C_n are appropriate constants, then one could demonstrate the equivalence of Eqs. (56) and (57).

Another difficulty is demonstrating rigorously the equivalence between the S-matrix as it is defined by Eqs. (47) and (48) of the stationary method, and its definition as a unitary matrix describing the state of the system at $t=\infty$ in terms of the state at $t=-\infty$; i.e., $\psi(\infty)=S\psi(-\infty)$. Actually, the important conclusions of this paper do not depend on a demonstration of the equivalence of these two definitions of the S-matrices.

However, many recent works in field theory have, in effect, depended on the assumption of the equivalence of these two definitions and it would be worthwhile to find a mathematically satisfactory demonstration of the equivalence.

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Imprisonment of Resonance Radiation in Gases. II

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This paper is a continuation of an earlier paper which treated the decay of resonance radiation in optically excited gases for the case of doppler-broadened radiation in plane-parallel enclosures. The treatment is here extended to a second type of enclosure geometry—infinite cylinders—and to a variety of spectral line shapes.

1. INTRODUCTION

THE phenomenon of imprisonment of resonance radiation in gases owes its existence to the selective absorbability of resonance lines by normal atoms of the emitting gas. Over a wide range of gas density this absorbability is so high that a resonance quantum emitted in the interior of a gas-filled enclosure has but a small chance of reaching the walls; hence, the eventual escape of a unit of atomic excitation energy from the enclosure generally takes place only after a large number of repeated emissions and absorptions. Under these conditions the radiation is said to be "imprisoned."

Perhaps the most direct way in which imprisonment manifests itself is in decay experiments with optically excited gases. In this type of experiment an enclosure of gas is irradiated with a beam of resonance radiation, which serves to excite some of the gas atoms to a given resonance state. The incident beam is then abruptly cut off, and the intensity of diffuse radiation, which is proportional to the concentration of atoms in the resonance state, is measured as a function of time. One observes essentially an exponential decay of the form $e^{-\gamma g t}$, where $1/\gamma$ is the radiative lifetime of an isolated atom and g, the "escape factor," is a dimensionless quantity characteristic of the imprisonment process. The quantity g may be regarded as the reciprocal of the number of emission and absorptions of an individual unit of atomic excitation prior to its escape from the enclosure.

In an experiment on the decay of the 6^3P_1 mercury resonance state, which combines optically with the ground state to emit the 2537A resonance line, Zemansky[1] observed values of g as low as 10^{-3}. The quantity g was also found to depend both on vapor density and enclosure geometry.

A theoretical study[2] of the decay problem was recently carried out by the author of the present paper. It was shown that g depends not only upon vapor density and enclosure geometry but also upon the shape of the resonance line. In particular, for an enclosure of the form of an infinite slab of thickness L and for a doppler-broadened resonance line, the following expression for g was obtained:

$$g=1.875/[k_0 L(\pi \log^{\frac{1}{2}} k_0 L)^{\frac{1}{2}}], \tag{1.1}$$

where k_0 is the absorption coefficient at the center of the resonance line. k_0, itself, is specified[3] in terms of the parameters of the system: gas density, gas temperature, wavelength of the line, and lifetime of the resonance state.

More recently,[4] measurements of the imprisonment of resonance radiation in mercury vapor over a wide range of vapor density were carried out at the Westinghouse Research Laboratories. In the region of density for which g had been evaluated—the doppler-broadening region—the agreement between theory and experiment was quite satisfactory.

[1] M. W. Zemansky, Phys. Rev. **29**, 513 (1927).
[2] T. Holstein, Phys. Rev. **72**, 1212 (1947), to be referred to hereafter as "I."
[3] A. C. G. Mitchell and M. W. Zemansky, *Resonance Radiation and Excited Atoms* (The Macmillan Company, New York, 1934) (to be referred to hereafter as "MZ") Chapter III, pp. 99–100 and Eq. (35).
[4] Alpert, McCoubrey, and Holstein, Phys. Rev. **76**, 1257 (1949).

In the course of preparation for these and other experiments to follow, calculations of g were performed for a second type of enclosure geometry—the infinite cylinder—and for a variety of spectral line shapes. These extensions of the theory constitute the subject matter of the present paper.

2. GENERAL METHOD

We present here a brief outline of the approach developed in I for the treatment of imprisonment phenomena with special emphasis on the decay problem.

The radiative transport of resonance excitation is formulated in terms of a Boltzmann-type integro-differential equation for the density of excited atoms, $n(\mathbf{r})$:

$$\partial n(\mathbf{r})/\partial t = -\gamma n(\mathbf{r}) + \gamma \int n(\mathbf{r}')G(\mathbf{r}', \mathbf{r})d\mathbf{r}'. \quad (2.1)$$

In (2.1) the integration goes over all volume elements, $d\mathbf{r}'$, of the enclosure. $G(\mathbf{r}', \mathbf{r})$, which represents the probability of radiation emitted at a point $\mathbf{r}'$ being absorbed in a unit volume element around the point $\mathbf{r}$, is given by the expression,

$$G(\mathbf{r}', \mathbf{r}) = -(1/4\pi\rho^2)(\partial T/\partial\rho), \quad (2.2)$$

where $\rho = |\mathbf{r}' - \mathbf{r}|$, and $T(\rho)$ is the probability of a resonance quantum traversing a distance ρ without being absorbed.

In evaluation of $T(\rho)$ it is necessary to take account of the rapid frequency variation of the absorption coefficient, $k(\nu)$, in the neighborhood of resonance. Denoting by $P(\nu)$ the frequency spectrum of the radiation emitted from a given volume element, we have for $T(\rho)$ the general expression,

$$T(\rho) = \int P(\nu)e^{-k(\nu)\rho}d\nu. \quad (2.3)$$

In the Appendix of Part I it is shown that the proportionality relation,

$$P(\nu) \propto k(\nu), \quad (2.4)$$

may be used when the shape of the resonance line is determined by either doppler- or pressure-broadening; in the latter case, it was assumed that $k(\nu)$ is given by the dispersion formula,

$$k(\nu) \propto 1/(1+[4\pi(\nu-\nu_0)/\gamma_p]^2), \quad (2.5)$$

where γ_p is proportional to the vapor density. In the present paper Eq. (2.4) will be assumed to hold for other types of pressure broadening; this extension is discussed in Appendix A.

The constant of proportionality in (2.4) is readily determined from the "integral absorption" relation,

$$\int k(\nu)d\nu = (\lambda_0^2 N/8\pi)(g_2/g_1)\gamma, \quad (2.6)$$

[see MZ pp. 95–96, Eq. (28); I, Eq. (2.5)] and from

the normalization requirement on $P(\nu)$: $\int P(\nu)d\nu = 1$. Thus,

$$k(\nu) \equiv \kappa P(\nu) = (\lambda_0^2 N/8\pi)(g_2/g_1)\gamma P(\nu), \quad (2.7)$$

where λ_0 is the wavelength at the center of the resonance line, N is the vapor density, and g_2, g_1 are the statistical weights of excited and ground states. Inserting Eq. (2.7) into Eq. (2.3), we readily evaluate $T(\rho)$ and, hence, $G(\mathbf{r}', \mathbf{r})$ for various line shapes.

The general solution of Eq. (2.1) may be written as a sum of product solutions of the form,

$$n_i(r)e^{-\gamma_{0i}t},$$

each of which satisfies an equation of the type,

$$(1-g_i)n_i(\mathbf{r}) = \int G(\mathbf{r}, \mathbf{r}')n_i(\mathbf{r}')d\mathbf{r}'. \quad (2.8)$$

As is discussed in I, the g's form a discrete positive-definite set of numbers; hence, after a sufficiently long time, only the "fundamental mode," with the lowest eigenvalue, is observed experimentally.

The evaluation of g and $n(r)$ for the fundamental mode may be achieved approximately by the Ritz variational method; the general procedure and its use in the derivation of Eq. (1.1) are presented in I.

3. LINE SHAPES FOR DOPPLER- AND PRESSURE-BROADENING

In this section we present a brief survey of the frequency variation of $k(\nu)$ for the cases of doppler- and pressure-broadening.

A. Doppler-Broadening

$k(\nu)$ is given by the relation,

$$k(\nu) = k_0 \exp[-((\nu-\nu_0)/\nu_0)^2(c/v_0)^2]. \quad (3.1)$$

In Eq. (3.1), ν_0 is the resonance frequency of an atom at rest, $v_0 = (2kT/M_{\text{atom}})^{\frac{1}{2}}$ and k_0 is given by the relation [I, Eq. (2.10)],

$$k_0 = \frac{\lambda_0^3 N}{8\pi} \frac{g_2}{g_1} \frac{\gamma}{\pi^{\frac{1}{2}}v_0}. \quad (3.2)$$

B. Pressure-Broadening

We include in our discussion two kinds of broadening: self-broadening (due to the interaction of an absorbing atom with others of the same kind) and foreign gas broadening. In the latter case, we assume that the foreign gas does not quench the resonance excitation of the absorbing atoms by collisions of the second kind.

As was stated in I, pressure broadening can often be represented by the dispersion distribution, which in its most general form reads:[5]

$$k(\nu) = k_p/\{1+[4\pi(\nu-\nu_0-\beta_p)/\gamma_p]^2\}. \quad (3.3)$$

[5] H. M. Foley, Phys. Rev. 69, 616 (1946); E. Lindholm, Arkiv. Mat. Astron. Fysik. 32A, No. 17 (1945).

Here β_p and γ_p are directly proportional to the density, N', of the broadening agent. (For self-broadening $N'=N$.) k_p is obtained from Eq. (2.6); thus,

$$k_p = \frac{\lambda_0^2 N}{2\pi} \frac{g_2}{g_1} \frac{\gamma}{\gamma_p}. \tag{3.4}$$

In discussing Eq. (3.3) we first point out that this line shape is generally unobservable in the immediate core of the line because of doppler-broadening. However, as shown in MZ, Appendix I, for frequencies whose distance $\Delta\nu$ from the center of the line is large compared to $\nu_0 v_0/c$, effects of doppler-broadening may be ignored; for such frequencies, since γ_p (as well as β_p) is generally small compared to $\nu_0 v_0/c_0$, Eq. (3.3) takes the simple form,

$$k(\nu) = k_p [\gamma_p / \{4\pi(\nu-\nu_0-\beta_p)\}]^2. \tag{3.3'}$$

In the case of self-broadening β_p is zero,[5] i.e., there is no frequency shift. Foreign gas broadening, on the other hand, usually produces a negative shift.

Deviations from the dispersion distribution occur at high pressures and for large $\Delta\nu$. According to Spitzer[6] these deviations first become appreciable when

$$\Delta\nu > \sim (1/2\pi)(v^n/C_n)^{1/(n-1)}. \tag{3.5}$$

Here, v is the average velocity of perturbing atoms relative to absorbing atoms; C_n and n are defined in terms of an assumed interaction law according to which the presence of a perturbing atom at a distance R from an absorbing atom, produces a change in the absorption frequency,

$$\Delta\nu_p = C_n/R^n. \tag{3.6}$$

(The C_n of Eq. (3.6) is essentially equal to Spitzer's interaction constant, q, divided by $\hbar$; it is not to be confused with the C_n of his equation, which is a dimensionless quantity of the order of magnitude unity.)

Another condition for the validity of the dispersion formula is that the mean free time between collisions be large compared to the time of a collision. This means that for the major fraction of time the absorption frequency of a given absorbing atom is negligibly perturbed. At high pressures such that the mean free time between collisions is so short that the absorption frequency is continually perturbed, the dispersion formula breaks down.

At high pressures or for $\Delta\nu$ larger than (3.5), pressure broadening is customarily treated from the standpoint of the *statistical theory*. The basic premise of this theory may be stated as follows. The system of absorbing atom plus perturbers possesses at any time t an instantaneous absorption frequency $\nu(t)$ equal to ν_0 plus frequency perturbations of the type (3.6). For a specified sequence of collisions of a given type, $\nu(t)$ is a known function of time. The statistical theory assumes that the absorption coefficient $k(\nu)$ is proportional to

that fraction of an arbitrarily large time interval, T, for which $\nu(t)$ is contained between ν and $\nu+d\nu$. Since an arbitrarily large number of collisions take place in time T, the general principles of statistical mechanics permit us to equate the defined time fraction to the probability of occurrence, $H(\nu)d\nu$, of those configurations of absorbing and perturbing atoms for which the absorption frequency lies between ν and $\nu+d\nu$. Furthermore, since $H(\nu)$ is by definition a normalized probability function, i.e., $\int H(\nu)d\nu$, we must have

$$P(\nu) = H(\nu), \tag{3.7}$$

since $P(\nu)$ is similarly normalized. The constant of proportionality between $k(\nu)$ and $H(\nu)$ is then the same as that of (2.7).

In simple applications of the statistical theory, one customarily assumes a frequency perturbation of the form (3.6); the following expressions for $P(\nu)$ and $k(\nu)$ are then readily obtained.[7]

$$P(\nu) = (4\pi N'/n)C_n^{3/n}/(\Delta\nu)^{3/n+1}, \tag{3.8}$$

$$k(\nu) = \kappa P(\nu) = \frac{\lambda_0^2}{2} \frac{g_2}{g_1} \frac{\gamma N N'}{n} \frac{C_n^{3/n}}{(\Delta\nu)^{3/n+1}}. \tag{3.9}$$

In a more exact treatment it is necessary to take account of the degeneracy of excited or ground states, which is always present in an actual case. The approach of a perturbing atom then entails not only a shift, but also a splitting of the absorption frequency. Hence, Eq. (3.6) must be replaced by a system of equations,

$$\Delta\nu_p^{(s)} = C_n^{(s)}/R^n, \tag{3.6'}$$

where s takes on a number of values equal to the multiplicity of the splitting. The statistical analysis then yields for $P(\nu)$ and $k(\nu)$ expressions of the form (3.8) and (3.9) wherein the interaction constant C_n is replaced by a more complicated term involving the $C_n^{(s)}$. In the derivation of (3.8) and (3.9) it is further assumed that binary collisions are alone of significance. As shown by Kuhn,[7] this condition is fulfilled for sufficiently large $\Delta\nu$, i.e., in the wings of a spectral line. The treatment of the present paper is restricted to this case.

In practice, two types of interaction are of special importance:

1. *Dipole-dipole interaction.*—This interaction exists when the colliding atoms are of the same kind, i.e., in the case of self-broadening. The resultant frequency perturbation is of the form,

$$\Delta\nu_p = C_3/R^3. \tag{3.10}$$

Inserting Eq. (3.10) into Eqs. (3.8) and (3.9), we obtain

$$P(\nu) = (4\pi N/3)[C_3/(\Delta\nu)^2] \tag{3.11}$$

$$k(\nu) = \frac{\lambda_0^2}{6} \frac{g_2}{g_1} \gamma N^2 \frac{C_3}{(\Delta\nu)^2}. \tag{3.12}$$

[6] Lyman Spitzer, Jr., Phys. Rev. **58**, 348 (1940).

[7] H. Kuhn, Phil. Mag. **18**, 987 (1934); Proc. Roy. Soc. (London) **A158**, 212 (1937).

We note that Eq. (3.12) exhibits the same dependence on $\Delta\nu$ and N as the dispersion formula (3.3') (with $\beta_p=0$). Furthermore, a treatment taking into account the degeneracy of excited and ground states, as outlined, shows (see a forthcoming paper by the present author[8]) that $P(\nu)$ and $k(\nu)$ are symmetrical with respect to the sign of the frequency deviation, $\Delta\nu$. Finally, as has already been demonstrated by Margenau and Watson,[9] the magnitudes of Eqs. (3.3') and (3.12) are of the same order. Thus, in the case of self-broadening, we may expect the dispersion distribution to be valid for $\Delta\nu$ much larger than the limit set by Spitzer's criterion (3.5), at least as long as the interaction is predominantly dipole-dipole in character.

2. *Van der waals interaction.*—This interaction, which gives a frequency perturbation of the form,

$$\Delta\nu_p = C_6/R^6, \tag{3.13}$$

represents the first term on the expansion, in powers of $1/R$, of the interaction between unlike atoms. For this case Eqs. (3.8) and (3.9) give

$$P(\nu) = \tfrac{2}{3}\pi N' C_6^{\frac{1}{2}}/(\Delta\nu)^{\frac{3}{2}}, \tag{3.14}$$

$$k(\nu) = \frac{\lambda_0^2}{12}\frac{g_2}{g_1}\frac{\gamma N N' C_6^{\frac{1}{2}}}{(\Delta\nu)^{\frac{3}{2}}}. \tag{3.15}$$

In contrast to the dipole-dipole case, the more exact analysis involving degeneracy considerations shows that $P(\nu)$ and $k(\nu)$ are asymmetrical with respect to the sign of $\Delta\nu$. Usually this asymmetry takes the form of enhanced intensity on the red side of the unperturbed frequency.[10]

Finally, it should be mentioned that in some cases of self-broadening in which the dipole-dipole interaction is weak to begin with, terms in higher powers of $1/R$, such as the van der waals $1/R^6$ term, may predominate for close approaches, which in the statistical theory determine the line shape at large $\Delta\nu$. Thus, Kuhn[10] finds that at 0.6A from the center of the 2537A mercury resonance line, on the red side,

$$k(\nu) \propto (\Delta\nu)^{-\frac{3}{2}},$$

whereas, on the violet side, $k(\nu)$ drops off with increasing $\Delta\nu$ much faster than the dispersion distribution. On the other hand, Gregory[11] finds that the self-broadening of each of the lines of the principal caesium doublet obeys the dispersion formula out to at least 20A from the line center.

The difference between these two cases may be understood[12] in terms of the magnitude of the dipole-dipole interaction, which is directly proportional to the

[8] T. Holstein, Phys. Rev. **76**, 457 (1949) (preliminary notice).
[9] H. Margenau and W. Watson, Revs. Modern Phys. **8**, 22 (1936).
[10] R. Minkowski, Z. Physik **93**, 731 (1935), H. Kuhn, Proc. Roy. Soc. (London) **A158**, 230 (1937).
[11] Chris Gregory, Phys. Rev. **61**, 465 (1942).
[12] V. Weisskopf, Physik Z. **34**, 1 (1933).

oscillator strength, f, of the resonance line. In the case of Hg 2537A, $f=0.029$, whereas for the D_1 and D_2 lines of Cs, $f=0.33$ and 0.67, respectively.

4. TRANSMISSION OF DOPPLER AND PRESSURE BROADENED RESONANCE RADIATION

The formulas which we obtain in this section for the transmission probability, $T(\rho)$, hold only for ρ large enough so that the core of the resonance line is quite thoroughly absorbed out; the main contribution to (2.3) is then contained in the wings of the line. A necessary and sufficient condition for the prevalence of this situation is

$$T(\rho) \ll 1. \tag{4.1}$$

Now, as has been shown in I for doppler-broadening, and as will be demonstrated more generally subsequently, the escape factor, g, is related to $T(\rho)$ by the order-of-magnitude equation,

$$g \sim T(d), \tag{4.2}$$

where d is the shortest linear dimension of the enclosure. Hence, the calculations of this section will apply to those cases in which the escape factor, is small compared to unity.

A. Doppler-Broadening

The expression for $T(\rho)$, obtained in I, is

$$T(\rho) = 1/k_0\rho(\pi \log k_0\rho)^{\frac{1}{2}} \tag{4.3}$$

where k_0 is given by (3.2).

B. Pressure-Broadening

1. *Dispersion distribution.*—We use here the asymptotic form (3.3') valid in the wings of the line. From Eqs. (2.7), (3.3'), and (3.4) we have

$$P(\nu) = \frac{4}{\gamma_p}\left[\frac{\gamma_p}{4\pi(\nu-\nu_0-\beta_p)}\right]^2. \tag{4.4}$$

Introducing (3.3') and (4.4) into (2.3) and transforming to a new variable $u=4\pi(\nu-\nu_0-\beta_p)/\gamma_p$ we obtain

$$T(\rho) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\left(\frac{1}{u^2}\right)\exp(-k_p\rho/u^2)du$$

$$= \frac{2}{\pi}\int_0^{\infty}\exp(-k_p\rho y^2)dy$$

$$= 1/(\pi k_p\rho)^{\frac{1}{2}}. \tag{4.5}$$

2. *Statistical distributions.*—We use here the general expression (3.8) for $P(\nu)$. We assume for the sake of simplicity that $P(\nu)$ differs from zero for only one sign of $\Delta\nu$. (It is immaterial whether we choose this sign positive or negative.) Then, changing the variable of

integration from $\Delta\nu$ to $P(\nu)$, we have

$$T(\rho)=\int_0^\infty Pe^{-\kappa\rho P}\left|\frac{d(\Delta\nu)}{dP}\right|dP$$

$$=\left(\frac{4\pi N'}{n}\right)^{n/(3+n)}C^{n/(3+n)}\left(\frac{n}{3+n}\right)$$

$$\times\int_0^\infty P^{-n/(3+n)}e^{-\kappa\rho P}dP$$

$$=\left(\frac{n}{3+n}\right)\left(\frac{4\pi N'}{n}\right)^{n/(3+n)}C^{3/(3+n)}$$

$$\times\left(-\frac{n}{3+n}\right)!(\kappa\rho)^{-3/(3+n)}. \quad (4.6)$$

Substituting the explicit expression for κ given by (2.7) into (4.6), we obtain

$$T(\rho)=\frac{\left(\dfrac{n}{3+n}\right)\left(\dfrac{4\pi N'}{n}\right)^{n/(3+n)}C^{3/(3+n)}\left(-\dfrac{n}{3+n}\right)!}{\left(\dfrac{\lambda_0^2 N}{8\pi}\dfrac{g_2}{g_1}\gamma\rho\right)^{3/(3+n)}}$$

or

$$T(\rho)\propto N'^{n/(3+n)}/N^{3/(3+n)}\rho^{3/(3+n)}. \quad (4.7)$$

For the purposes of the next section, in which we calculate escape factors for all these cases, we express $T(\rho)$ in the general asymptotic form,

$$T(\rho)=a_m/\rho^m; \quad 0<m\leq 1, \quad (4.8)$$

where a_m is a function of N, N', and atomic constants. Equation (4.8) embraces all the pressure-broadening cases discussed here. In the case of doppler-broadening, the presence of the logarithmic term in (4.3) does not permit an exact representation of $T(\rho)$ by (4.8). However, as we have already observed in I, the logarithmic term can, in most instances, be treated as constant; for example, in that paper, we replace $\log k_0\rho$ by $\log\frac{1}{2}k_0L$. This approximation will also be used in the present paper in the calculations for infinite slabs; in the case of cylinders, we replace $\log k_0\rho$ by $\log k_0R$, where R is the radius of the cylinder. With these approximations, the transmission of doppler-broadened radiation is represented by (4.8) with $m=1$.

5. ESCAPE FACTORS FOR DOPPLER- AND PRESSURE-BROADENED RADIATION IN INFINITE SLABS AND CYLINDERS

In this section we employ the variational method described in I to calculate the escape factor, g. The variational expression for g reads [see I, Eq. (3.11)]:

$$g=\frac{\int n^2(\mathbf{r})E(\mathbf{r})d\mathbf{r}+\frac{1}{2}\int\int[n(\mathbf{r})-n(\mathbf{r}')]^2G(\mathbf{r},\mathbf{r}')d\mathbf{r}d\mathbf{r}'}{\int n^2(\mathbf{r})d\mathbf{r}}, \quad (5.1)$$

where

$$E(\mathbf{r})=1-\int G(\mathbf{r},\mathbf{r}')d\mathbf{r}'. \quad (5.2)$$

The problem is to find the minimum value of g and the function $n(\mathbf{r})$ for which this minimum is obtained.

A. Infinite Slab

We choose a cartesian system of coordinates whose z axis is perpendicular to the slab; the origin is chosen so that the plane $z=0$ bisects the region. We assume that $n(\mathbf{r})$ is a function of z alone; it is then possible to effect a considerable simplification in Eq. (5.1) by integrating over the other two variables, x and y.

The quantities which have to be evaluated are $\int G(\mathbf{r},\mathbf{r}')dx'dy'$ and $E(\mathbf{r})$. From (2.2) and (4.8) we have

$$G(\mathbf{r},\mathbf{r}')=ma_m/4\pi\rho^{m+3}, \quad (5.3)$$

where $\rho=|\mathbf{r}-\mathbf{r}'|$. We then obtain

$$G(\mathbf{r},\mathbf{r}')dx'dy'$$

$$=\frac{ma_m}{4\pi}\int\int_{-\infty}^{+\infty}\frac{dx'dy'}{[(x-x')^2+(y-y')^2+(z-z')^2]^{(m+3)/2}}$$

$$=\frac{ma_m}{2(m+1)}\frac{1}{|z-z'|^{m+1}}. \quad (5.4)$$

The calculation of $E(\mathbf{r})$ is not to be performed by a direct evaluation of $\int G(\mathbf{r},\mathbf{r}')d\mathbf{r}'$, since important contributions to the latter integral occur for small values of ρ, for which the asymptotic expression (5.3) is not valid. One gets around this difficulty by observing that the integral of $G(\mathbf{r},\mathbf{r}')$ over an infinite region is unity; this follows from the definition of $G(\mathbf{r},\mathbf{r}')$ given in the text after Eq. (2.1) as well as from (2.2). We then immediately obtain

$$E(\mathbf{r})=\int_{\text{Ext}}G(\mathbf{r},\mathbf{r}')d\mathbf{r}', \quad (5.5)$$

where the subscript "Ext" denotes integration over all volume elements outside the gas enclosure. In the plane case

$$E(z)=\frac{ma_m}{2(m+1)}\left[\int_{\frac{1}{2}L}^\infty\frac{dz'}{|z-z'|^{m+1}}+\int_{-\infty}^{-\frac{1}{2}L}\frac{dz'}{|z-z'|^{m+1}}\right]$$

$$=\frac{a_m}{2(m+1)}\left[\frac{1}{(\frac{1}{2}L-z)^m}+\frac{1}{(\frac{1}{2}L+z)^m}\right] \quad (5.6)$$

where L is the thickness of the slab. Substituting Eqs. (5.5) and (5.6) into Eq. (5.1) and introducing the variable,

$$\xi = 2z/L, \tag{5.7}$$

we obtain

$$g = \frac{g_m{}^P}{2^{1-m}} \left[\int_{-1}^{+1} n^2(\xi) \left\{ \frac{\frac{1}{2}}{(1-\xi)^m} + \frac{\frac{1}{2}}{(1+\xi)^m} \right\} d\xi \right.$$

$$\left. + \frac{m}{4} \int\!\!\int_{-1}^{+1} \frac{[n(\xi) - n(\xi')]^2}{|\xi - \xi'|^{1+m}} d\xi d\xi' \right] \Big/ \int_{-1}^{+1} n^2(\xi) d\xi \tag{5.8}$$

where

$$g_m{}^P = 2a_m/(1+m)L^m. \tag{5.8a}$$

(The superscript "P" denotes "plane case" as contracted with the superscript "C" to be used later to denote "cylindrical case.")

We now have to consider the type of function to be used for $n(\xi)$ in the variational calculation of g. In I we found that results good to a few percent can be

TABLE I. g_0, g, and g_{01} in units of a_m/L^m for typical values of m.

m	$\frac{1}{3}$	$\frac{1}{2}$	1
g_0	1.125	1.333	∞
g_1	1.085	1.198	1.875
g_{01}	1.077	1.150	1.875

expected for the two parameter "parabolic" function

$$n_{01}(\xi) = a_0 n_0(\xi) = a_1 n_1(\xi)$$

$$= a_0 + a_1(1 - \xi^2). \tag{5.9}$$

Introducing (5.9) into (5.8), we obtain

$$g/g_m{}^P \equiv \alpha = \sum_{i,j=0,1} a_i a_j K_{ij}{}^P \Big/ \sum_{i,j=0,1} a_i a_j H_{ij}{}^P \tag{5.10}$$

where

$$K_{ij}{}^P = \frac{1}{2^{1-m}} \int_{-1}^{+1} n_i(\xi) n_j(\xi) \left[\frac{\frac{1}{2}}{(1-\xi)^m} + \frac{\frac{1}{2}}{(1-\xi)^m} \right] d\xi$$

$$+ \frac{1}{2^{1-m}} \frac{m}{4} \int\!\!\int_{-1}^{+1} \frac{[n_i(\xi) - n_i(\xi')][n_j(\xi) - n_j(\xi')]}{|\xi - \xi'|^{1+m}} d\xi d\xi' \tag{5.11a}$$

and

$$H_{ij}{}^P = \int_{-1}^{+1} n_i(\xi) n_j(\xi) d\xi. \tag{5.11b}$$

The Ritz variational procedure consists in minimizing g with respect to arbitrary variations in the a_i, i.e.,

$$\partial g/\partial a_i = 0; \quad i = 0, 1. \tag{5.12}$$

Equation (5.12) yields two homogeneous equations for the a_i

$$(K_{00} - \alpha H_{00})a_0 + (K_{01} - \alpha H_{01})a_1 = 0$$
$$(K_{01} - \alpha H_{01})a_0 + (K_{11} - \alpha H_{11})a_1 = 0 \tag{5.13}$$

the nontrivial solution of which requires the vanishing of the determinant of the coefficients, i.e.,

$$\begin{vmatrix} K_{00} - \alpha H_{00} & K_{01} - \alpha H_{01} \\ K_{01} - \alpha H_{01} & K_{11} - \alpha H_{11} \end{vmatrix} = 0. \tag{5.14}$$

From Eq. (5.14) we obtain two values for α, and, hence, for g; since we are seeking an absolute minimum, only the smaller of these is significant. Once α is known, we may obtain a_0 and a_1 to within an arbitrary constant from (5.13).

The calculation of the $K_{ij}{}^P$ and $H_{ij}{}^P$ is straightforward.[13] The results are:

$$K_{00}{}^P = 1/(1-m) \qquad\qquad H_{00}{}^P = 2$$

$$K_{01}{}^P = 4/(2-m)(3-m) \qquad H_{01}{}^P = 4/3 \tag{5.15}$$

$$K_{11}{}^P = 16/(2-m)(3-m)(5-m) \qquad H_{11}{}^P = 16/15.$$

Inserting these expressions into (5.14) and solving for $\alpha = g/g_m{}^P$ we obtain the "two-parameter" approximation to g.

$$g_{01} = \frac{3a_m/L^m[10 - m - m^3 - m(101 - 40m + 2m^2 + m^4)^{\frac{1}{2}}]}{(1 - m^2)(2 - m)(3 - m)(5 - m)}. \tag{5.16}$$

One may also calculate "one-parameter" approximations, g_0 and g_1 by taking $n(\xi) = a_0$ and $n(\xi) = a_1(1 - \xi^1)$, respectively. The results are:

$$g_0 = g_m{}^P K_{00}{}^P/H_{00}{}^P = a_m/(1 - m^2)L^m \tag{5.17}$$

$$g_1 = g_m{}^P K_{11}{}^P/H_{11}{}^P$$
$$= 30a_m/(1 + m)(2 - m)(3 - m)(5 - m)L^m. \tag{5.18}$$

In Table I we present the numerical values of g_0, g_1, and g_{01} in units of a_m/L^m for $m = \frac{1}{3}$, $\frac{1}{2}$, and 1. From Table I, it can be seen that g_1 and g_{01} do not differ by more than a few percent. Furthermore, for $m = \frac{1}{3}$ and $\frac{1}{2}$, the discrepancy between g_0 and g_{01} does not exceed 15 percent. We thus see that, with the exception of the g_0 for $m = 1$, the one-parameter results are not very different from the more accurate two-parameter values; hence, it is to be expected that the three-parameter function, derived from $n_{01}(\xi)$ by the addition of a term of the type, e.g., $a_2(1 - \xi^4)$ will, at best, improve the value for g by only a few percent.

We should point out here that actually g_0 does not become infinite for $m = 1$. As shown in I, p. 1220, the asymptotic form of $T(\rho)$ given by (4.8) or (4.3) is not

[13] The calculation of the double-integral term of $K_{11}{}^P$ is facilitated by the introduction of new integration variables, $u = \xi + \xi'$, $v = \frac{1}{2}(\xi - \xi')$.

accurate enough for the evaluation of $K_{00}{}^P$. The detailed treatment presented in I gives for $K_{00}{}^P$ the value $2 \log k_0 L$ (Eq. (4.7) of I must be multiplied by 4, since the $n_0(\xi)$ of the present paper is twice the $n_0(\xi)$ of I). Hence,

$$g_0 = K_{00}{}^P/H_{00}{}^P = \log k_0 L.$$

The value of g_{01} is also affected by the modification of $K_{00}{}^P$. From I, (4.18), we have

$$g_{01} = 1.875 \left(1 - \frac{\frac{1}{16}}{\log k_0 L - 5/4} \right), \qquad (5.19)$$

which, for typical values of $\log k_0 L$ (~ 4 to 5), differs from the number 1.875, given in Table I, by only a few percent.

Having obtained g_{01}, or α, we may substitute into Eq. (5.13) and solve for $n_{01}(\xi)$ to within an arbitrary constant. With use of (5.15) we obtain a set of values for the ratio a_0/a_1 (Table II). The two-parameter functions $n_{01}(\xi)$ corresponding to these ratios are plotted as full curves in Fig. 1; here the arbitrary multiplicative constant has been chosen to make $n_{01}(0)$ equal to unity for all m.

As pointed out the correct value of K_{00} for $m=1$ is $2 \log k_0 L$ instead of infinity. When this correction is taken into account, one obtains

$$\frac{a_0}{a_1} = \frac{\frac{3}{8}}{\log k_0 L - 15/8} \qquad (5.20)$$

which, for a typical value of $\log k_0 L$, e.g., 5, gives

$$a_0/a_1 = 0.12. \qquad (5.20')$$

The density function $n_{01}(\xi)$ corresponding to (5.20') is represented by the dashed curve of Fig. 1.

B. Infinite Cylinder

As in the case of the infinite slab, the starting point of the analysis is the variational expression (5.1) for g, together with the auxiliary relation (5.5) for $E(r)$. Introducing cylindrical coordinates, z, φ, and r, with $-\infty \leqq z \leqq \infty$, $0 \leqq \varphi \leqq 2\pi$, and $0 \leqq r \leqq R$ (the radius of the cylinder), we have

$$d\mathbf{r} = dz d\sigma; \quad d\mathbf{r}' = dz' d\sigma' \qquad (5.21)$$

where $d\sigma$ and $d\sigma'$ are elements of cross-sectional area

$$d\sigma = r dr d\varphi; \quad d\sigma' = r' dr' d\varphi'. \qquad (5.22)$$

In view of the symmetry of the problem, we may take $n(\mathbf{r})$ to be a function of the radial coordinate, r, alone. The integrations with respect to z and z' then proceed readily. We have, from (5.3)

$$\int_{-\infty}^{+\infty} G(\mathbf{r}, \mathbf{r}') dz' = \frac{m a_m}{4\pi} \int_{-\infty}^{+\infty} \frac{dz'}{\rho^{m+3}}$$

$$= m a_m \int_{-\infty}^{+\infty} \frac{dz'}{[q^2 + (z-z')^2]^{(m+3)/2}}$$

TABLE II. The ratio a_0/a_1 for typical values of m.

m	$\frac{1}{3}$	$\frac{1}{2}$	1
a_0/a_1	0.89	0.31	0

with

$$q = [r^2 + r'^2 - 2rr' \cos(\varphi - \varphi')]^{\frac{1}{2}}. \qquad (5.23)$$

Carrying out the integration, we obtain[14]

$$\int_{-\infty}^{+\infty} G(\mathbf{r}, \mathbf{r}') dz' = C_m/q^{m+2} \qquad (5.24a)$$

where

$$C_m = \frac{m a_m}{2^{3+m}} \frac{(m+1)!}{[(m+1)/2]!}. \qquad (5.24b)$$

Introducing (5.24a) into (5.5), we obtain

$$E(\mathbf{r}) = \int_{\text{Ext}} (C_m/q^{m+2}) d\sigma' \qquad (5.5')$$

where the subscript "Ext" denotes integration over the cross-sectional area exterior to a circle of radius R. (When this subscript is absent, the integration is to be taken over the area of the circle, itself.)

From the form of (5.5') and (5.24a) we see that the integrands of all the terms in (5.1) are independent of z; hence, integration with respect to this variable may be ignored and we obtain

$$g = \frac{\left[\iint_{\text{Ext}} n^2(r)(C_m/q^{m+2}) d\sigma' d\sigma + \frac{1}{2} \iint [n(r) - n(r')]^2 (C_m/q^{m+2}) d\sigma d\sigma' \right]}{\int n^2(r) d\sigma}. \qquad (5.1')$$

For the variational treatment of (5.1'), we choose for $n(r)$ a two-parameter "parabolic" function,

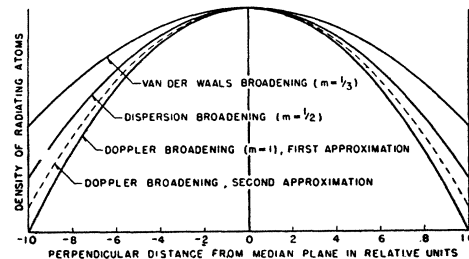$$n(r) = a_0 n_0(r) + a_1 n_1(r) = a_0 + a_1(1 - r^2/R^2) \qquad (5.25)$$

FIG. 1. Spatial distributions of radiating atoms in an infinite slab for different spectral line shapes.

[14] E. Jahnke and F. Emde, Tables of Functions (Dover Publications, New York, 1945), Fourth Edition, p. 21, first equation.

TABLE III. g_0, g_1, and g_{01} in units of a_m/R^m for typical values of m.

m	$\frac{1}{3}$	$\frac{1}{2}$	1
g_0	0.964	1.376	∞
g_1	0.893	1.143	1.60
g_{01}	0.880	1.115	1.60

and obtain

$$g= \sum_{i,j=0,1} a_i a_j K_{ij}{}^C / \sum_{ij}' a_i a_j H_{ij}{}^C \qquad (5.26)$$

with

$$K_{ij}{}^C = \int\int_{Ext} n_i(r)n_j(r)(C_m/q^{m+2})d\sigma'd\sigma$$

$$+\tfrac{1}{2}\int\int [n_i(r)-n_i(r')][n_j(r)-n_j(r')]$$

$$\times(C_m/q^{m+2})d\sigma d\sigma' \qquad (5.27)$$

and

$$H_{ij}{}^C = \int n_i(r)n_j(r)d\sigma. \qquad (5.28)$$

The $H_{ij}{}^C$ are readily obtained; the more tedious evaluation of the $K_{ij}{}^C$ is relegated to Appendix B. The results may be expressed as follows. If

$$K_{ij}{}^C = \frac{\pi^2}{2^{3+m}}\frac{a_m}{R^{m-2}}\frac{(1+m)!}{[\tfrac{1}{2}(1+m)]!^2}\frac{(1-m)!}{[1-\tfrac{1}{2}m]!^2}k_{ij} \qquad (5.29)$$

and

$$H_{ij}{}^C = \pi R^2 h_{ij} \qquad (5.30)$$

we have

$$k_{00}=(2-m)/(1-m) \qquad h_{00}=1$$

$$k_{01}=4/(4-m) \qquad h_{01}=\tfrac{1}{2} \qquad (5.31)$$

$$k_{11}=16/(4-m)(6-m) \qquad h_{11}=\tfrac{1}{3}.$$

These expressions for the k_{ij} are fully equivalent to the appendix relations (6B), (8B), and (13B) for the $K_{ij}{}^C$. In the text version, the constant C_m has been eliminated with use of (5.24b).

The two-parameter escape factor may now be determined by the same variational procedure as was used in the plane case. Introducing the notation,

$$g_m{}^C = \frac{\pi}{2^{2+m}}\frac{(1+m)!}{[\tfrac{1}{2}(1+m)]!^2}\frac{(1-m)!}{[1-\tfrac{1}{2}m]!^2}\frac{a_m}{R^m} \qquad (5.32)$$

we find

$$g_{01}=g_m{}^C\left[\frac{24-8m-m^3-m(304-96m+16m^2+m^4)^{\frac{1}{2}}}{(1-m)(4-m)(6-m)}\right]. \qquad (5.33)$$

For the purposes of comparison we give here the one-parameter escape factors

$$g_0=K_{00}{}^C/H_{00}{}^C=g_m{}^C[1-\tfrac{1}{2}m]/(1-m) \qquad (5.34)$$

and

$$g_1=K_{11}{}^C/H_{11}{}^C=g_m{}^C24/(4-m)(6-m). \qquad (5.35)$$

In Table III we present the numerical values of g_0, g_1, and g_{01} in units of a_m/R^m for $m=\tfrac{1}{3}$, $\tfrac{1}{2}$, and 1.

6. SUMMARY OF RESULTS

In the foregoing sections, the escape factor g has been computed for a variety of line shapes and for two different enclosure geometries. These results will now be summarized.

A. Doppler-Broadening

Here

$$k(\nu) = k_0 \exp[- \{(\nu-\nu_0)/\nu_0\}(c/v_0)^2] \qquad (6.1)$$

where k_0 is given by (3.2). We have, for plane-parallel geometry

$$g=1.875/k_0L(\pi \log k_0L)^{\frac{1}{2}} \qquad (6.2)$$

whereas, for the cylindrical case

$$g=1.60/k_0R(\pi \log k_0R)^{\frac{1}{2}}. \qquad (6.3)$$

B. Impact-Broadening

$$k(\nu) = \frac{k_p}{1+[4\pi(\nu-\nu_0-\beta p)/\gamma_p]^2} \qquad (6.4)$$

where k_p is given by (3.4). The expressions for g then read

$$g=1.150(\pi k_p L)^{-\frac{1}{2}} \qquad (6.5)$$

and

$$g=1.115(\pi k_p R)^{-1}. \qquad (6.6)$$

C. Statistical-Broadening

Starting from the binary interaction law,

$$\Delta\nu_p = \frac{C_n}{R^n} \qquad (6.7)$$

one obtains

$$k(\nu) = \kappa \frac{4\pi N'}{n}\frac{C_n{}^{3/n}}{(\Delta\nu)^{(1+3/n)}}, \qquad (6.8)$$

where N and N' are the densities of absorbing and perturbing atoms and where κ is given by (2.7). We then have for the two geometries,

$$g=\frac{3a_m/L^m[10-m-m^3-m(101-40m+2m^2+m^4)^{\frac{1}{2}}]}{(1-m^2)(2-m)(3-m)(5-m)} \qquad (6.9)$$

and

$$g=\frac{\pi}{2^{2+m}}\frac{(1+m)!}{[\tfrac{1}{2}(1+m)]!}\frac{(1-m)!}{(1-\tfrac{1}{2}m)!}\frac{a_m}{R^m}$$

$$\times\left[\frac{24-8m-m^3-m(304-96m+16m^2+m^4)^{\frac{1}{2}}}{(1-m)(4-m)(6-m)}\right]. \qquad (6.10)$$

In these expressions, $m = 3/(3+n)$ and, from Eq. (4.6)

$$a_m = \left(\frac{n}{3+n}\right)\left(-\frac{n}{3+n}\right)!C^{3/(3+n)}$$

$$\times \left(\frac{4\pi N'}{n}\right)^{n/(3+n)}\left(\frac{\lambda_0^2 N}{8\pi}\frac{g_2}{g_1}\gamma\right)^{-3/(3+n)}.$$

APPENDIX A

One of the fundamental assumptions of our theory is the proportionality relationship (2.2),

$$P(\nu) \propto k(\nu), \tag{1A}$$

between the emission spectrum of a given excited atom, $P(\nu)$, and the absorption coefficient $k(\nu)$. The latter quantity is determined by the nature of the absorbing atoms, their thermal motion and their interaction with other atoms. The emission spectrum, on the other hand, depends not only on atomic parameters, but also on the mechanism of excitation. In the case of natural broadening, for example, excitation of a given atom by a "monochromatic" beam (whose spectral width is small compared to the natural width) produces an emission spectrum identical with that of the incident beam. In order to obtain $P(\nu)$ proportional to $k(\nu)$ it is necessary to excite the atom in a "nonresonant" manner. This is accomplished either by collisions with material particles— atoms, ions, or electrons—or by optical excitation with "white" radiation, i.e., radiation whose spectrum is broad compared to the absorption spectrum.[14]

A similar although less drastic situation prevails in the case of doppler-broadening. Here, monochromatic stimulation evokes an emission spectrum which, while not itself monochromatic, still deviates greatly from the typical doppler absorption line shape. As in the natural broadening case, an exact proportionality between $P(\nu)$ and $k(\nu)$ is obtained only with nonresonant excitation. Thus, in both cases the emission spectrum depends appreciably upon the spectrum of the exciting radiation.

Now, in an imprisonment experiment the excitation agent is the imprisoned radiation itself. The latter, in turn, consists of radiation emitted by excited atoms throughout the enclosure. The spectrum of the exciting radiation is thus not known *a priori;* its determination forms an integral part of the whole imprisonment analysis. An analysis of this type for the doppler case and a plane-parallel enclosure (of width L) was carried out in Part B of the Appendix of I (pp. 1229–1233). It was found that over most of the enclosure —exclusive of a small peripheral region—the spectral intensity of imprisoned radiation is essentially constant in a domain of frequencies for which $k(\nu)L > 1$. Outside this range the intensity drops off rapidly to negligibly small values. With a distribution of this type for the spectrum of the exciting radiation, $P(\nu)$ and g were calculated. Comparison of the results with (3.1) and (1.1) demonstrated the approximate applicability of (1A) to doppler-broadened resonance radiation.

In the case of impact pressure broadening, as was shown in Part A of the Appendix of I (pp. 1227–1229), $P(\nu)$ is almost independent of the mechanism of excitation. Namely, even with monochromatic excitation, $P(\nu)$ turns out to be essentially proportional to $k(\nu)$. In this appendix we present a semi-quantitative argument for the applicability of (1A) to the case of statistical broadening; a detailed quantitative discussion may be found in a Westinghouse Research Laboratories report.[15]

As pointed out in Sec. III of this paper, the fundamental quantity of the statistical theory is the instantaneous emission-absorption frequency, $\nu(t)$, of a given atom. $\nu(t)$ differs from the unperturbed frequency, ν_0, of an *isolated* atom because of collisions with neighboring atoms. In the course of time, $\nu(t)$ will range

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[15] Westinghouse Research Laboratories, scientific paper 1501, available upon written request.

over a continuum of frequencies. The occurrence probability defined (see Sec. III) as the fraction of time for which $\nu(t)$ is contained between ν and $\nu+d\nu$, gives directly both the absorption, coefficient and the emission spectrum arising from white excitation.

Let us suppose, in accordance with the procedure used in the Appendix of I, that the atoms are excited by a monochromatic beam of frequency ν_1. The statistical theory states that a given atom absorbs energy from the beam only at times t_i such that $\nu(t_i) = \nu_1$. Now, after each such absorption, the atom radiates with an intensity proportional to $\exp[-\gamma(t-t_i)]$; the spectrum of the emitted radiation is thus determined essentially by the variation of $\nu(t)$ over a time interval $\sim 1/\gamma$. The question of the dependence of $P(\nu)$ upon the excitation frequency may then be put in the following form. To what extent is the variation of $\nu(t)$ over the time interval, $1/\gamma$, dependent upon the circumstance that the initial value of $\nu(t)$ is ν_1, i.e., that the atom acquires its excitation energy when $\nu(t) = \nu_1$?

Now, as in Sec. III, we assume that the time dependence of $\nu(t)$ is determined mainly by binary collisions. During each of these $\nu(t)$ is equal to ν_0 plus a perturbation of the form (3.6). Excluding the very special case of $\nu_1 = \nu_0$, we observe that absorption takes place only during a collision, in fact, only during those collisions in which $\nu(t)$ attains the value ν_1. This stipulation implies a nonrandom situation in the time interval immediately subsequent to excitation. Namely, the "absorption" collision, during which the newly excited atom begins radiating, is not a random encounter but is selected in a special way. (For example, if $\nu_1 - \nu_0$ is large, absorption takes place only in those encounters in which the distance of closest approach is sufficiently small as determined by (3.6).) Hence, the possibility of a correlation between $\nu(t)$ and ν_1 during the absorption collision cannot be ruled out. On the other hand, when the absorption collision is terminated, the further density of $\nu(t)$ becomes the responsibility of new binary encounters, which are completely uncorrelated with each other, or with the absorption collision. Thus, the excited atom "remembers" the excitating frequency at most for an interval of the order of a "time of collision," τ. We thus arrive at the qualitative conclusion that, if $\tau \ll 1/\gamma$, the influence of the exciting frequency on the emission spectrum is small. Now this condition is almost always obeyed in practice ($\tau \sim$ collision radius/atomic velocity $\sim < 10^{-10}$ sec, $1/\gamma \sim 10^{-8}$ sec); the emission spectrum is thus determined predominantly by collisions whose parameters are uncorrelated with the absorption frequency and, hence, must obey (1A).

APPENDIX B

In the evaluation of the integrals in (5.27) use is made of the relation,

$$1/q^p = [1/(2-p)^2]\nabla^2(1/q^{p-2}) = [1/(2-p)]^2\nabla'^2(1/q^{p-2}), \tag{1B}$$

where ∇^2 and ∇'^2 are the two-dimensional laplacian operators involving differentiation with respect to r, φ and r', φ', respectively. Thus

$$K_{00}{}^C = \int\int_{\mathrm{Ext}}(C_m/q^{m+2})d\sigma'd\sigma = \int\int_{\mathrm{Ext}}(C_m/m^2)\nabla'^2 q^{-m}d\sigma'd\sigma$$

which, with the aid of Gauss' theorem, can be transformed into

$$K_{00}{}^C = -(C_m/m^2)\int\int_s(\partial/\partial r')q^{-m}ds'd\sigma$$

$$= -(C_m/m^2)\int_s ds'(\partial/\partial r')\int q^{-m}d\sigma, \tag{2B}$$

where the subscript "s" indicates a line integration over the circumference of the cross-sectional circle. By further use of (1B) and Gauss' theorem, we obtain

$$K_{00}{}^C = -\frac{C_m}{m^2(2-m)^2J}\int_s\int_s(\partial/\partial r)(\partial/\partial r')q^{-m+2}dsds'. \tag{3B}$$

The double line integrals of the type exemplified by (3B) can

readily be evaluated; in fact,

$$\int\int_s \int_s (\partial/\partial r)(\partial/\partial r')q^{-p}dsds' = \frac{\pi^2 p^2}{R^p}\frac{(-p)!}{(-\frac{1}{2}p)!^2}\frac{p+2}{p+1} \quad (4B)$$

as we shall now verify.

Differentiating with respect to r and r' and setting both these variables equal to R, we obtain for the left-hand side of (4B) the expression,

$$\int_0^{2\pi}\int_0^{2\pi}\{p(1+p\sin^2[\tfrac{1}{2}(\varphi-\varphi')])\}/$$
$$(2R)^{p+2}\sin^{p+2}[\tfrac{1}{2}(\varphi-\varphi')]\}R^2 d\varphi d\varphi'.$$

One now introduces the transformation $\varphi'=\varphi+\theta$ and integrates with respect to φ, obtaining

$$\frac{2\pi p}{2^{p+2}R^p}\int_0^{2\pi}\frac{1+p\sin^2\tfrac{1}{2}\theta}{\sin^{p+2}\tfrac{1}{2}\theta}d\theta = \frac{2\pi p}{(2R)^p}\int_0^{\pi/2}\frac{1+p\sin^2 X}{\sin^{p+2}X}.$$

The remaining integral is evaluated with the aid of the formula quoted in reference 12; minor algebraic manipulation of the resultant factorial terms then yields (4B).

Similarly one may readily show that

$$\int\int_s\int_s (\partial/\partial r)q^{-p}dsds' = \int\int_s\int_s (\partial/\partial r')q^{-p}dsds' = -\frac{2\pi^2 p}{R^{p-1}}\frac{(-p)!}{(-\frac{1}{2}p)!^2}. \quad (5B)$$

Equation (5B) will be used in the evaluation of the other $K_{ij}{}^C$.

Inserting (4B) into (3B), we obtain, after elementary algebraic operations,

$$K_{00}{}^C = \frac{2\pi^2 C_m}{mR^{m-2}}\frac{(1-m)!}{(1-\frac{1}{2}m)!^2}\frac{1-\frac{1}{2}m}{1-m}. \quad (6B)$$

To calculate $K_{01}{}^C$ we first apply (1B) and Gauss' theorem to the "primed" integration variables, obtaining

$$K_{01}{}^C = -\frac{C_m}{m^2}\int_s ds'\frac{\partial}{\partial r'}\int\frac{1-(r/R)^2}{q^m}d\sigma$$

We then perform two integrations-by-parts with respect to the unprimed variables, wherein we integrate $1/q^m$ and differentiate $1-(r/R)^2$ with the aid of (1B) and Green's formula,

$$\int(u\nabla^2 v - v\nabla^2 u)d\sigma = \int_s[u(\partial v/\partial n) - v(\partial u/\partial n)]ds. \quad (7B)$$

This procedure yields

$$K_{01}{}^C = \frac{-(2/R)C_m}{(2-m)^2 m^2}\int\int_s\int_s \frac{\partial}{\partial r'}\left[\frac{1}{q^{m-2}} - \frac{(2/R)}{(4-m)^2}\frac{\partial}{\partial r}\frac{1}{q^{m-4}}\right]dsds'.$$

The integrals to be evaluated are of the forms of (4B) and (5B). We obtain

$$K_{01}{}^C = \frac{4\pi^2 C_m}{mR^{m-2}}\frac{(1-m)!}{(1-\frac{1}{2}m)!^2}\frac{1}{4-m}. \quad (8B)$$

The first term of $K_{11}{}^C$ is evaluated in much the same way except that three partial integrations have to be performed with respect to the unprimed variables. One finds

$$K_{11}{}^{C(1)} \equiv \int E(r)[1-(r/R)^2]d\sigma$$

$$= \frac{8C_m}{m^2(2-m)^2(4-m)^2 R^2}\int\int_s\int_s\frac{\partial}{\partial r'}$$
$$\times\left[\frac{\partial}{\partial r}\left\{\frac{1}{q^{m-4}} - \frac{8}{(6-m)^2 R^2}\frac{1}{q^{m-6}}\right\} - \frac{4}{R}\frac{1}{q^{m-4}}\right]dsds'$$

$$K_{11}{}^{C(1)} = \frac{\pi^2 C_m}{mR^{m-2}}\frac{(2-m)!}{(2-\frac{1}{2}m)!^2}\frac{8}{6-m}. \quad (9B)$$

Finally, the second term of $K_{11}{}^C$,

$$K_{11}{}^{C(2)} \equiv \left(\frac{C_m}{2R^4}\right)\int\int\left[\frac{(r^2-r'^2)}{q^{m+2}}\right]d\sigma d\sigma'$$

$$= \frac{C_m}{2R^4}\int_0^R\int_0^R\int_0^{2\pi}\int_0^{2\pi}\frac{(r^2-r'^2)rr'drdr'd\varphi d\varphi}{[r^2+r'^2-2rr'\cos(\varphi-\varphi')]^{\frac{1}{2}m+1}}$$

is evaluated as follows. We express r' and φ' in terms of new variables, $x=r'/r$ and $\theta=\varphi'-\varphi$, and integrate with respect to φ obtaining

$$K_{11}{}^{C(2)} = \frac{\pi C_m}{R^4}\int_0^R dr\int_0^{R/r}dx\int_0^{2\pi}\frac{r^{5-m}(1-x^2)^2 drxdxd\theta}{(1+x^2-2x\cos\theta)^{\frac{1}{2}m+1}}.$$

The form of the integrand of this expression indicates the desirability of inverting the order of the integrations over x and r. Performing this inversion and integrating with respect to r, we obtain

$$K_{11}{}^{C(2)} = \frac{\pi C_m}{(6-m)R^{m-2}}\left[\int_0^{2\pi}\int_0^1\frac{(1-x^2)^2 xdxd\theta}{[1+x^2-2x\cos\theta]^{\frac{1}{2}m+1}}\right.$$
$$\left. + \int_0^{2\pi}\int_1^\infty\frac{(1-x^2)^2 dxd\theta}{x^{5-m}[1+x^2-2x\cos\theta]^{\frac{1}{2}m+1}}\right]. \quad (10B)$$

With the aid of the transformation $x=1/y$, we can readily establish the equality of the two double integrals in (10B); we then have

$$K_{11}{}^{C(2)} = \frac{2\pi C_m}{(6-m)R^{m-2}}\int_0^1\int_0^{2\pi}\frac{(1-x^2)^2 xdxd\theta}{[1+x^2-2x\cos\theta]^{\frac{1}{2}m+1}}$$

or

$$K_{11}{}^{C(2)} = (2\pi C_m/(6-m)R^{m-2})\int_{s_1}[(1-x^2)^2/q^{m+2}]d\sigma \quad (11B)$$

where $q=(1+x^2-2x\cos\theta)^{\frac{1}{2}}$ and the subscript s_1 indicates integration with respect to a circle of radius unity. The technique for evaluating (11B) is the same as has already been employed; three integrations-by-parts yield

$$K_{11}{}^{C(2)} = \frac{16\pi^2 C_m}{m^2(2-m)^2(6-m)R^{m-2}}$$
$$\times\int_0^{2\pi}\left[\frac{\partial}{\partial x}\left\{\frac{1}{q^{m-2}} + \frac{8}{(4-m)^2}\frac{1}{q^{m-4}}\right\} - \frac{4}{q^{m-2}}\right]_{x=1}d\theta.$$

One now utilizes Eqs. (4B) and (5B), obtaining

$$K_{11}{}^{C(2)} = \frac{4\pi^2 C_m}{(2-m)(6-m)R^{m-2}}\frac{(2-m)!}{(2-\frac{1}{2}m)!^2}. \quad (12B)$$

Adding (9B) and (12B), we then have

$$K_{11}{}^C = \frac{16\pi^2 C_m}{m(6-m)(4-m)R^{m-2}}\frac{(1-m)!}{(1-\frac{1}{2}m)!^2}. \quad (13B)$$

Substituting the text Eqs. (5.24B) into Eqs. (6B), (8B), and (13B), we obtain the results listed in the text Eqs. (5.29) and (5.31).