$$

where $\psi(X_1)=\psi_{\alpha(1)}(x_1)$, $\phi(X_1)=\phi_{\alpha(1)}(x_1)$ and all the repeated variables and indices are integrated and summed over, respectively. The convention $\theta(0)=\frac{1}{2}$ in the definition (4) guarantees the charge symmetry of Eq. (6). Wick's theorem[2] and Eq. (5) yield

$$\langle T[\psi(X_1)\phi(X_1)\cdots\psi(X_n)\phi(X_n)]\rangle_0 =(-i)^n\epsilon^{i(1)\cdots i(n)}K^F(X_1X_{i(1)})\cdots K^F(X_nX_{i(n)}), \quad (7)$$

and, consequently,

$$\langle S\rangle_0=D_0(e). \quad (8)$$

Having established contact with the Fredholm theory, we may now exploit the known properties of $D_0(e)$ to find another representation for $\langle S\rangle_0$. The integral equation for $f(x)$,

$$f=f_0+\lambda Kf, \quad (9)$$

is solved in the Fredholm theory by

$$f_0=f-\lambda[D_0(\lambda)]^{-1}D_1(\lambda)f_0. \quad (10)$$

Moreover,[3] we have

$$\mathrm{Tr}\,D_1(\lambda)=-D_0'(\lambda), \quad (11)$$

where the prime denotes differentiation with respect to the argument. Comparing (10) with the Liouville-Neumann solution,

$$f_0=f-\lambda[1-\lambda K]^{-1}Kf_0, \quad (12)$$

and making use of Eq. (11) and of $D_0(0)=1$, we immediately deduce the expressions

$$D_0(\lambda)=\exp[\mathrm{Tr}\log(1-\lambda K)], \quad (13)$$

$$D_1(\lambda)=K[1-\lambda K]^{-1}\exp[\mathrm{Tr}\log(1-\lambda K)] =K\exp[\sum_{n=1}^{\infty}n^{-1}\lambda^n(K^n-\mathrm{Tr}\,K^n)] =K\prod_{n=1}^{\infty}\exp[n^{-1}\lambda^n(K^n-\mathrm{Tr}\,K^n)]. \quad (14)$$

For $\langle S\rangle_0$ we get

$$\langle S\rangle_0=\exp[\mathrm{Tr}\log(1-eK^F)]. \quad (15)$$

Expression (15) could be derived more simply by an elegant method due to Glauber,[4] but the connection with the Fredholm theory is then somewhat obscured.

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1 R. Courant and D. Hilbert, *Methoden der Mathematischen Physik*, second edition, Vol. I, Chapter III.
2 G. C. Wick, Phys. Rev. **80**, 268 (1950).
3 See reference 1, Eq. (77).
4 R. J. Glauber, Harvard University thesis, June, 1949. Also "Some notes on multiple boson processes," to be published.

Isotope Shift in the Spectrum of Os I and the Magnetic Moment of Os[189]

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IN order to determine the ratio of the distances between the components due to even isotopes in the spectrum of osmium and the magnetic moment of Os^{189}, the hyperfine structure (hfs) of Os I[1] was studied, using a water-cooled hollow cathode discharge tube and a Fabry-Perot etalon.

The hfs of any clearly resolved line was found to consist of six components. Four of them, which are spaced equidistantly to a rough approximation, are due to the even isotopes (186, 188, 190, 192), and the remaining two are due to the odd isotope 189. The number of components due to Os^{189} in each line and their intensity ratio were found to be in harmony with the assumption of the nuclear spin $\frac{1}{2}$. Kawada[2] had previously shown that Os^{189} might have probably a spin of $\frac{1}{2}$.

The result of the measurements is given in Table I. In each

TABLE I. Displacement effect of the even osmium isotopes and the splitting of Os^{189} in the spectrum of Os I (in unit of 10^{-3} cm^{-1}).

		Even isotopes			Os^{189}
λ(A)	Combination[a]	$\Delta\nu$ (186–188)	$\Delta\nu$ (188–190)	$\Delta\nu$ (190–192)	Doublet distance
4794	$d^6s^2\,{}^5D_3-d^6sp\,{}^7D_3$		51.4	48.7	236
4447	$(14)_5-(53)_5$		100.3	95.2	···
4420	$d^6s^2\,{}^5D_4-d^6sp\,{}^7D_4$	67	57.5	55.4	350
4261	$d^6s^2\,{}^5D_4-d^6sp\,{}^7D_5$	67	64.5	58.8	538
4135	$d^6s^2\,{}^5D_3-d^6sp\,{}^7P_4$	58	54.4	47.9	333
4112	$d^6s^2\,{}^5D_1-(36)_2$		58.4	55.6	···
3876	$(9)_4-(53)_5$	~100	87.0	84.4	···
3752	$d^6s^2\,{}^5D_2-(35)_3$?		51	49	371
	Mean ratio	1.1_8 :	1.05 :	1	

[a] See reference 1.

line listed the heavier isotopes have greater wavelength, as expected from the indicated electron configuration of the terms. The