PHYSICAL REVIEW VOLUME 83, NUMBER 5 SEPTEMBER 1, 1951

Constraints in Covariant Field Theories*

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(Received May 7, 1951)

In this paper we have considered certain problems which arise when one attempts to cast a covariant field theory into a canonical form. Because of the invariance properties of the theory, certain identities exist between the canonical field variables. To insure that the canonical theory is equivalent to the underlying lagrangian formalism one must require that these identities, once satisfied, will remain satisfied through the course of time. In general, this will be true only if additional constraints are set between the canonical variables. We have shown that only a finite number of such constraints exist and that they form a function group. Our proof rests essentially on the possibility of constructing a generating function for an infinitesimal canonical transformation that is equivalent to an invariant infinitesimal transformation on the lagrangian formalism.

Once a hamiltonian is obtained by one of the procedures outlined in previous papers of this series, and the constraints have all been found, the consistent, invariant canonical formulation of the theory is completed. The main results of the paper have been formulated in such a manner as to make them applicable to a fairly general type of invariance. In the last sections we have applied these results to the cases of gauge and coordinate invariance. In the latter case a hamiltonian, corresponding to a quadratic lagrangian, has been constructed in a parameter-free form; and in both cases the constraints, together with the poisson bracket relations between them, have been obtained explicitly. As was to be expected, two constraints were found for a gauge-invariant theory and eight for a coordinate-invariant theory.

I. INTRODUCTION

IN this paper, we shall complete the examination of the constraints which are met with in the hamiltonian formulation of theories possessing invariance properties. This examination had begun in earlier papers,[1–3] with particular emphasis on the type of covariance met with in the general theory of relativity, and is in some respects similar to the results obtained by L. Rosenfeld.[4,5] All theories dealt with in physics possess some covariance properties, i.e., the laws of the theory take the same form in more than one representation. In general, different representations of the same situation may be converted into one another by means of so-called transformation equations, and we say that a given theory is covariant with respect to such-and-such a transformation group. These transformation groups fall in general into three distinct classes: The group may contain only a finite or at least discrete number of transformations, such as the symmetric group (of permutations), of importance in spectroscopy, and the space group of crystallography. The group may be a continuous set, the members of which can be identified by a finite number of continuously variable parameters, a so-called Lie group; among these groups are the group of orthogonal transformations and the Lorentz group. Finally, the group may again be a continuous set, but so that the number of parameters required to represent it is no longer finite but equivalent to one or several arbitrary functions; examples for this class are the group of canonical transformations, the group of unitary transformations in Hilbert space, the group of gauge transformations in electrodynamics, and the group of coordinate transformations in the general theory of relativity. Our attention in this paper will be focused on covariance of a theory with respect to groups belonging to the third class.

More particularly, we shall be concerned with field theories whose laws can be derived from a variational principle. The Euler-Lagrange equations will possess a certain number of so-called identities which depend directly on the group of transformations with respect to which the theory is invariant. The Euler-Lagrange equations of the theory will be covariant with respect to certain transformations if the lagrangian integral transforms by adding at most a surface integral, which is immaterial for a variational principle. (For continuous groups, it is sufficient to consider the infinitesimal transformations from which the whole group can be constructed.) But this change is an integral over the product of the variational derivatives of the lagrangian and the infinitesimal changes of the field variables. At each point in space-time, this integral must then be a complete divergence (or zero), even when the variational derivatives of the lagrangian are different from zero, i.e., even when the field equations are not satisfied. This condition will lead to identities between the field variables and their derivatives which, depending on the character of the group, will take the form of integrals or will hold at each point separately. In the third class of transformation groups, the identities are generally point-to-point.

For certain groups the identities can be converted into the form of a pure divergence, and hence lead to

* This work was supported by the ONR under Contract 6N-onr-24806.
[1] P. G. Bergmann, Phys. Rev. **75**, 680 (1949), referred to as I.
[2] P. G. Bergmann and J. H. M. Brunings, Revs. Modern Phys. **21**, 480 (1949), referred to as II.
[3] Bergmann, Penfield, Schiller, and Zatzkis, Phys. Rev. **80**, 81 (1950), referred to as III.
[4] L. Rosenfeld, Ann. Physik **5**, 113 (1930).
[5] L. Rosenfeld, Ann. inst. Henri Poincaré **2**, 25 (1932).

conservation laws. These are "strong" conservation laws (I); they are satisfied even if the field equations are not, e.g., when there are singularities in the field. In the case of the gauge invariance of electrodynamics, we have one such law, the conservation of charge, while coordinate covariance leads to the four laws of conservation of energy and linear momentum. It is the existence of the strong conservation laws which leads to equations of motion for the singularities of the field without requiring separate assumptions such as the Lorentz force.

When the theory is carried from the lagrangian over into the hamiltonian (or canonical) formalism, it turns out that the various momentum densities, defined in the customary manner, are not algebraically independent of each other, but satisfy a number of relations, free of time derivatives, which we shall call primary constraints in analogy to the customary terminology of mechanics; these relationships are analogous to the first subsidiary condition of quantum electrodynamics. Unless the momentum densities satisfy these constraints, there will be no conceivable set of "velocities" (time derivatives of the field variables) consistent with the momentum densities.

In the canonical formalism, the field equations are replaced by the first-order canonical field equations. Not all solutions of the canonical field equations satisfy the primary constraints, nor is it sufficient to satisfy the primary constraints on one initial hypersurface to assure that the equations of motion will preserve them elsewhere. But if we were to start with a solution of the lagrangian field equations and translate this solution into the canonical formalism, both the canonical field equations and the primary constraints would be satisfied everywhere. Hence, they are at least compatible with each other. Compatibility, while necessary, is not sufficient. We must discover the totality of all conditions to be satisfied on an initial hypersurface which will guarantee that solutions of the canonical field equations will satisfy the primary constraints everywhere. We shall show that these additional conditions, called the secondary constraints, are finite in number and that their exact number depends on the details of the transformation laws for the field variables and for the lagrangian.

In this paper, we shall derive a number of results for an assumed transformation law for the field variables somewhat more general than that postulated in I and II (though including that law as a special case). Furthermore, we shall carry the argument through without using the parameters introduced in II; their introduction into the theory can be accomplished without difficulty, but contributes little to the developments presented here.

II. INVARIANT TRANSFORMATIONS

In what follows, we shall consider field theories in which the field equations can be derived from a four-dimensional variational principle:

$$S=\int L(y_A,\, y_{A,\sigma})d^4x, \qquad (2.1)$$

$$\delta S=0.$$

The lagrangian density L is to be a function of the field variables $y_A(x^1,\cdots,x^4)$ and their first derivatives $y_{A,\sigma}$. We shall assume that the field equations go over into themselves if the field variables are subjected to any one of a group of transformations; we shall call the members of this group invariant transformations, a designation that is meaningful only with respect to a particular theory or class of theories. We shall further assume that the group structure permits the identification of its members, including the infinitesimal transformations, by a finite number of arbitrary functions. Well-known examples of this type of transformation group are the gauge transformations of electrodynamics, whose infinitesimal transformation law can be written in the form

$$\delta\phi_\mu=\xi_{,\mu} \qquad (2.2)$$

and the coordinate transformations, whose infinitesimal (substantive, see I) transformation law taken the form

$$\bar{\delta}y_A=F_{A\mu}{}^{B\nu}y_B\xi^\mu{}_{,\nu}-y_{A,\mu}\xi^\mu, \qquad (2.3)$$

provided the field variables transform according to a linear homogeneous law (Christoffel symbols, for instance, do not). Accordingly, we shall adopt as our general transformation law for the field variables

$$\bar{\delta}y_A={}^0f_{Ai}\xi^i+{}^2f_{Ai}{}^\mu\xi^i{}_{,\mu}+\cdots+{}^Pf_{Ai}{}^{\mu\cdots\tau}\xi^i{}_{,\mu\cdots\tau}, \qquad (2.4)$$

$$\bar{\delta}x^\rho=0.$$

For the sake of generality, we shall not assume any particular geometric or physical significance for the arbitrary functions ξ^i, the "descriptors," which appear in this infinitesimal transformation law. The coefficients ${}^sf_{Ai}{}^{\mu\cdots\tau}$ are some functions of the field variables and their derivatives, generally such that the sum of the orders of differentiation in the various factors of any one term in Eq. (2.4) does not exceed the finite number P. For the two laws (2.2) and (2.3), this total order P equals one, and these two laws are, moreover, homogeneous with respect to the order of differentiation. To assure group character, we must and shall require that the commutator of two expressions (2.4) with two different sets of descriptors will be an expression of the same kind (with the same coefficients f), with a third set of descriptors.

III. IDENTITIES

By a method similar to that employed in I we find that the field variables satisfy a number of identities. We require that the lagrangian density, in the face of a transformation of the form (2.4), shall add only a pure divergence. If this is the case then

$$\bar{\delta}L = L^A\bar{\delta}y_A + [(\partial L/\partial y_{A,\rho})\bar{\delta}y_A]_{,\rho} = Q^\rho{}_{,\rho}, \tag{3.1}$$

where the L^A are the field equations, i.e.,

$$L^A \equiv (\partial L/\partial y_A) - (\partial L/\partial y_{A,\rho})_{,\rho}. \tag{3.2}$$

$\bar{\delta}L$ will be a divergence only if

$$L^A\,{}^0f_{Ai} - (L^{A1}\,f_{Ai}{}^\mu)_{,\mu} + \cdots + (-1)^P(L^A\,{}^Pf_{Ai}{}^{\mu\cdots\tau})_{,\mu\cdots\tau} \equiv 0. \tag{3.3}$$

These equations are the generalized "Bianchi identities" of the theory. There are as many of them as there are descriptors in the transformation law. In such identities, the terms containing the highest derivatives of the field variables must vanish by themselves. These terms are of the form

$$L^{A\rho B\sigma}\,{}^Pf_{A\,i}{}^{\mu\cdots\tau}y_{B,\rho\sigma\mu\cdots\tau} \equiv 0, \tag{3.4}$$

where $L^{A\rho B\sigma}$ is shorthand for $\frac{1}{2}[(\partial^2L/\partial y_{A,\rho}\partial y_{B,\sigma}) + (\partial^2L/\partial y_{A,\sigma}\partial y_{A,\rho})]$. Symmetrizing with respect to $\rho\sigma\mu\cdots\tau$, we have

$$\{L^{A\rho B\sigma}\,{}^Pf_{Ai}{}^{\mu\cdots\tau}\}_{(\rho\sigma\mu\cdots\tau)} \equiv 0, \tag{3.5}$$

$(\rho\sigma\mu\cdots\tau)$ indicating that the expression is symmetrized with respect to these indices. Of particular interest for what follows are those expressions in which all Greek indices are set equal to 4. They are

$$L^{A4B4}u_{Bi} \equiv 0, \tag{3.6}$$

where

$$u_{Bi} = {}^Pf_{Ai}{}^{4\cdots4}. \tag{3.7}$$

IV. HAMILTONIAN FORMALISM

Let us now specialize our problem by assuming that our lagrangian is quadratic in the differentiated quantities; i.e., it has the form

$$L = \Lambda^{A\rho B\sigma}y_{A,\rho}y_{B,\sigma}. \tag{4.1}$$

Momentum densities canonically conjugate to the y_A are defined in the customary manner to be

$$\pi^A \equiv \frac{\partial L}{\partial \dot{y}_A} = 2\Lambda^{A4B4}\dot{y}_B + 2\Lambda^{A4Bs}y_{B,s}. \tag{4.2}$$

If we now try to solve these equations for the $y_{A,4}$ in terms of the π^A, we find that we cannot do so directly because of the fact that Λ^{A4B4} is a singular matrix, as shown by Eq. (3.6), and hence has no inverse in the ordinary sense. However, we can circumvent this difficulty by introducing the quasi-inverse (III) to Λ^{A4B4} by the equations

$$\begin{aligned} E_{AB}\Lambda^{B4C4}E_{CD} &= E_{AD}, \\ \Lambda^{A4B4}E_{BC}\Lambda^{C4D4} &= \Lambda^{A4D4}. \end{aligned} \tag{4.3}$$

These equations do not determine E_{AB} completely, but only up to a linear combination of the null vectors of the matrix Λ^{A4B4}, the u_{Bi}. It can be shown easily that

$$\begin{aligned} \Lambda^{A4C4}E_{AB} &= \delta_B{}^C - v^{Ci}u_{Bi}, \\ v^{Ci}E_{CA} = 0, &\quad v^{Ci}u_{Ci} = \delta_i{}^j, \end{aligned} \tag{4.4}$$

where the expressions v^{Ci} depend on the choice of E_{AB}. If we now multiply Eq. (4.2) by E_{AB} and introduce as a convenient abbreviation

$$\pi^A - 2\Lambda^{A4Bs}y_{B,s} = \bar{\pi}^A, \tag{4.5}$$

we find

$$\dot{y}_B = \tfrac{1}{2}E_{BC}\bar{\pi}^C + 2u_{Bi}v^{Ci}\dot{y}_C. \tag{4.6}$$

It is not difficult to see that the expressions $\Lambda^{A4B4}\dot{y}_B$ entering into Eq. (4.2) and the expressions $v^{Bi}\dot{y}_B$ of Eq. (4.6) are completely independent of each other, i.e., all of them can be given arbitrary values without giving rise to inconsistencies. As a result, the $\bar{\pi}^A$ determine the $\dot{y}_B$ only incompletely, and the extent of this incomplete determination is expressed accurately by a set of equations

$$\dot{y}_B = \tfrac{1}{2}E_{BC}\bar{\pi}^C + u_{Bi}w^i \tag{4.7}$$

with arbitrary functions w^i.

If we multiply Eqs. (4.2) by u_{Ai}, we find

$$g_i \equiv u_{Ai}\bar{\pi}^A = 0, \tag{4.8}$$

a set of algebraic relations to be satisfied by the canonical momentum densities together with the field variables y_A. These relations are the primary constraints.

If we wish to form a hamiltonian and to reformulate the whole set of field equations in terms of a canonical formalism, it can be shown that for the quadratic lagrangian density (3.1) the corresponding hamiltonian density H is

$$H = \tfrac{1}{4}E_{AB}\bar{\pi}^A\bar{\pi}^B - \Lambda^{ArBs}y_{A,r}y_{B,s} + w^ig_i, \tag{4.9}$$

where the functions w^i are arbitrary and identical with those of Eq. (4.7). This result, first obtained by Penfield[6] for the coordinate formalism (which dispenses with the introduction of the parameters), is complete in the sense that it describes the full extent of the arbitrariness remaining in the choice of hamiltonian.[7]

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[6] R. Penfield, Ph.D. dissertation, Syracuse University, 1950 (to be published).

[7] Although this result has been obtained using a quadratic lagrangian, it is quite general. The identities (3.6) can be written in the form,

$$u_{Bi}\partial\pi^B/\partial\dot{y}_c = 0$$

and, since by definition u_{Bi} contains no differentiated field variables, we can integrate them and obtain, as primary constraints

$$u_{Bi}\pi^B - K_i(y, y_{,s}) \equiv 0.$$

We then have the problem of solving the set of equations

$$\pi^A = \partial L/\partial \dot{y}_A$$

together with the above constraints. Since we actually have

Naturally, other forms of the hamiltonian density can be obtained by canonical transformations; but the arguments appearing in such new expressions will no longer have the significance of the original field variables y_A and the momentum densities defined by Eq. (4.2). It follows in particular that transformations of the form (2.4) will change the expression (4.9) at most by adding to it further linear combinations of the primary constraints, i.e., by leading to new arbitrary functions w^i. To avoid misunderstandings, it should also be noted that whenever we speak of arbitrary functions, we imply that they are arbitrary in our configuration or phase space; i.e., they may depend on the coordinates as well as the field variables and their derivatives and, in the canonical formalism, on the momentum densities and their derivatives as well.

V. CONTACT TRANSFORMATIONS AND INFINITESIMAL TRANSFORMATIONS

Let us now return to the infinitesimal transformations defined by Eqs. (2.4). We may interpret this transformation law in a number of different spaces. We shall call the function space of all functions $y_A(x^1, x^2, x^3)$ the configuration space of the theory and denote it by S_c. In such a formulation, t appears as a parameter outside the function space, S_c, and the change of the physical system in the course of time appears as the motion of the representative point. Obviously, the transformation (2.4) is not a transformation in S_c, since it involves on its right-hand side $\dot{y}_A, \ddot{y}_A, \cdots$. We might introduce the function space of which the elements are $y_A, \dot{y}_A, \cdots \partial^P y_A/\partial t^P$, and call that space the configuration-plus-"velocity" space, S_{cv}. In that space, the transformation law for the y_A is properly defined, but then the transformation law for $\dot{y}_A$ will involve $P+1$ time derivatives, and so forth indefinitely. A possible way out of this difficulty is to introduce a different type of space, the space of the functions $y_A(x^1, \cdots, x^4)$. A whole trajectory of a representative point in S (its motion as a function of t) becomes a single point in this new space, $\sum_c$. In this new space, the transformation law (2.4) is properly defined, and with the arbitrary functions ξ^i, constitutes a group.

We now ask the following question: if the y_A are subjected to a transformation of the form (2.4), what effect does this have on the corresponding canonical formulation? To answer this question, we shall first introduce as the phase space $\sum$ (without subscript) the function space of all functions $y_A(x^1, \cdots, x^4)$, $\pi^A(x^1, \cdots, x^4)$. Most of its points will not correspond to possible points in $\sum_c$. But $\sum$ is the space in which all canonical transformations

$$\delta y_A = \mathfrak{d}\mathcal{C}/\mathfrak{d}\pi^A, \quad \delta\pi^A = -\mathfrak{d}\mathcal{C}/\mathfrak{d}y_A, \qquad \delta\mathcal{H} = \partial\mathcal{C}/\partial t \tag{5.1}$$

are defined. There is, however, a space which is isomorphic with the space $\sum_c$, although not identical with it; it is a subspace, which we shall designate by $\sum_l$ (lagrangian subspace), consisting of all points of $\sum$ that satisfy the conditions (4.2). We shall call a one-to-one mapping between $\sum_c$ and $\sum_l$ in which the functions y_A of corresponding points are simply the same functions (and they define π^A in $\sum_l$ uniquely), the "identification." Under the identification, each possible transformation in $\sum_c$ corresponds uniquely to one in $\sum_l$, and vice versa. All these transformations can be expanded into canonical transformations in $\sum$ by the following method (and possibly in other ways as well): First, replace the original field variables by the transformed variables in the lagrangian of the variational principle. Then transform this formally new lagrangian theory into a hamiltonian theory. The new hamiltonian theory is equivalent to the original one and, therefore, connected with it by a canonical transformation. We shall call the class of canonical transformations which correspond to transformations in $\sum_c$ "canonical transformations in $\sum_l$," because they map the points of the original $\sum_l$ on the points of the new $\sum_l$. The two subspaces $\sum_l$ are in general not determined by identical Eqs. (4.2), because the lagrangian will have changed its form.

Of special interest are the invariant infinitesimal transformations, Eq. (2.4). We shall call the corresponding infinitesimal canonical transformations "(infinitesimal) invariant canonical transformations." Because of the invariant character of these transformations, we may adopt the same formal expression for the lagrangian density (2.1) before and after the transformation, and we shall do so, because with this choice the identity of the subspace $\sum_l$ remains unaffected by the corresponding canonical transformations.

Given an invariant infinitesimal coordinate transformation, we can expand it into a canonical transformation in $\sum$ in infinitely many ways. If we have obtained one such transformation and its generating functional, we can add to the generator any expression at least quadratic in the primary constraints. These additional terms will have no effect in the subspace $\sum_l$, but they will affect the transformation equations elsewhere in $\sum$.

To each invariant canonical transformation belongs only one invariant transformation in $\sum_c$, because such a transformation leads uniquely to a transformation law in $\sum_l$ and hence in $\sum_c$. The degree of arbitrariness in going from $\sum_c$ to $\sum_l$ and thence to $\sum$ is completely

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fewer equations than variables, a particular solution can always be found. Let it be of the form,

$$\dot{y}_A{}^0 = \Gamma_A(\pi, y, y_{,s}).$$

It is easy to see that

$$\dot{y}_A = \dot{y}_A{}^0 + w^i u_{Bi}$$

is also a solution. Hence, we can, after forming a hamiltonian according to the equation,

$$H = \pi^A \dot{y}_A{}^0 - L(y, \dot{y}^0)$$

add to it a term of the form $w^i(u_{Bi}\pi^B - K_i)$ and still obtain the same canonical equations.

exhausted by the addition of terms quadratic in g_i and has no effect on the transformation law in $\sum_l$, which is uniquely determined by the transformation law in $\sum_c$. Any nonconstant addition to the generator of a canonical transformation other than terms at least quadratic in the primary constraints would have an effect on the transformation law in $\sum_l$.

We can conclude that the invariant canonical transformations form a group. Because of its importance, this argument will be presented at some length. First of all, the commutator of two invariant transformations in $\sum_c$ must itself be an invariant transformation; under the identification, the corresponding transformations in $\sum_l$ must form a group, too. The only difficulty arises in connection with the expansion into canonical transformations in $\sum$. Let A and B represent two invariant transformations in $\sum_l$, and A' and B' expansions of A and B, respectively, into canonical transformations in $\sum$. Let C be the commutator of A and B in $\sum_l$, and C' some canonical expansion of C in $\sum$. On the other hand, let C'' be the commutator of A' and B' in $\sum$. Obviously, both C' and C'' are canonical transformations by definition, but they might not be identical. Actually, for the proof of group property, it will be necessary and sufficient to show that C'' is an invariant transformation and an expansion of C, that it differs, in other words, from C' at most by terms at least quadratic in the primary constraints. Inasmuch as A' and B' map $\sum_l$ on itself, their commutator must have the same property, and this mapping must be C. Therefore, C'' must be a canonical expansion of C, and that was the burden of the proof. A corollary is that the generators of the infinitesimal invariant canonical transformations form a function group in the terminology of Lie; i.e., their poisson brackets are again generators of infinitesimal invariant canonical transformations.

We now turn our attention to these generators and their relation to the hamiltonian (or hamiltonians) of the theory. We know from the form of the infinitesimal transformations in $\sum_c$ that the generators will be homogeneous linear in the descriptors and their derivatives. The question arises as to the order of the derivatives that appear in the generators. They will, of course, contain no higher derivatives than the transformation laws for the y_A and π^A in $\sum_c$. We have assumed that the order of differentiation of the descriptors in the transformation laws of the y_A is P. In order to find the order of differentiation for the transformation laws of the π^A we proceed as follows. From Eqs. (3.2) we have that

$$\bar{\delta}\pi^A = 2\delta\Lambda^{A4B4}\dot{y}_B + 2\Lambda^{A4B4}\bar{\delta}\dot{y}_A + \cdots. \tag{5.2}$$

According to Eq. (2.4) the $\dot{y}_A$ transform as

$$\bar{\delta}\dot{y}_A = {}^0f_{Ai}\xi^i + \cdots + (-1)^P\, {}^Pf_{Ai}{}^{\mu\cdots\nu}\xi^i{}_{,\mu\cdots\nu}. \tag{5.3}$$

Upon substitution of this expression into (5.2) we see that because of Eq. (3.6), time derivatives of the descriptors appear only up to order P. Space derivatives may appear to a higher order, but that does not concern us, since they can be removed by an integration by parts. In fact, if we restrict ourselves to infinitesimal transformations whose descriptors differ from zero only in a finite spatial domain (a "patch"), we may integrate by parts and assert that the generating density can be brought into the simple form

$$\mathfrak{C} = {}^0A_i\xi^i + {}^1A_i\dot{\xi}^i + \cdots + {}^PA_i\frac{\partial^P\xi^i}{(\partial t)^P}. \tag{5.4}$$

In this form we can ascertain the values for the poisson brackets of the coefficients ${}^nA_i$ by writing down the correct expression for the commutator of two infinitesimal transformations, for, since

$$\bar{\delta}^*(\bar{\delta}y_A) - \bar{\delta}(\bar{\delta}^*y_A) = \bar{\delta}^{**}y_A, \tag{5.5}$$

where $\bar{\delta}$, $\bar{\delta}^*$, and $\bar{\delta}^{**}$ represent transformations of the form (2.4), one has immediately

$$(\mathfrak{C}, \mathfrak{C}^*) = \mathfrak{C}^{**}; \tag{5.6}$$

i.e., the poisson bracket between two generators of the form (5.4) is again of the same form. Hence one can conclude that the poisson bracket between any two functions of the set ${}^nA_i$ must be expressible in terms of members of this set: these ${}^nA_i$ form a function group.

To investigate the poisson bracket relations of the ${}^nA_i$ with the hamiltonian, we proceed as follows. We know from the results of Sec. IV that in the face of an infinitesimal invariant transformation in $\sum_c$ the hamiltonian can change as a function of its arguments only by terms of the form

$$\delta'\mathcal{H} = \int \delta' w^i g_i d^3x, \tag{5.7}$$

the $\delta' w^i$ being some functions of the descriptors of the transformation. (Note that $\bar{\delta}$ represents the change produced in a dynamical variable by an infinitesimal transformation, at a point with the same coordinate values, which, depending on the nature of the transformation, may or may not be the same point, while δ' represents the change in $\mathcal{H}$ and w^i as a function of their arguments, i.e., of the y_A and π^A.) Under the corresponding canonical transformation in $\sum$ the hamiltonian changes by an amount

$$\delta'\mathcal{H} = (\mathfrak{C}, \mathcal{H}) + \frac{\partial\mathfrak{C}}{\partial t} = \int \delta' w^i g_i d^3x, \tag{5.8}$$

where $\mathfrak{C}$ is the generator of the transformation. If we restrict ourselves, for the time being, to descriptors that do not depend on the canonical variables, but only on the coordinates, we can write

$$\int \delta' w^i g_i d^3x = \int \left({}^0A_i \xi^i + \cdots + {}^PA_i \frac{\partial^{P+1}\xi^i}{(\partial t)^{P+1}} \right) d^3x$$
$$+ \int ({}^0A_i, \mathcal{H}) \xi^i d^3x + \cdots + \int ({}^PA_i, \mathcal{H}) \frac{\partial^P \xi^i}{(\partial t)^P} d^3x. \quad (5.9)$$

Now under an infinitesimal transformation $\dot{y}_A$ changes by an amount

$$\bar{\delta}\dot{y}_A = \cdots + u_{Ai} \frac{\partial^{P+1}\xi^i}{(\partial t)^{P+1}}, \quad (5.10)$$

while, according to Eq. (4.7) the change in $\dot{y}_A$ due to the change in the hamiltonian as given by Eq. (5.7) is

$$\bar{\delta}\dot{y}_A = \cdots + u_{Ai} \delta' w^i. \quad (5.11)$$

In each of the above expressions we have explicitly written down only those terms containing $(P+1)$st time derivatives of the descriptors. If Eqs. (5.10) and (5.11) are to agree, as they must if we operate in $\sum_l$, the dependence of $\delta' w^i$ on the $(P+1)$st time derivative of the descriptors is simply

$$\delta' w^i = \cdots + \partial^{P+1}\xi^i / (\partial t)^{P+1} \quad (5.12)$$

and, therefore,

$$g^i = {}^PA_i.$$

However, since $\delta'\mathcal{H}$ is equal to some linear combination of the g_i and hence of the ${}^PA_i$, we can conclude further from Eq. (5.9) that the poisson brackets of all the ${}^nA_i$ with $\mathcal{H}$ can be expressed in terms of the ${}^nA_i$. In fact, since the left-hand side of Eq. (5.9) is nothing more than a linear combination of the ${}^PA_i$, the right-hand side must also have this form. Now, since the ξ^i and all of their time derivatives are independent of each other, the only way in which this can come about is for all of the coefficients of different ξ^i, both with regards to index and differential order, to be linear combinations of the ${}^nA_\rho$. Comparing terms, we find that

$$({}^0A_i, \mathcal{H}) = \text{linear combination of } g_i,$$
$${}^{n-1}A_i + ({}^nA_i, \mathcal{H}) = \text{linear combination of } g_i,$$
$$n = 1, \cdots, P. \quad (5.13)$$

In particular, we see that the ${}^nA_i$, together with the hamiltonian, form a function group.

These results are the ones needed to establish the total number of constraints that must be set if the canonical equations are to be solved and if the primary constraints are to remain satisfied. A necessary and sufficient condition is that the poisson brackets of the primary constraints with the hamiltonian vanish. In case they do not vanish identically, they must be set equal to zero, and the requirement then becomes that the poisson bracket of these expressions with the hamiltonian vanish, and so on until a point is reached where no new constraints are being obtained. We see that the ${}^nA_i$ (or rather certain linear combinations of them) are just the constraints which must be set equal to zero and, because of Eq. (5.13), only a finite number of them exist. This number is at most $(P+1)$ times the number of descriptors necessary to define our invariant transformation group.

VI. APPLICATION TO ELECTROMAGNETIC THEORY

Here we assume a transformation law of the form (2.1). Recasting this into the form of (2.3), we have

$$\bar{\delta}\phi_\mu = \delta_\mu{}^\nu \xi_{,\nu}. \quad (6.1)$$

The lagrangian of the theory is

$$L = (1/8\pi)(\eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma})\phi_{\mu,\rho}\phi_{\nu,\sigma}. \quad (6.2)$$

The field equations are

$$L^\mu = -(1/4\pi)(\eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma})\phi_{\nu,\rho\sigma}. \quad (6.3)$$

The Bianchi identity is

$$-(1/4\pi)(\eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\nu\mu}\eta^{\rho\sigma})\phi_{\nu,\mu\rho\sigma} \equiv 0. \quad (6.4)$$

The equation which corresponds to (3.6) is simply

$$\eta^{44}\eta^{\nu4} - \eta^{\nu4}\eta^{44} \equiv 0. \quad (6.5)$$

The momenta canonically conjugate to the ϕ_μ are

$$\pi^\mu = (1/4\pi)(\eta^{\mu\sigma}\eta^{\nu4} - \eta^{\mu\nu}\eta^{4\sigma})\phi_{\nu,\sigma}. \quad (6.6)$$

We see immediately that the primary constraint [there is just one, since there is just one descriptor in the definition of the infinitesimal transformation law (6.1)] is simply

$$\pi^4 \equiv 0. \quad (6.7)$$

The generating density which corresponds to Eq. (5.4) can be written down immediately. It is

$$\mathfrak{C} = \pi^s{}_{,s}\xi + \pi^4\dot{\xi}. \quad (6.8)$$

At first glance it might appear that this generator is incomplete, since it produces no change in the π^μ. However, examination of the transformation law for the π^μ produced by combining Eqs. (6.1) and (6.6) shows that this is just the case.

The A's for this theory are then just π^4 and $\pi^s{}_{,s}$. Direct calculation then verifies the assertion that together with the hamiltonian they form a function group. In fact, we have that

$$\begin{aligned} &(\pi^4, \pi^s{}_{,s}) \equiv 0, \\ &(\pi^4, \mathcal{H}) = \pi^s{}_{,s}, \\ &(\pi^s{}_{,s}, \mathcal{H}) = 0. \end{aligned} \quad (6.9)$$

Equation (4.6) is thus verifiable directly.

VII. APPLICATION TO COORDINATE-COVARIANT THEORIES

Although the results of the preceding application are well known, it was inserted in order to familiarize the reader with the methodology employed. The application to arbitrary coordinate transformations is not as trivial, and the results are not as well known. We will assume a transformation law of the form (2.3). Here the ξ^μ are the infinitesimal coordinate changes. A thorough discussion of the resulting Bianchi identities has been given in I and will not be repeated here. We merely set down the main result for lagrangians of the form (3.1). For such a langrangian, Eq. (3.6) becomes

$$\Lambda^{A4B4}F_{A\mu}{}^{B4}y_B \equiv 0. \tag{7.1}$$

For this type of invariance Eqs. (4.2) to (4.6) can be taken over intact. The primary constraints (four in number, since there are four descriptors in our transformation law) become

$$g_\mu \equiv F_{A\mu}{}^{B4}y_B\bar{\pi}^A = 0. \tag{7.2}$$

The generating density corresponding to Eq. (5.4) is not as easily obtained as was the case for electromagnetic theory. After some considerable calculations it was found to have the form

$$C = g_\mu\dot{\xi}^\mu - \Big\{ L_\mu - (F_{A\mu}{}^{Br}y_B v^{A\alpha}g_\alpha)_{,r} - \delta_\mu{}^r v^{A\alpha}g_\alpha y_{A,r} + \tfrac{1}{2}F_{C\mu}{}^{B4}y_B\left(\frac{\partial v^{N\alpha}}{\partial y_C} - \frac{\partial v^{C\alpha}}{\partial y_N}\right)E_{ND}\bar{\pi}^D - \delta_\mu{}^4 v^{A\alpha}g_\alpha\dot{y}_A \Big\}\xi^\mu, \tag{7.3}$$

where the new quantities which appear are defined as

$$\begin{aligned} L_\mu &= (g_\mu, \mathcal{H}) \\ &= \frac{1}{2}\left\{F_{A\mu}{}^{C4}E_{CD} - \tfrac{1}{2}F_{C\mu}{}^{B4}y_B\frac{\partial E_{AD}}{\partial y_C}\right\}\bar{\pi}^A\bar{\pi}^D \\ &\quad + F_{C\mu}{}^{B4}y_B\Gamma^{4rC,AE}y_{E,r}E_{AD}\bar{\pi}^D \\ &\quad + F_{C\mu}{}^{Br}y_B\Lambda^{A4C4}(E_{AD}\bar{\pi}^D)_{,r} \\ &\quad + F_{C\mu}{}^{B4}y_B\Gamma^{rsC,AE}y_{A,r}y_{E,s} - 2F_{C\mu}{}^{B4}\Lambda^{ArCs}y_B y_{A,rs}, \end{aligned} \tag{7.4}$$

$$\Gamma^{\mu\nu A,BC} \equiv \frac{\partial\Lambda^{B\mu C\nu}}{\partial y_A} - \frac{\partial\Lambda^{A\mu C\nu}}{\partial y_B} - \frac{\partial\Lambda^{B\mu A\nu}}{\partial y_C},$$

and where the $v^{C\alpha}$ are defined by the equation

$$E_{AB}\Lambda^{B4C4} + v^{C\mu}F_{A\mu}{}^{B4}y_B = \delta_A{}^C. \tag{7.5}$$

The appearance of the $\dot{y}_A$ in the expression for the generator appears at first sight unfortunate, but is unavoidable because the transformation law for field variables in configuration space (including the transport term) depends explicitly on $\dot{y}_A$ [see Eq. (2.3)]. While in $\sum_t$ there exist relationships between y_A, $\dot{y}_A$, and π^A at each point of space-time, these relationships are not sufficient to express $\dot{y}_A$ uniquely as functions of the canonical variables alone. Consequently, there may exist two different fields of $y_A(t)$ in $\sum_t$ which on one spacelike hypersurface have identical values of y_A and π^A everywhere, but nevertheless different $\dot{y}_A$. Thus the transformation law under the canonical transformation cannot be equivalent in $\sum_t$ to the infinitesimal coordinate transformation for all conceivable fields $y_A(t)$ unless the lack of unique determination of the $\dot{y}_A$ by the canonical variables is reflected in their explicit appearance in the generating function.

These "velocities" are all multiplied by primary constraints, and thus in $\sum_t$, at least, transformation laws can be formulated without knowledge of the "poisson bracket" between a velocity and other canonical variables. However, elsewhere in $\sum$, and also for the determination of commutators, these poisson brackets are needed, and we have, as a general rule,

$$(\dot{y}_A, F) = \partial(y_A, F)/\partial t. \tag{7.6}$$

This rule is valid because $\dot{y}_A$ may be interpreted as the difference between two slightly different y_A's, and the poisson bracket is a linear operation with respect to either one of its two components.

We are now in a position to assert that g_μ and L_μ are the only constraints which will appear in a theory because of coordinate invariance, and furthermore, that without or together with the hamiltonian they form a function group. By examining the commutator between various subgroups of transformation laws (2.3) together with the corresponding poisson brackets between the corresponding generators, one can verify that

$$\begin{aligned} (g_\mu(x), g_\nu(x')) &= (\delta_\nu{}^4\delta_\mu{}^\sigma - \delta_\mu{}^4\delta_\nu{}^\sigma)g_\sigma\delta(x-x'), \\ (\bar{L}_\mu(x), \bar{L}_\nu(x')) &= (\delta_\nu{}^r\delta_\mu{}^\sigma - \delta_\mu{}^r\delta_\nu{}^\sigma)\bar{L}_\sigma(\delta(x-x'))_{,r}, \\ (\bar{L}_\mu(x), g_\nu(x')) &= -\delta_\mu{}^4\bar{L}_\nu\delta(x-x') + (\delta_\nu{}^r\delta_\mu{}^\sigma - \delta_\mu{}^r\delta_\nu{}^\sigma)g_\sigma(\delta(x-x'))_{,r}, \end{aligned} \tag{7.7}$$

where $\bar{L}_\mu$ is the coefficient of ξ^μ in Eq. (7.3). All of the above results can, of course, be checked by direct calculation, and one finds that the results are the same. However, owing to the great amount of labor involved in such a calculation and the fact that the calculations cannot be readily carried over to the case of a non-quadratic lagrangian, they will not be presented here.

VIII. CONCLUSION

With this present paper, the canonical formulation of covariant field theories with quadratic lagrangian has been completed. According to our present results, a field-theoretical situation is completely determined if on a three-dimensional hypersurface the canonical variables are given in such a manner that all the constraints (which are finite in number at each point) are satisfied and a hamiltonian has been chosen for all of space-time. In that case, the solutions are unique and do not permit even a coordinate transformation. A coordinate transformation which leaves the initial situation unchanged is equivalent to the adoption of a new hamiltonian away from the initial hypersurface.

In attempting to quantize this type of theory, we can simplify our problem by first carrying out a canonical transformation that converts the primary constraints into canonical momentum densities. As for the secondary (and higher, if necessary) constraints, it appears that their conversion is neither easy in practice nor desirable, because in singular regions the secondary constraints may not be satisfied. Once we have found a proper formulation for the singular regions, the "sources" and "sinks" of the vacuum field, the examination of the quantized covariant theory can be undertaken in earnest.

PHYSICAL REVIEW VOLUME 83, NUMBER 5 SEPTEMBER 1, 1951

Photon Counter Measurements of Solar X-Rays and Extreme Ultraviolet Light

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(Received May 10, 1951)

Data telemetered continuously from photon counters in a V-2 rocket, which rose to 150 km at 10:00 A.M. on September 29, 1949, showed solar 8A x-rays above 87 km, and ultraviolet light around 1200A and 1500A above 70 km and 95 km, respectively. The results indicated that solar soft x-rays are important in E-layer ionization, that Lyman α-radiation of hydrogen penetrates well below E-layer, and that molecular oxygen is rapidly changed to atomic above 100 km.

A V-2 ROCKET, fired in September 1949, carried a set of photon counter tubes which were sensitive to light in the soft x-ray and extreme ultraviolet regions of the spectrum. Each tube responded to a relatively narrow portion of the spectrum in one of four bands covering 0–10A, 1100–1350A, 1425–1650A, and 1725–2100A. The experiment provided an uninterrupted telemetered recording of solar radiation intensities within these wavelength bands throughout the flight of the rocket. Intense solar x-ray emission was detected at altitudes above 87 kilometers. In the atmospheric window near λ 1200A, the photon counters responded strongly above a level of about 70 kilometers. Most of the solar radiation near the peak of the Schumann-Runge absorption band of molecular oxygen, was absorbed between the levels of approximately 95 and 115 km. Only one tube was flown which was sensitive to the longer wavelengths between the oxygen and ozone bands ($\sim$2000A). Its counting rate rose sharply when the rocket reached an altitude of 7 kilometers, but the tube was too sensitive to provide any useful data above 20 kilometers. These measurements support the ideas: that E-layer ionization is directly related to soft x-ray emission from the corona; that Lyman α-radiation penetrates the atmosphere well below E-layer; and that the transition from molecular to atomic oxygen takes place at altitudes near 100 km.

FLIGHT DETAILS

The rocket (V-2, No. 49) was fired at 10:00 A.M., M.S.T. on September 29, 1949 at the White Sands Proving Ground, New Mexico. The altitude of the sun was 43 degrees. No unusual solar activity was noted at the time of the flight. The telemetering record was continuous over the entire flight period of 336 seconds, during which time the rocket soared to a height of 150 km. Fuel cutoff was made at 64 seconds after take-off. For the first 60 seconds the rocket was in stable flight, after which time it developed a slow, steady roll of approximately 12 seconds average period. The roll persisted until the warhead was blown off at 336 seconds.

EXPERIMENTAL DETAILS

An assortment of 6 photon counter tubes was contained in two pressurized boxes, each unit comprising in effect a photon counter spectroscope. The boxes were located on opposite sides of the warhead, with exposed window areas parallel to the surface of the warhead. In addition to the counter tubes, each box also contained one control tube, sensitive only to cosmic rays, and one photocell for determining the roll orientation of the rocket with respect to the sun.

Each tube consisted of a chrome-iron cathode cylinder, $\frac{3}{4}$ inch in diameter and 2 inches long, and an anode wire, 0.025 inch in diameter. The cathode also served as the envelope of the tube. Glass caps, which supported the anode wire, were sealed to the steel cylinder at each end. A circular aperture $\frac{3}{16}$ in. in diameter was provided in a flat recess, milled midway along the length of the cathode. This aperture was covered by