# Nonstatic Solutions of Einstein's Field Equations for Spheres of Fluids Radiating Energy

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The energy tensor for a mixture of matter and outflowing radiation is derived, and a set of equations following from Einstein's field equations are written down whose solutions would represent nonstatic radiating spherical distributions. A few explicit analytical solutions are obtained, which describe a distribution of matter and outflowing radiation for $r\leq a(t)$, an ever-expanding zone of pure radiation for $a(t)\leq r\leq b(t)$ and empty space beyond $r=b(t)$. Since $db(t)/dt$ is almost equal to 1 and $da(t)/dt$ is negative, the solutions obtained represent contracting distributions, but the contraction is not gravitational because $m/r$ is a constant on the boundary $r=a(t)$, $m$ being the mass. The contraction is a purely relativistic effect, the corresponding newtonian distributions being equilibrium distributions. It is hoped that the scheme developed here will be useful in working out solutions which would help in a clear understanding of the initial or the final stages of stellar evolution.

## I. INTRODUCTION

IT is well known that during the initial stages of stellar evolution, and during the final stages when the thermonuclear sources of energy are exhausted, the conditions within the interior of the star are such that the general relativistic corrections to the newtonian theory become important. So a description of the stellar interior based on the relativistic gravitational theory would provide a clearer picture of the processes during these stages of stellar evolution. Taking appropriate static solutions of Einstein's field equations, Oppenheimer and Volkoff[1] have found that equilibrium configurations of massive neutron cores do not exist for masses greater than $0.7\odot$. Therefore, for description, of the final evolutionary stages of heavier stars nonstatic solutions of the field equations have to be considered. Oppenheimer and Snyder[2] have succeeded in getting a nonstatic solution representing the continued gravitational contraction of a stellar body after all its sources of thermonuclear energy are exhausted. But in obtaining this solution the radial pressure of the stellar matter and the gravitational effect of any escaping radiation were neglected. The importance of nonstatic solutions in understanding certain aspects of the earlier stages of stellar evolution has been recently emphasized by Klein.[3] Thus it appears that a study of the field equations of general relativity with a view to getting solutions representing nonstatic spherically symmetric distributions is a necessary first step toward a clear understanding of the problems of the earlier and the later stages of stellar evolution.

A nonstatic distribution would be radiating energy, and so it would be surrounded by an ever-expanding zone of radiation. If this radiating distribution together with its radiation is to form an isolated system, beyond the zone of pure radiation we must have empty space given by the Schwarzschild's exterior solution. In the following we shall develop a scheme for working out solutions of Einstein's field equations which will sustain the above picture of a nonstatic distribution. Using a line-element we have

$$ds^2=-e^{\lambda}dR^2-R^2(d\theta^2+\sin^2\theta d\varphi^2)+e^{\nu}dT^2,\quad \lambda=\lambda(R,T),\quad \nu=\nu(R,T). \tag{1.1}$$

We shall work out solutions which will describe (1) a "mixture" of matter and outflowing radiation for $R\leq R_i(T)$, (2) a zone of pure outflowing radiation for $R_i(T)\leq R\leq R_e(T)$, and (3) empty space for $R\geq R_e(T)$, the line-element being then given by the well-known Schwarzschild's exterior solution,

$$ds^2=-(1-2M/R)^{-1}dR^2-R^2(d\theta^2+\sin^2\theta d\varphi^2)+(1-2M/R)dT^2, \tag{1.2}$$

$M$, a constant, being the total mass of the distribution and the radiated energy. We shall call the solution for $R\leq R_i(T)$ the interior solution and the one holding good for $R_i(T)\leq R\leq R_e(T)$ the exterior solution. In this notation the boundary $R=R_i(T)$ will be called the interior boundary and $R=R_e(T)$ will be called the exterior boundary. We begin with the method of finding the interior solution.

## II. THE ENERGY-TENSOR FOR THE INTERIOR FIELD

Following Tolman[4] we express the energy-momentum tensor for the combined field of matter and radiation in the form

$$T^{\mu\nu}=T^{\mu\nu}{}_{(me)}+T^{\mu\nu}{}_{(em)}. \tag{2.1}$$

Here $T^{\mu\nu}{}_{(me)}$ stands for the mechanical energy tensor, i.e., the tensor due to matter. $T^{\mu\nu}{}_{(em)}$ stands for the electromagnetic tensor, i.e., the tensor due to radiation. The matter comprising the distribution may be regarded as a perfect fluid so that

$$T^{\mu\nu}{}_{(me)}=(p+\rho)v^\mu v^\nu-pg^{\mu\nu}; \tag{2.2}$$

$$v^\mu v_\mu=1; \tag{2.3}$$

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[1] J. R. Oppenheimer and G. M. Volkoff, Phys. Rev. **55**, 374 (1939).

[2] J. R. Oppenheimer and H. Snyder, Phys. Rev. **56**, 455 (1939).

[3] O. Klein, Arkiv Mat. Astron. Fysik **34A**, 19 (1947).

[4] R. C. Tolman, *Relativity Thermodynamics and Cosmology* (Oxford University Press, New York, 1934), p. 261.

and for the directional flow of radiation in empty space, the author has proved elsewhere[5] the formula

$$T^{\mu\nu}{}_{(em)}=\sigma w^{\mu}w^{\nu}, \tag{2.4}$$

$$w^{\mu}w_{\mu}=0, \quad (w^{\mu})_{,\nu}w^{\nu}=0, \tag{2.5}$$

$\sigma$ being the density of the flowing radiation. This simple formula can obviously be taken over to the case of radiation flowing through a medium; the presence of the medium will affect only the law of variation of the scalar $\sigma$. Thus we can write the final expression for the energy tensor (2.1) as

$$T^{\mu\nu}=(p+\rho)v^{\mu}v^{\nu}-pg^{\mu\nu}+\sigma w^{\mu}w^{\nu}, \tag{2.6}$$

with the further Eqs. (2.3) and (2.5) for the vectors $v^{\mu}$ and $w^{\mu}$.

We shall now work out the consequences of the fundamental equations

$$(T_{\mu}{}^{\nu})_{,\nu}=0. \tag{2.7}$$

For this purpose we choose a co-moving frame of reference, i.e., a frame of reference relative to which the fluid comprising the distribution is at rest. In such a reference-frame the line-element showing spherical symmetry cannot be of the form (1.1) but has to be of the general form

$$\begin{gathered} ds^2=-e^{\alpha}dr^2-r^2e^{\beta}(d\theta^2+\sin^2\theta d\varphi^2)+e^{\gamma}dt^2, \\ \alpha=\alpha(r,t), \quad \beta=\beta(r,t), \quad \gamma=\gamma(r,t). \end{gathered} \tag{2.8}$$

Using this line-element and the form (2.6) of the energy tensor with the additional restrictions

$$v^1=v^2=v^3=0, \quad v^4=e^{-\gamma/2} \tag{2.9}$$

due to the co-moving nature of the reference-frame, Eqs. (2.7) lead to

$$-p'-\tfrac{1}{2}(p+\rho)\gamma'+(\sigma w^{\nu})_{,\nu}w_1=0 \tag{2.10}$$

and

$$\dot{\rho}+(p+\rho)(\dot{\beta}+\dot{\alpha}/2)+(\sigma w^{\nu})_{,\nu}w_4=0. \tag{2.11}$$

Here a prime denotes a differentiation with regard to $r$, while an overhead dot denotes that with regard to $t$. On elimination of $\sigma$ between these two equations we find

$$p'+\tfrac{1}{2}(p+\rho)\gamma'-e^{(\alpha-\gamma)/2}[\dot{\rho}+(p+\rho)(\dot{\beta}+\dot{\alpha}/2)]=0. \tag{2.12}$$

Equation (2.12) replaces the familiar equation

$$p'+(p+\rho)\gamma'/2=0$$

for the static distributions. We shall now show that (2.11) leads, at the newtonian level of approximation, to the well-known equation expressing the equilibrium of a stellar configuration[6]

$$p'+p_r'=-G\rho M(r)/r^2, \tag{2.13}$$

$p_r$ being the radiation pressure; while Eq. (2.11) will be a purely relativistic equation having no newtonian counterpart. It has already been proved that[7] in space not occupied by matter

$$\sigma w_{\mu}w^{\nu}=-F_{\mu\alpha}F^{\nu\alpha}+\tfrac{1}{4}\delta_{\mu}{}^{\nu}F_{\alpha\beta}F^{\alpha\beta}, \tag{2.14}$$

where $F^{\mu\nu}$ is the skew-symmetric electromagnetic field tensor, so that

$$(\sigma w^{\nu})_{,\nu}=0.$$

When the radiation traverses a medium, it is weakened by absorption and so (2.14) will not hold good. If $\sigma$ is the density of flowing radiation after it has traversed a length $s$ of the medium and $\sigma_0$ the density that would be obtained if there were no medium present, we may write

$$\sigma=\sigma_0e^{-f}$$

so that

$$\sigma w_{\mu}w^{\nu}=e^{-f}[-F_{\mu\alpha}F^{\nu\alpha}+\tfrac{1}{4}\delta_{\mu}{}^{\nu}F_{\alpha\beta}F^{\alpha\beta}], \tag{2.15}$$

where $f=f(r,t)$ may be regarded as the "optical depth" of the medium[8] which is connected with its mass absorption coefficient $\kappa$ through

$$df/ds\equiv w^1(\partial f/\partial r)+w^4\partial f/\partial t=\kappa\rho.$$

With this law of absorption of the medium, (2.10) immediately transforms to

$$-p'-\tfrac{1}{2}(p+\rho)\gamma'-\kappa\rho\sigma w_1=0.$$

Again, for quasi-static processes, writing

$$p_r'=-\kappa\rho L(r)/4\pi r^2$$

we can finally get

$$p'+p_r'=-(p+\rho)\gamma'/2, \tag{2.16}$$

where $L(r)=-4\pi r^2\sigma w_1$ measures the luminosity of the distribution at the newtonian level of approximation. Equation (2.16) readily gives the familiar equation (2.13). We are thus able to recover the newtonian equation of stellar equilibrium from our energy-momentum tensor (2.6). This gives an increased confidence in the form (2.6) of the energy tensor for a mixture of matter and radiation.

## III. GENERAL METHOD OF FINDING THE INTERIOR SOLUTION

Using the co-moving frame of reference introduced in the last section we write the line-element as

$$\begin{gathered} ds^2=-e^{\alpha}dr^2-r^2e^{\beta}(d\theta^2+\sin^2\theta d\varphi^2)+e^{\gamma}dt^2, \\ \alpha=\alpha(r,t), \quad \beta=\beta(r,t), \quad \gamma=\gamma(r,t). \end{gathered} \tag{3.1}$$

The components of $T_{\mu}{}^{\nu}$ can now be calculated in terms of the functions $\alpha$, $\beta$, $\gamma$ with the help of the field equa-

[5] P. C. Vaidya, Proc. Indian Acad. Sci., to be published.

[6] S. Chandrasekhar, *Introduction to the Study of Stellar Structure* (Chicago University Press, Chicago, 1939), pp. 213, 214.

[7] V. V. Narlikar and P. C. Vaidya, Nature **159**, 642 (1947).

[8] See reference 6, p. 191. For simplicity we are considering here the case of homogeneous radiation.

tions of Einstein. We find these components to be

$$8\pi T_1^1 = -e^{-\alpha}\left[\frac{\beta'^2}{4}+\frac{\beta'\gamma'}{2}+\frac{\beta'+\gamma'}{r}+\frac{1}{r^2}\right] +\frac{e^{-\beta}}{r^2}+e^{-\gamma}\left[\frac{\partial^2\beta}{\partial t^2}+\frac{3}{4}\dot{\beta}^2-\frac{\dot{\beta}\dot{\gamma}}{2}\right]; \quad (3.2)$$

$$8\pi T_2^2 = 8\pi T_3^3 = -e^{-\alpha}\left[\frac{\beta''}{2}+\frac{\gamma''}{2}+\frac{\beta'^2}{4}+\frac{\gamma'^2}{4}-\frac{\alpha'\beta'}{4} +\frac{\beta'\gamma'}{4}-\frac{\alpha'\gamma'}{4}-\frac{\alpha'}{2r}+\frac{\beta'}{r}+\frac{\gamma'}{2r}\right] +e^{-\gamma}\left[\frac{\partial^2\alpha/\partial t^2}{2}+\frac{\partial^2\beta/\partial t^2}{2}+\frac{\dot{\alpha}^2}{4}+\frac{\dot{\beta}^2}{4} +\frac{\dot{\alpha}\dot{\beta}}{4}-\frac{\dot{\alpha}\dot{\gamma}}{4}-\frac{\dot{\beta}\dot{\gamma}}{4}\right]; \quad (3.3)$$

$$8\pi T_4^4 = -e^{-\alpha}\left[\beta''+\frac{3}{4}\beta'^2-\frac{\alpha'\beta'}{2}+\frac{3\beta'}{r}-\frac{\alpha'}{r}+\frac{1}{r^2}\right] +\frac{e^{-\beta}}{2}+e^{-\gamma}\left[\frac{\dot{\alpha}\dot{\beta}}{2}+\frac{\dot{\beta}^2}{4}\right]; \quad (3.4)$$

$$8\pi T_1^4 = -e^{-\gamma}\left[\dot{\beta}'-\frac{\dot{\beta}\gamma'}{2}+(\dot{\beta}-\dot{\alpha})\frac{\beta'}{2}+(\dot{\beta}-\dot{\alpha})\frac{1}{r}\right]. \quad (3.5)$$

But the form (2.6) of the tensor $T_\mu{}^\nu$, *viz.*

$$T_\mu{}^\nu = (p+\rho)v_\mu v^\nu - pg_\mu{}^\nu + \sigma w_\mu w^\nu \quad (3.6)$$

with

$$v^1=v^2=v^3=0, \quad v^4=e^{-\gamma/2} \quad (3.7)$$

and

$$w^2=w^3=0, \quad w_\mu w^\mu=0, \quad (w^\mu)_{;\nu}w^\nu=0, \quad (3.8)$$

gives

$$T_1^1=-p+\sigma w_1w^1, \quad T_2^2=T_3^3=-p, \quad T_4^4=\rho+\sigma w_4w^4, \quad T_1^4=\sigma w_1w^4. \quad (3.9)$$

Hence we can find the physical quantities $p$, $\rho$, $\sigma$, $v^\mu$, and $w^\mu$ in terms of the functions $\alpha$, $\beta$, and $\gamma$:

$$p=-T_2^2; \quad \rho=T_1^1+T_4^4-T_2^2; \quad (3.10)$$

$$\sigma=-e^{(3\alpha-\gamma)/2}(w^1)^2T_1^4; \quad (3.11)$$

$$w^4=e^{(\alpha-\gamma)/2}w^1;$$

$$\frac{\partial w^1}{\partial r}+e^{(\alpha-\gamma)/2}\frac{\partial w^1}{\partial t}+w^1\left[\frac{\alpha'+\gamma'}{2}+e^{(\alpha-\gamma)/2}\dot{\alpha}\right]=0. \quad (3.12)$$

The last equation (3.12) follows from (3.8). Once $\alpha$, $\beta$, $\gamma$ are known as functions of $r$ and $t$, Eqs. (3.2) to (3.6) determine the components of $T_\mu{}^\nu$ as functions of $r$ and $t$ and then (3.10), (3.11), and (3.12) determine the march of the physical variables $\rho$, $p$, $\sigma$, etc., through the distribution at any time $t$. Equations (3.9) lead to the following relation between the components of $T_\mu{}^\nu$:

$$T_1^1-T_2^2=e^{(\gamma-\alpha)/2}T_1^4. \quad (3.13)$$

Substituting the values of $T_1^1$, $T_2^2$ and $T_1^4$ from (3.2), (3.3), and (3.5) we find that (3.13) gives

$$\frac{\beta''}{2}+\frac{\gamma''}{2}+\frac{\gamma'^2}{4}-\frac{\alpha'\beta'}{4}-\frac{\beta'\gamma'}{4}-\frac{\gamma'\alpha'}{4}-\frac{\alpha'}{2r}-\frac{\gamma'}{2r}-\frac{1}{r^2} +\frac{e^{\alpha-\beta}}{r^2}+e^{\alpha-\gamma}\left[\frac{\partial^2\beta/\partial t^2}{2}-\frac{\partial^2\alpha/\partial t^2}{2}-\frac{\dot{\alpha}^2}{4}+\frac{\dot{\beta}^2}{2} -\frac{\dot{\alpha}\dot{\beta}}{4}-\frac{\dot{\beta}\dot{\gamma}}{4}+\frac{\dot{\alpha}\dot{\gamma}}{4}\right]+e^{(\alpha-\gamma)/2}\left[\dot{\beta}'-\frac{\dot{\beta}\gamma'}{2} +(\dot{\beta}-\dot{\alpha})\frac{\beta'}{2}+(\dot{\beta}-\dot{\alpha})\frac{1}{r}\right]=0. \quad (3.14)$$

It may be noted that this relation is identical with Eq. (2.13) of the last section. Equation (3.14) is one relation between the three functions $\alpha$, $\beta$, and $\gamma$. This relation, it will be remembered, is a consequence of the co-moving nature of the coordinate system used. Two more relations are necessary in order to determine the three functions $\alpha$, $\beta$, and $\gamma$. These relations are not supplied by the gravitational theory but are given by the physical nature of the distributions under consideration. They are: (1) the equation of state of the material of the distribution usually expressed as a relation between $p$ and $\rho$; (2) the law of energy generation within the distribution. Given these two relations we must use (3.14) with them and determine the forms of the three functions $\alpha$, $\beta$, and $\gamma$; and then, as stated above, the complete march of the physical variables within the distribution can be determined. This is then the general scheme of working out the internal solution.

But even in the static case (where $\sigma=0$) the problem treated in this direct way presents a formidable mathematical task and explicit solutions are not known for any but the simplest equations of state ($\rho=0$ and $\rho=a$ constant). We therefore adopt the indirect procedure suggested by Tolman[9] of solving (3.14) together with two assumed relations between $\alpha$, $\beta$, and $\gamma$ and then seeing what equation of state and what type of energy-generating system the solution so obtained possesses. But there is one important point to be noted in applying Tolman's method to our case. The interior boundary is obtained by finding the first value of $r$ at which the fluid pressure $p$ vanishes. But as the coordinates in use are co-moving, the radius of the interior boundary must be constant so that the two assumed relations between $\alpha$, $\beta$, and $\gamma$ must be such as would make the resulting $p$ vanish for a constant value of $r$. This considerably limits the choice of the two relations to be assumed. In the

[9] R. C. Tolman, Phys. Rev. 55, 364 (1939).

TABLE I. Relations among the constants in the nine mutually independent interior solutions.

| Solution | $p$ and $m$ | $u$ and $l$ | $n$ | $s+1$ | $Ck$ | $D^2$ |
|---|---|---|---|---|---|---|
| I | $p=s$<br>$m=n$ | $u=s+1$<br>$l=n$ | 1 | — | $2(s+1-q)$ | $-2(s+1)(s+1-2q)$ |
| II | $p=s$<br>$m=n$ | $u=s+1$<br>$l=n$ | 0 | — | — | $2(s+1)(s+1-Ck)$ |
| III | $p=s$<br>$m=n$ | $u=s+1$<br>$l=n-1$ | — | $q(2n^2-1)/n$ | $2nqB$ | $2q^2(1-2n^2)/n^2$ |
| IV | $p=s-2q$<br>$m=n-1$ | $u=s+1$<br>$l=n-1$ | — | 0 | $Bq(3-4n)/(n-1)$ | $4q^2(2n^2-1)$ |
| V | $p=s-q$<br>$m=n-\frac{1}{2}$ | $u=s+1$<br>$l=n$ | $(3\pm\sqrt{5})4$ | — | $s+1-q$ | $2q[(4n-1)q-(2n-1)(s+1)]$ |
| VI | $p=s-q$<br>$m=n-\frac{1}{2}$ | $u=s+1$<br>$l=n$ | $(3\pm\sqrt{5})/4$ | 0 | — | $8n^2q^2+4Cknq$ |
| VII | $p=s$<br>$m=n-\frac{1}{2}$ | $u=s+1-2q$<br>$l=n-1$ | 1 | — | $s+1-q$ | $2B(s+1)^2-4Bq^2$ |
| VIII | $p=s$<br>$m=n-\frac{1}{2}$ | $u=s+1-2q$<br>$l=n-1$ | — | $2nq$ | $(1-2n)q$ | $4Bq^2(2n^2-1)$ |
| IX | $p=s$<br>$m=n-\frac{1}{2}$ | $u=s+1-2q$<br>$l=n-1$ | — | 0 | $(1-2n)q$ | $-4Bq^2$ |

next section we show that a large number of solutions of Eq. (3.14) can be obtained by choosing particular forms of the functions $\alpha$, $\beta$, and $\gamma$:

## IV. PARTICULAR INTERIOR SOLUTIONS

We try to solve Eq. (3.14) with the following forms of the functions $\alpha$, $\beta$, and $\gamma$:

$$e^{-\alpha}=A^2e^{2kt}r^{-2s}(B+r^{2q})^{2n};\tag{4.1}$$

$$e^{-\beta}=A^2D^2e^{2kt}r^{-2p}(B+r^{2q})^{2m};\tag{4.2}$$

$$e^{-\gamma}=A^2C^2e^{2kt}r^{-2u}(B+r^{2q})^{2l}.\tag{4.3}$$

All the twelve new symbols introduced here stand for constants whose values are to be suitably determined.[10] When these forms of $\alpha$, $\beta$, $\gamma$ are used in the differential equation (3.14), it will be found that, in general, seven restrictions are imposed on these twelve parameters so that seven of them get determined in terms of the rest. Now it happens that Eq. (3.14) leaves a fairly wide choice of the seven constants to be determined by it and as many as 39 different solutions of that equation can be obtained by this method. But some of these solutions are mutually transformable by simple transformations of the radial coordinate. The accompanying table gives the nine sets of relations between these constants representing the nine mutually independent solutions of (3.14). In Table I the second and the third columns give the values of the four parameters $p$, $m$, $u$, and $l$ and the last four columns give the values of the other four parameters $n$, $s+1$, $Ck$, and $D^2$. As only seven parameters are determined by (3.14), there is always one dash in one of the last four columns for every solution. The physical contents of all these solutions have been investigated and it is found that they can be grouped together into four different groups. We put on record here one solution of each of these four physically distinct types.

[10] There will be no occasion for any confusion between the parameter $p$ introduced in (4.2) and the fluid pressure $p$.

### Solution III

We have

$$e^{\alpha}=\frac{e^{-2kt}r^{2s}}{A^2(B+r^{2q})^{2n}},\quad e^{\beta}=\frac{e^{\alpha}}{D^2},\quad e^{\gamma}=r^2(B+r^{2q})^2\frac{e^{\alpha}}{C^2},$$

with $s+1=q(2n^2-1)/n$, $Ck=2nqB$, $n^2D^2=2q^2(1-2n^2)$.

We then have

$$8\pi p=\frac{e^{-\alpha}q^2(1-2n)}{r^2n^2}\cdot\frac{(1-2n)r^{2q}+(1+2n)B}{r^{2q}+B},$$

$$8\pi\rho=\frac{e^{-\alpha}q^2}{r^2n^2}\cdot\frac{(1-4n^2)r^{2q}+(1+4n^2)B}{r^{2q}+B},$$

$$8\pi T_1{}^4=-\frac{e^{-\alpha}4CBq^2}{r^3}\cdot\frac{(2n-1)r^{2q}+(2n^2-1)B}{(r^{2q}+B)^3},$$

$$w^1=Ee^{(-\alpha-\gamma)/2},$$

$E$ is an arbitrary constant. This solution can give a finite constant radius of the interior boundary where $p$ will vanish. If $r=a$ is to be the interior boundary, the solution will be of physical significance only if

$$\text{(i)}\quad p=0 \text{ for } r=a; \tag{4.4}$$

$$\text{(ii)}\quad p>0 \text{ for } r<a; \tag{4.5}$$

$$\text{(iii)}\quad \rho>0 \text{ for } r\leq a. \tag{4.6}$$

These restrictions are satisfied by this solution if $B=-a^{2q}(1-2n)/(1+2n)$, $q$ is negative, and $0<n<\frac{1}{2}$. For $\frac{1}{4}\leq n<\frac{1}{2}$, $\rho-3p$ will be nonnegative for $r\leq a$. It is now found that $T_1{}^4$ is negative for $r\leq a$ so that the product $\sigma w^1$ is positive and we find

$$8\pi\sigma=\frac{e^{3(\gamma-\alpha)/2}4CBq^2}{E^2r^3}\cdot\frac{(2n-1)r^{2q}+(2n^2-1)B}{(r^{2q}+B)^3},$$

so that $\dot{\sigma}=0$. This last property is common to all our solutions.

## Solution VIII

We have

$$e^{\alpha}=\frac{e^{-2kt}r^{2s}}{A^2(B+r^{2q})^{2n}},\quad e^{\beta}=(B+r^{2q})\frac{e^{\alpha}}{D^2},$$

$$e^{\gamma}=r^{2-4q}(B+r^{2q})^2e^{\alpha}/C^2$$

with $s+1=2nq$, $Ck=(1-2n)q$, $D^2=4Bq^2(2n^2-1)$.

We then have

$$8\pi p=4q^2(1-n)\frac{e^{-\alpha}}{r^2}\cdot\frac{nr^{2q}+(1-n)B}{r^{2q}+B},$$

$$8\pi\rho=4q^2(n-1)\frac{e^{-\alpha}}{r^2}\cdot\frac{3nr^{2q}+(n+1)B}{r^{2q}+B},$$

$$8\pi T_1{}^4=-\frac{4q^2(1-2n)(n-1)CBe^{-\alpha}}{r^{3-4q}(B+r^{2q})^3},$$

$$w^1=Ee^{-(\alpha+\gamma)/2}.$$

This solution can give a finite constant radius $r=a$ of the interior boundary; and the conditions (4.4), (4.5), (4.6) are satisfied if $B=na^{2q}/(n-1)$, $q$ is positive, and $n$ does not lie in the range $(-1,+1)$. It is now found that $T_1{}^4$ is positive for $r\leq a$. This makes $\sigma w^1$ negative; and since we have begun with $w^1$ positive, this means that the solution gives negative values of $\sigma$ within the sphere $r=a$. It appears, therefore, that the solution has no physical significance. But then, as given by the differential equation (3.12), $w^1$ always gets determined with an arbitrary multiplying constant and so the negative sign of the product $\sigma w^1$ can as well be attributed to $w^1$ making $\sigma$ positive. Of course, once we accept a negative value of $w^1$, the whole problem of solving Einstein's field equations has to be studied afresh. The distribution then would not be radiating energy, but it would be absorbing energy. The results of this paper will no longer hold because an absorbing distribution will not be isolated. This particular solution leads to quite new types of distributions—radiation absorbing distributions—allowed by the gravitational theory of Einstein. These new types of distributions are at present being investigated.[11]

## Solution IX

We have

$$e^{\alpha}=\frac{e^{-2kt}}{A^2r^2(B+r^{2q})^{2n}},\quad e^{\beta}=(B+r^{2q})\frac{e^{\alpha}}{D^2},$$

$$e^{\gamma}=r^{2-4q}(B+r^{2q})^2e^{\alpha}/C^2,$$

with $Ck=(1-2n)q$, $D^2=-4q^2B$.

We then have

$$8\pi p=\frac{e^{-\alpha}}{r^2}\cdot\frac{4q^2B}{r^{2q}+B},$$

$$8\pi\rho=-\frac{e^{-\alpha}\cdot 4q^2B}{r^2(B+r^{2q})},$$

$$8\pi T_1{}^4=\frac{e^{-\alpha}4Cq^2(1-2n)}{r^{3-4q}}\cdot\frac{nr^{2q}+B}{(r^{2q}+B)^3}.$$

Here we have $\rho+p=0$. If the cosmological constant $\Lambda$ is introduced, it will be found that DeSitter's static cosmological solution is a particular case of this solution obtained by putting $n=\frac{1}{2}$ here.

## Solution IV

We have

$$e^{\alpha}=\frac{e^{-2kt}}{A^2r^2(B+r^{2q})^{2n}},\quad e^{\beta}=(B+r^{2q})^2r^{-4q}\frac{e^{\alpha}}{D^2},$$

$$e^{\gamma}=r^2(B+r^{2q})^2e^{\alpha}/C^2,$$

with $Ck=B(3-4n)q/(n-1)$, $D^2=4q^2(2n^2-1)$.

We then have

$$8\pi p=\frac{4q^2e^{-\alpha}}{r^2(B+r^{2q})^2}[(n-1)^2r^{4q}+Br^{2q}-B^2(2n-1)(6n-5)/4(n-1)^2],$$

$$8\pi\rho=\frac{4q^2e^{-\alpha}}{r^2(B+r^{2q})^2}[(n^2-1)r^{4q}+Br^{2q}(1-6n)+3B^2(2n-1)(6n-5)/4(n-1)^2],$$

$$8\pi T_1{}^4=\frac{4Cq^2e^{-\alpha}}{r^3(B+r^{2q})^3}B(3-4n)r^{2q}.$$

This solution can give a finite boundary $r=a$ and the conditions (4.4), (4.5), (4.6) are satisfied by (1) $q$ is

[11] P. C. Vaidya, Nature **166**, 565 (1950).

negative, (2) $B=a^{2q}/x$ or $B=-a^{2q}/y$, where

$$x,\, y=\tfrac{1}{2}[(12n^2-16n+6)^{\frac{1}{2}}\mp 1]/(n-1)^2,$$

and (3) $1<n\leq 2$ when $B$ is negative and $1<n\leq z$ when $B$ is positive, where $z$ is that root of the cubic

$$45z^3-123z^2+106z-31=0$$

which lies between 1 and 2. Under these conditions we have $\rho-3p\geq 0$ for $r\leq a$.

It is found that when $B$ is positive, $T_1{}^4$ is negative, so that $w^1$ is positive and the distribution is a radiating one. It is further found that in this case $\rho-3p$ does not strictly decrease from $r=0$ to the value of $\rho$ on the boundary, but that it attains a non-negative minimum value at a point within the distribution. When $B$ is negative, $T_1{}^4$ is positive, so that the solution then represents a radiation-absorbing distribution.

## V. THE EXTERIOR SOLUTION

The exterior solution of a radiating star describing a distribution of pure outflowing radiation has already been completely discussed elsewhere,[12] and we put down here the results of this solution that are useful for our present work. The line-element describing this field is

$$ds^2=\frac{-1}{1-2m/R}dR^2-R^2(d\theta^2+\sin^2\theta d\varphi^2)+\frac{\dot{m}^2}{f^2}\left(1-\frac{2m}{R}\right)dT^2, \quad (5.1)$$

where $m=m(R,T)$, $\dot{m}=\partial m/\partial T$, and $f=f(m)$ is an arbitrary function of $m$ given by

$$(\partial m/\partial R)(1-2m/R)=f(m). \quad (5.2)$$

This solution is true for $R_i(T)\leq R\leq R_e(T)$, and it is continuous at $R=R_e(T)$ with the Schwarzschild's exterior solution

$$ds^2=-(1-2M/R)^{-1}dR^2-R^2(d\theta^2+\sin^2\theta d\varphi^2)+(1-2M/R)dT^2.$$

The continuity of $g_{\mu\nu}$ at $R=R_e(T)$ determines the function $R_e(T)$ and the arbitrary function $\varphi(T)$ of $T$ which occurs when (5.2) is solved as a differential equation for $m$. It is found that $R_e(T)$ is given by

$$(dR_e/dT)=1-2M/R_e. \quad (5.3)$$

The function $f(m)$ is left arbitrary and will be determined by the conditions at the interior boundary $R=R_i(T)$. The energy-momentum tensor describing this zone of pure radiation is given by

$$T_\mu{}^\nu=\sigma w_\mu w^\nu,\quad w_\mu w^\mu=0,\quad (w^\mu)_{;\nu}w^\nu=0$$

and we have

$$\sigma=F(m)/4\pi r^2,\quad w^1=[f(m)/F(m)]^{\frac{1}{2}}, \quad (5.4)$$

$F(m)$ being another arbitrary function of $m$.

[12] P. C. Vaidya, Current Sci. (India) 12, 183 (1943); see reference 5. See also H. Mineur, Ann. école normale supér. Sér. 3, 5, 1 (1933).

Equation (5.2) for the function $m(R,T)$ can be readily solved in the particular case

$$f(m)=f=\text{a constant}.$$

If $f<\frac{1}{8}$ we find that

$$(m-xR)^j(m-yR)^h=\varphi(T) \quad (5.5)$$

satisfies (5.2) if

$$x,\, y=\tfrac{1}{4}(1\pm(1-8f)^{\frac{1}{2}}),\quad j,\, h=\tfrac{1}{2}(1\mp(1-8f)^{-\frac{1}{2}}).$$

The continuity of $g_{\mu\nu}$ at the exterior boundary determines $\varphi(T)$ to be

$$\varphi(T)=(M-xR_e)^j(M-yR_e)^h.$$

We are now ready to make the connection of our interior solution with this exterior solution over the boundary.

## VI. CONNECTION BETWEEN THE INTERIOR AND THE EXTERIOR SOLUTIONS

Our interior solutions have been obtained with reference to the co-moving coordinates characterized by the line-element

$$ds^2=-e^\alpha dr^2-r^2e^\beta(d\theta^2+\sin^2\theta d\varphi^2)+e^\gamma dt^2, \quad (6.1)$$

while the exterior solution is developed in the coordinates $(R,T)$ of the line-element,

$$ds^2=-e^\lambda dR^2-R^2(d\theta^2+\sin^2\theta d\varphi^2)+e^\nu dT^2. \quad (6.2)$$

The first problem then is to transform the coordinates $(r,t)$ of (6.1) to the coordinates $(R,T)$ of (6.2). As the general features of all the radiating distributions obtained here are the same, we illustrate the method of making the connection by taking a particular case of Solution III and connecting it with the exterior solution.

### Detailed Consideration of a Particular Case of Solution III

We shall choose $n=\frac{1}{4}$ so that $\rho_c/p_c=3$, $\rho_c$ being the central density and $p_c$ being the central pressure. The solution then is

$$e^\alpha=\frac{e^{-2kt}\cdot r^{-7q}}{A^2r^2(B+r^{2q})^{\frac{1}{2}}},\quad e^\beta=\frac{e^\alpha}{28q^2},\quad e^\gamma=\frac{e^\alpha r^2(B+r^{2q})^2}{C^2}.$$

$$8\pi p=\frac{12q^2}{r^2}e^{-\alpha}\cdot\frac{r^{2q}-a^{2q}}{3r^{2q}-a^{2q}},$$

$$8\pi\rho=\frac{4q^2}{r^2}e^{-\alpha}\cdot\frac{9r^{2q}-5a^{2q}}{3r^{2q}-a^{2q}}, \quad (6.3)$$

$$8\pi\sigma=\frac{q^2a^{2q}}{18E^2C^2}(12r^{2q}-7a^{2q}),$$

$$v^1=v^2=v^3=0,\quad v^4=e^{-\gamma/2},$$

$$w^2=w^3=0,\ w^1=Ee^{-(\alpha+\gamma)/2},\ w^4=Ee^{-\gamma},\ Ck=-qa^{2q}/6.$$

$B$ is given in terms of the boundary radius by $B=-a^{2q}/3$ and we must take $q$ to be negative.

It will be noted that the solution corresponds to a sphere of fluid of infinite density and pressure at the center, having at that point the ratio which would hold for disordered radiation or for particles of such high kinetic energy that their rest mass may be neglected in comparison with their total mass. Other ratios could of course be obtained with a different choice for the parameter $n$. The constant $B$ is negative. As $B$ approaches zero through negative values, the ratio of the pressure and the density approach $\frac{1}{3}$ throughout, the constant $k$ approaches zero, and the sphere tends to be static and larger without limit.

We now transform this solution to the line-element (6.2). Let the equations of transformation be

$$R=R(r,t), \quad T=T(r,t).$$

The law of transformation of the tensor $g^{\mu\nu}$ leads to

$$e^{-\lambda}=R'^2e^{-\alpha}-\dot{R}^2e^{-\gamma}, \tag{6.4}$$

$$R^2=r^2e^{\beta}, \tag{6.5}$$

$$e^{-\nu}=-T'^2e^{-\alpha}+\dot{T}^2e^{-\gamma}, \tag{6.6}$$

$$0=-R'T'e^{-\alpha}+\dot{R}\dot{T}e^{-\gamma}. \tag{6.7}$$

Equation (6.5) immediately gives $R(r,t)$. Equation (6.7) can be integrated to find $T(r,t)$. It gives

$$(3)^{-7}r^{96q}(24r^{2q}-7a^{2q})^8(28A^2q^2)^{14}R^{28}\Psi(T)=1, \tag{6.8}$$

$\Psi(T)$ being an arbitrary function of $T$. The boundary $R=R_i(T)$ of the distribution is now easily obtained by putting $r=a$ in (6.8). We then find

$$(3)^{-7}(17)^8(28A^2q^2)^{14}a^{112q}R_i^{28}\Psi(T)=1, \tag{6.9}$$

so that

$$(1/\Psi)(\partial\Psi/\partial T)+(28/R_i)(\partial R_i/\partial T)=0. \tag{6.10}$$

The transformations of coordinates can now be completed. We find that

$$e^{\lambda}=7(3r^{2q}-a^{2q})/3(4r^{2q}-a^{2q}),$$

$$e^{\nu}=[112(3r^{2q}-a^{2q})^2a^{-4q}+e^{\lambda}](\partial\psi/\partial T)^2(R/28\psi)^2,$$

$$8\pi p=(9-14e^{-\lambda})/7R^2, \quad 8\pi\rho=(11-14e^{-\lambda})/7R^2,$$

$$8\pi\sigma=\frac{q^2a^{4q}}{18E^2C^2}\,\frac{16-21e^{-\lambda}}{7e^{-\lambda}-4}, \quad w^1=-\frac{3}{7q}\,\frac{EC}{a^{2q}}\,\frac{7e^{-\lambda}-4}{R},$$

$$v^1=-(7e^{-\lambda}-4)/2(28)^{\frac{1}{2}}.$$

Over the boundary $R=R_i(T)$ we have

$$e^{-\lambda}=9/14, \quad e^{\nu}=(448+14/9)(dR_i/dT)^2, \quad p=0,$$
$$8\pi\rho=2/7R_i^2, \quad 8\pi\sigma=5q^2a^{4q}/18E^2c^2, \quad v^1=-1/4(28)^{\frac{1}{2}}.$$

As $v^1$ is negative, it is clear that $dR_i/dT$ is negative and so the distribution is a contracting one. We now connect this solution with the exterior solution (5.1). The connection is made through the continuity of $g_{\mu\nu}$, $p$, and $\sigma$ at $R=R_i(T)$. The continuity of $p$ is already obtained by making $p=0$ on $R=R_i$. The continuity of $e^{\lambda}$ gives

$$m_i/R_i=5/28 \quad \text{or} \quad m(R_i,T)=(5/28)R_i, \tag{6.11}$$

so that $m/R$ is a constant on the boundary. Equation (6.11) is an equation to determine $\Psi(T)$ once the form of the function $m(R,T)$ is known. The continuity of $e^{\nu}$ gives

$$[1/f(m_i)](\partial m/\partial T)_i=(14\times 17/9)(dR_i/dT). \tag{6.12}$$

We now show that (6.11) and (6.12) together determine $f(m)$ as a constant. For that purpose we determine $(\partial m/\partial T)_i$, i.e., the value of $\partial m/\partial T$ when $R=R_i$, by differentiating (6.11) with regard to $T$:

$$(\partial m/\partial R_i)(dR_i/dT)+(\partial m/\partial T)_i=(5/28)(dR_i/dT). \tag{6.13}$$

But on using (5.2) we find that

$$\partial m/\partial R_i=(\partial m/\partial R)_i=(14/9)f(m_i),$$

and therefore (6.13) gives

$$f(m_i)=5/(28)^2.$$

Now $m_i$ is not a constant, but $f(m_i)$ is a constant. This is possible only if the arbitrary function $f(m)$ is a constant. To complete the process of using the continuity of $g_{\mu\nu}$ at $R=R_i(T)$ we must now determine $m(R,T)$ of the exterior solution so that (6.11) may determine the arbitrary function $\Psi(T)$. As $f(m)$ is a constant $=5/(28)^2$, the form of the function $m(R,T)$ is the one given by (5.4). We find that

$$\begin{aligned}&(m-xR)^j(m-yR)^h=(M-xR_e)^j(M-yR_e)^h;\\ &x,y=\tfrac{1}{4}(1\pm(93/98)^{\frac{1}{2}}); \quad j,h=\tfrac{1}{2}(1\mp(98/93)^{\frac{1}{2}}).\end{aligned} \tag{6.14}$$

$M$ is a constant and $R_e=R_e(T)$ is the exterior boundary determined by

$$dR_e/dT=1-2M/R_e.$$

On the internal boundary $R=R_i(T)$ (6.14) and (6.11) give

$$R_i(5-28x)^j(5-28y)^h=28(M-xR_e)^j(M-yR_e)^h,$$

so that the arbitrary function $\Psi(T)$ now gets determined by

$$\Psi(T)=K(M-xR_e)^{-28j}(M-yR_e)^{-28h},$$
$$K=(3)^7(28)^{14}(17)^{-8}(Aq)^{-28}a^{-112q}(5-28x)^{28j}(5-28y)^{28h}.$$

Thus the continuity of $g_{\mu\nu}$ across $R=R_i(T)$ determines the function $f(m)$ of the exterior solution and the function $\Psi(T)$ of the interior solution. The only arbitrary function left undetermined in the scheme of our solutions is the function $F(m)$ of (5.4). This is now determined by the continuity of $\sigma$, the density of the flowing radiation across $R=R_i(T)$. For this continuity leads to

$$F(m_i)/4\pi R_i^2=5q^2a^{4q}/144\pi E^2c^2,$$

which is readily satisfied by

$$F(m)=[(14qa^{2q})^2/45E^2c^2]m^2.$$

This completes the connection of the solutions at the interior boundary.

## VII. CONCLUSION

Here we have given the first complete nonstatic solution, holding from $R=0$ to $R=\infty$, representing an isolated spherically symmetric nonstatic distribution radiating out energy. There are two boundaries, the interior one and the exterior one, but we have been able to give a line-element whose coefficients are continuous throughout from $R=0$ to $R=\infty$. It has already been proved by the author[5] that in the case of a distribution with the line-element

$$ds^2=-e^{\lambda}dR^2-R^2(d\theta^2+\sin^2\theta d\varphi^2)+e^{\nu}dT^2,$$

the continuity of $g_{\mu\nu}$ alone is sufficient to insure that the total mass of the isolated distribution is conserved.

The solution discussed in detail in the last section shows that the corresponding material distribution is contracting, $dR_i/dT$ being negative. But $m_i/R_i$ is a constant, $m_i$ being the mass of the distribution as observed from its external gravitational field. So the newtonian gravitational potential energy of the interior distribution remains constant and the contraction therefore is not gravitational. It seems to be a purely relativistic effect. The radiating distribution loses energy and so its mass decreases. But if $m_i/R_i$ is to remain a constant, then $R_i$ must also decrease. This may explain the contraction.

Some of the particular solutions derived here have suggested the existence of a new class of solutions of Einstein's field equation. These solutions would represent non-isolated spherically symmetric distributions absorbing energy from the cosmos. These are at present being investigated.

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# On the Ionization Yields of Fission Fragments*†

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The disagreement between the distributions in fission fragment mass and ionization is attributed to a variation in ionization yield with fragment mass and to a dispersion arising from such effects as neutron recoil and instrumental errors of ionization measurement. After the effects of dispersion in the available data of fragments from $U^{235}$ slow neutron fission are taken into account, the variation in ionization yield is estimated from the remaining disagreement in distributions. For the most probable fission asymmetry, the energy-ionization ratio of light fragments is found to be approximately 3.7 percent less than for heavy fragments.

## I. INTRODUCTION

A RECENT analysis[1] of the ionization produced by slow, heavy particles indicates that appreciable kinetic energy is lost to recoiling gas atoms having a reduced ionization efficiency. The resulting increase in the energy-ionization rate $dE/dI$ as the heavy particle is stopped accounts for the ionization defect, the difference between the actual energy of the particle and the energy determined from the total ionization on the basis of $w=dE/dI$ of fast particles. Since the ionization defect increases with the mass of the particle, it is expected that $w_H$, the energy-ionization ratio obtained for complete stopping of a heavy fission fragment, is greater than $w_L$ of a light fragment.

Any direct measure of $w_L$ and $w_H$ would require measurements of both the ionization and energy of individual fragments. Because fission fragments do not all have the same kinetic energy, mass and effective charge, energy measurements of individual fragments by such means as calorimetry, magnetic deflection or electrostatic deflection have not been feasible. From the momentum condition of fission and the measured mass distribution,[2] however, the distribution in the *relative* kinetic energies of the complementary light and heavy fragments can be determined quite accurately. In this investigation, a comparison of this energy ratio distribution with the corresponding ionization ratio distribution[3,4] is used to determine indirectly $w_L/w_H$ for the slow neutron fission of $U^{235}$.

## II. COMPARISON OF DISTRIBUTIONS

### Method

Using double "back-to-back" ionization chambers, both Brunton and Hanna[3] and Deutsch and Ramsey[4] have made coincidence measurements of the number of ion pairs $I_L$ and $I_H$ (subscripts $L$ and $H$ always refer

* Work performed in the Ames Laboratory of the AEC.
† Details of this analysis are contained in ISC 98.
‡ Now at Los Alamos Scientific Laboratory, Los Alamos, New Mexico.
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[2] Plutonium Project, Revs. Modern Phys. 18, 513 (1946).
[3] D. C. Brunton and G. C. Hanna, Can. J. Research A28, 190 (1950).
[4] M. Deutsch and M. Ramsey, MDDC 945 (1946) (unpublished).