Fredholm Structures in Positron Theory*

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THE integral equation for the matter field in positron theory—the theory of a quantized Dirac field interacting with a given external electromagnetic field—

$$\psi(x)=\psi_0(x)+e\int K^F(xx')\psi(x')dx' \tag{1}$$

was investigated from the point of view of the Fredholm method.[1] The symbol K^F is defined as

$$K_{\alpha\beta}{}^F(xx')\equiv iS_{\alpha\sigma}{}^F(xx')\gamma_{\sigma\beta}{}^\mu A^\mu(x'), \tag{2}$$

where $S^F(x)$ is the well-known Feynman propagation function. It was found that the Fredholm determinant $D_0(e)$ derived from (2) is equal to the vacuum expectation value of the Heisenberg S-operator. A set of operators $\Delta_0(e)$, $\Delta_1(e)$, $\cdots\Delta_n(e)\cdots$ [where $\Delta_0(e)=D_0(e)$], whose matrix elements $\langle x_1x_2\cdots x_n|\,\Delta_n(e)\,|y_1y_2\cdots y_n\rangle$ are determinantal representations of the nth particle parts of the S matrix, was also constructed. The set of operators $\{\Delta_n(e)\}$ is isomorphic to a set of operators $\{D_n(e)\}$ whose matrix elements $\langle x_1x_2\cdots x_n|\,D_n(e)\,|y_1y_2\cdots y_n\rangle$ are the nth Fredholm minors constructed on K^F. The set $\{\Delta_n(e)\}$ has many but not all of the functional properties of $\{D_n(e)\}$. The situation is somewhat similar to that obtaining between an ordinary exponential function and the chronologically ordered exponential of Dyson.

The properties of $\{D_n(e)\}$ have been extensively investigated in classical analysis. One can therefore utilize the isomorphism between $\{\Delta_n(e)\}$ and $\{D_n(e)\}$ to construct various representations of $\langle S\rangle_n$. The determinantal representation of $\{\Delta_n(e)\}$ are interesting in so far as they clearly exhibit the operation of the exclusion principle for real and virtual particles.

We shall illustrate the method by considering the somewhat trivial case of $\Delta_0(e)$ where the isomorphism degenerates into an equality $\Delta_0(e)=D_0(e)$.

We write

$$H\equiv ieT(\psi_\alpha(x)\phi_\alpha(x)). \tag{3}$$

Here T is the ordering operator of Wick[2]

$$T(\psi_\alpha(x)\phi_\alpha(x'))=\theta(x^0-x'^0)\psi_\alpha(x)\phi_\alpha(x')-\theta(x'^0-x^0)\phi_\alpha(x')\psi_\alpha(x); \tag{4}$$

$\theta(x)$ is the usual step function: $\theta(|x|)=1$, $\theta(-|x|)=0$ for $x\neq 0$; $\theta(0)=\frac{1}{2}$. The letter $\phi_\alpha(x)$ stands for $\bar\psi_\beta(x)\gamma_{\beta\alpha}{}^\mu A^\mu(x)$. We also have

$$\langle T(\psi_\alpha(x)\phi_\beta(x'))\rangle_0=-iK_{\alpha\beta}{}^F(xx'). \tag{5}$$

$\langle S\rangle_0$ may then be written as

$$\langle S\rangle_0=\sum_{n=0}^{\infty}[n!]^{-1}(-ie)^n\langle T[\psi(X_1)\phi(X_1)\cdots\psi(X_n)\phi(X_n)]\rangle_0, \tag{6}$$

where $\psi(X_1)=\psi_{\alpha(1)}(x_1)$, $\phi(X_1)=\phi_{\alpha(1)}(x_1)$ and all the repeated variables and indices are integrated and summed over, respectively. The convention $\theta(0)=\frac{1}{2}$ in the definition (4) guarantees the charge symmetry of Eq. (6). Wick's theorem[2] and Eq. (5) yield

$$\langle T[\psi(X_1)\phi(X_1)\cdots\psi(X_n)\phi(X_n)]\rangle_0 =(-i)^n\epsilon^{i(1)\cdots i(n)}K^F(X_1X_{i(1)})\cdots K^F(X_nX_{i(n)}), \tag{7}$$

and, consequently,

$$\langle S\rangle_0=D_0(e). \tag{8}$$

Having established contact with the Fredholm theory, we may now exploit the known properties of $D_0(e)$ to find another representation for $\langle S\rangle_0$. The integral equation for $f(x)$,

$$f=f_0+\lambda Kf, \tag{9}$$

is solved in the Fredholm theory by

$$f_0=f-\lambda[D_0(\lambda)]^{-1}D_1(\lambda)f_0. \tag{10}$$

Moreover,[3] we have

$$\mathrm{Tr}\,D_1(\lambda)=-D_0'(\lambda), \tag{11}$$

where the prime denotes differentiation with respect to the argument. Comparing (10) with the Liouville-Neumann solution,

$$f_0=f-\lambda[1-\lambda K]^{-1}Kf_0, \tag{12}$$

and making use of Eq. (11) and of $D_0(0)=1$, we immediately deduce the expressions

$$D_0(\lambda)=\exp[\mathrm{Tr}\log(1-\lambda K)], \tag{13}$$

$$D_1(\lambda)=K[1-\lambda K]^{-1}\exp[\mathrm{Tr}\log(1-\lambda K)]$$
$$=K\exp[\sum_{n=1}^{\infty}n^{-1}\lambda^n(K^n-\mathrm{Tr}\,K^n)]$$
$$=K\prod_{n=1}^{\infty}\exp[n^{-1}\lambda^n(K^n-\mathrm{Tr}\,K^n)]. \tag{14}$$

For $\langle S\rangle_0$ we get

$$\langle S\rangle_0=\exp[\mathrm{Tr}\log(1-eK^F)]. \tag{15}$$

Expression (15) could be derived more simply by an elegant method due to Glauber,[4] but the connection with the Fredholm theory is then somewhat obscured.

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[1] R. Courant and D. Hilbert, *Methoden der Mathematischen Physik*, second edition, Vol. I, Chapter III.
[2] G. C. Wick, Phys. Rev. **80**, 268 (1950).
[3] See reference 1, Eq. (77).
[4] R. J. Glauber, Harvard University thesis, June, 1949. Also "Some notes on multiple boson processes," to be published.

Isotope Shift in the Spectrum of Os I and the Magnetic Moment of Os^{189}

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IN order to determine the ratio of the distances between the components due to even isotopes in the spectrum of osmium and the magnetic moment of Os^{189}, the hyperfine structure (hfs) of Os I[1] was studied, using a water-cooled hollow cathode discharge tube and a Fabry-Perot etalon.

The hfs of any clearly resolved line was found to consist of six components. Four of them, which are spaced equidistantly to a rough approximation, are due to the even isotopes (186, 188, 190, 192), and the remaining two are due to the odd isotope 189. The number of components due to Os^{189} in each line and their intensity ratio were found to be in harmony with the assumption of the nuclear spin $\frac{1}{2}$. Kawada[2] had previously shown that Os^{189} might have probably a spin of $\frac{1}{2}$.

The result of the measurements is given in Table I. In each line listed the heavier isotopes have greater wavelength, as expected from the indicated electron configuration of the terms. The

TABLE I. Displacement effect of the even osmium isotopes and the splitting of Os^{189} in the spectrum of Os I (in unit of 10^{-3} cm^{-1}).

		Even isotopes			Os^{189}
		$\Delta\nu$	$\Delta\nu$	$\Delta\nu$	Doublet
λ(A)	Combination[a]	(186–188)	(188–190)	(190–192)	distance
4794	$d^6s^2\,{}^5D_3-d^6sp\,{}^7D_3$		51.4	48.7	236
4447	$(14)_5-(53)_5$		100.3	95.2	···
4420	$d^6s^2\,{}^5D_4-d^6sp\,{}^7D_4$	67	57.5	55.4	350
4261	$d^6s^2\,{}^5D_4-d^6sp\,{}^7D_5$	67	64.5	58.8	538
4135	$d^6s^2\,{}^5D_3-d^6sp\,{}^7P_4$	58	54.4	47.9	333
4112	$d^6s^2\,{}^5D_1-(36)_2$		58.4	55.6	···
3876	$(9)_4-(53)_5$	~100	87.0	84.4	···
3752	$d^6s^2\,{}^5D_2-(35)_3$?		51	49	371
	Mean ratio	1.1[8] :	1.05 :	1	

[a] See reference 1.

ratio $\Delta\nu(186\text{–}188):\Delta\nu(188\text{–}190):\Delta\nu(190\text{–}192)$ can be regarded as constant within experimental error for all lines except λ4135, whose structure was possibly disturbed by the overlapping of another line of the combination $(13)_3$–$(60)_3$, though this was not well confirmed. The mean value of the ratio, excluding λ4135, is given in the last row of Table I. This ratio is not in agreement with the measurements of Kawada;[2] but the present investigation might be considered the more accurate, owing to the improved resolving power of the spectroscopic apparatus.

The ratios of the isotopic displacements of even isotopes in the heavy elements with even atomic number $Z(Z=74-82)$, are summarized in Table II. The magnitude of the shift decreases as

TABLE II. Ratio of spacings between consecutive even isotopes in heavy elements.

	Z \ N	126	-	124	-	122	-	120	-	118	-	116
Pb[a,b]	82			1	:	0.90						
Hg[a]	80					1	:	0.99	:	0.88		
Pt[c]	78									1	:	0.97

	Z \ N	116	-	114	-	112	-	110	-	108	-	106
Os	76			1	:	1.05	:	1.1[8]				
W[d]	74							1	:	1.13	:	1.02

[a] K. Murakawa and S. Suwa, J. Phys. Soc. Japan **5**, 429 (1950); **5**, 382 (1950).
[b] F. E. Geiger, doctoral thesis, university of Wisconsin (1950).
[c] S. Tolansky and E. Lee, Proc. Roy. Soc. (London) **A158**, 110 (1937).
[d] J. A. Vreeland and K. Murakawa, Phys. Rev. **83**, 229 (1951).

the neutron number N decreases from $N=126$ to 116, increases from $N=116$ to 108, and again begins to decrease from $N=108$. This indicates a variation in the effect caused by adding two neutrons to a nucleus.

The magnetic moment of Os^{189} can be calculated from the hfs of λ4420 and λ4261 (see Table I). Neglecting any splitting of the final term $d^6s^2\ {}^5D_4$ of Os^{189}, the coupling constant of the 6s electron, $a(6s)$, was found to be 0.79 cm^{-1}. Putting this value and $I=\frac{1}{2}$ into the well-known Goudsmit-Fermi-Segrè formula,

$$\mu(Os^{189})=+0.6\pm0.1 \text{ nm}$$

was obtained.

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[1] W. Albertson, Phys. Rev. **45**, 304 (1934).
[2] T. Kawada, Proc. Phys. Math. Soc. Japan **20**, 653 (1938).

Quantum Corrections to Transport Properties at High Temperatures

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IN the quantum theory of transport phenomena[1] the following formulas are given in Boltzmann statistics for the "cross sections," $\phi^{(1)}$ for diffusion, thermal diffusion, and viscosity of mixtures, and $\phi^{(2)}$ for viscosity and thermal conductivity:

$$\phi^{(1)}(g)=\frac{2g}{k^2}\sum_{l=0}^{\infty}(l+1)\sin^2(\eta_{l+1}-\eta_l), \tag{1a}$$

$$\phi^{(2)}(g)=\frac{2g}{k^2}\sum_{l=0}^{\infty}\frac{(l+1)(l+2)}{(2l+3)}\sin^2(\eta_{l+2}-\eta_l), \tag{1b}$$

in which $k=\mu g/\hbar$, μ and g are the reduced mass and the initial relative velocity of a pair of colliding molecules, and $\eta_l(k)$ is the phase-shift corresponding to the orbital quantum number, l. At very low temperatures formulas analogous to Eq. (1b), written for Fermi-Dirac and Bose-Einstein statistics, have been used for the calculation of the viscosity of the light and heavy isotopes of helium[2,3] and for light and heavy hydrogen,[4] by numerical evaluation of the phase shifts. At higher temperatures this method is not feasible because of the large number of phase-shifts required. Consequently, it is important that another method of attack be developed for those temperatures for which the deviations from classical behavior are small.

In this temperature region we have succeeded in expressing the cross sections (and hence all of the transport coefficients) as power series in Planck's constant; this was accomplished by using Kahn's[5] expressions for the phase shifts, which are derived from the W.K.B. method:

$$\eta_l(k)=\eta_l^{I}(k)+\eta_l^{II}(k)+\cdots, \tag{2}$$

$$\eta_l^{I}(k)=(\pi/2)bk-kr_m+k\int_{r_m}^{\infty}[(1-F)^{-\frac{1}{2}}-1]dr, \tag{2a}$$

$$\eta_l^{II}(k)=(\pi/16bk)-(1/24k)\int_{r_m}^{\infty}[(F'''/F')-(F''/F')^2][1-F]^{-\frac{1}{2}}dr, \tag{2b}$$

in which the variable, b, defined by $\hbar^2l(l+1)=\mu^2g^2b^2$, has been introduced; r_m is the distance of closest approach in a collision; $F(r)=[\varphi(r)/(\frac{1}{2}\mu g^2)]+[b^2/r^2]$, $F'=dF/dr$, etc., and $\varphi(r)$ is the intermolecular potential function.

The differences in the phase-shifts which occur in Eqs. (1a) and (1b) can be expressed in terms of the various derivatives of η_l with respect to b, with the result that

$$[\eta_{l+1}-\eta_l]=\tfrac{1}{2}\chi+\tfrac{1}{4}b\chi'(\hbar/\mu gb)+\tfrac{1}{16}[\chi-\psi+b\psi'+(4/3)b^2\chi''](\hbar/\mu gb)^2+O(\hbar^3), \tag{3a}$$

$$[\eta_{l+2}-\eta_l]=\chi+b\chi'(\hbar/\mu gb)+\tfrac{1}{8}[\chi-\psi+b\psi'+(16/3)b^2\chi''](\hbar/\mu gb)^2+O(\hbar^3), \tag{3b}$$

in which the primes on χ and ψ indicate differentiation with respect to b; χ, defined as $(2/k)(\partial\eta^I/\partial b)$, or

$$\chi(b,g)=\pi-2b\int_{r_m}^{\infty}[1-F]^{-\frac{1}{2}}r^{-2}dr, \tag{4a}$$

is the classical angle of deflection, and $\psi=16bk\eta^{II}$ or

$$\psi(b,g)=\pi-\tfrac{2}{3}b\int_{r_m}^{\infty}[(F'''/F')-(F''/F')^2][1-F]^{-\frac{1}{2}}dr. \tag{4b}$$

The final expressions for the cross sections may be obtained by substituting Eqs. (3a, b) into Eqs. (1a, b), developing $\sin(\eta_{l+n}-\eta_l)$ in a power series in $(\hbar/\mu gb)$, and transforming the summation over into an integration over b [using $dl=db(\mu g/\hbar)\{1-\frac{1}{8}(\hbar/\mu gb)^2+\cdots\}$]. The first term, proportional to $\hbar^0$, is the classical expression for $\phi^{(n)}$. The terms proportional to odd powers of $\hbar$ may be seen to vanish by partial integrations; we may finally write

$$\phi^{(n)}=\phi_{cl}^{(n)}+\phi_{qu}^{(n)}, \tag{5}$$

where

$$\phi_{cl}^{(n)}=g\int_0^{\infty}[1-\cos^n\chi]bdb, \tag{5a}$$

$$\phi_{qu}^{(1)}=(\hbar^2/8\mu^2g)\int_0^{\infty}[\chi-\psi+b\psi'+\tfrac{1}{3}b^2\chi'']\sin\chi(db/b)+O(\hbar^4), \tag{5b}$$

$$\phi_{qu}^{(2)}=(\hbar^2/8\mu^2g)\int_0^{\infty}\{[\chi-\psi+b\psi'+(4/3)b^2\chi'']\sin2\chi-\sin^2\chi\}(db/b)+O(\hbar^4). \tag{5c}$$

Chapman and Cowling[1] have shown that the transport properties may be expressed directly in terms of a set of quantities, $\Omega^{(n)}(s)$, which are functions of the temperature and are defined in terms of the quantities $\phi^{(n)}$ calculated above:

$$\Omega^{(n)}(s)=\pi^{\frac{1}{2}}\int_0^{\infty}\exp(-\gamma^2)\gamma^{2s+2}\phi^{(n)}d\gamma, \tag{6}$$

where $\gamma^2=\mu g^2/2\kappa T$, κ being Boltzmann's constant. Consequently, substitution of the expression (5) into Eq. (6) leads to a series development of each of the transport coefficients in a series of powers of $\hbar^2/\mu\kappa T$, the coefficients of which are functions of T and