tions $\delta\Pi^b(x')$, commute or anticommute with $\phi^a(x)$, $\Pi^a(x)$ for all x and x' on a given σ, where the relation of anticommutativity holds when both a and b refer to components of half-integral spin fields. The consistency of this statement with the general commutation relations that have already been deduced from it is easily verified. By subjecting the canonical variables in Eq. (2.81) to independent variations, we obtain

$$[\phi^a(x), \delta\phi^b(x')]_\pm = [\Pi^a(x), \delta\phi^b(x')]_\pm = 0,$$
$$[\phi^a(x), \delta\Pi^b(x')]_\pm = [\Pi^a(x), \delta\Pi^b(x')]_\pm = 0, \quad (3.25)$$

which is valid for all x, x' on σ. In addition, Eq. (2.81) properly states that all physical quantities commute at distinct points of σ.

We conclude that the connection between the spin and statistics of particles is implicit in the requirement of invariance under coordinate transformations.[10]

[10] The discussion of the spin and statistics connection by W. Pauli [Phys. Rev. 58, 716 (1940)] is somewhat more negative in character, although based on closely related physical requirements. Thus, Pauli remarks that Bose-Einstein quantization of a half-integral spin field implies an energy that possesses no lower bound, and that Fermi-Dirac quantization of an integral spin field leads to an algebraic contradiction with the commutativity of physical quantities located at points with a spacelike interval. Another postulate which has been employed, that of charge symmetry [W. Pauli and F. J. Belinfante, Physica 7, 177 (1940)], suffices to determine the nature of the commutation relations for sufficiently simple systems. As we have noticed, it is a consequence of time reflection invariance. The comments of Feynman on vacuum polarization and statistics [Phys. Rev. 76, 749 (1949)] appear to be an illustration of the charge symmetry requirement, since a contradiction is established when the charge symmetrical concept of the vacuum is applied to a Bose-Einstein spin ½ field, or to a Fermi-Dirac spin 0 field.

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Diffusion of High Energy Gamma-Rays through Matter. III. Refinement of the Solution of the Diffusion Equation*

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In Part I of the present series of papers an approximate equation was derived governing the diffusion of high energy gamma-rays through matter. In Part II an approximate solution of this diffusion equation was obtained in the energy region where the total gamma-ray cross section was substantially independent of energy. In the present paper, by consideration of the methods employed in obtaining the solution in II, a refinement of the solution is carried out which reduces the errors introduced by the approximations made both in the energy distribution and the angular distribution of the multiply-scattered gamma-rays. The solution is also modified to take into partial account the effect of small variations of the total gamma-ray cross section with energy. An upper and lower bound on the solution is obtained when the cross section is independent of energy.

I. INTRODUCTION

In the first (I) of the present series of papers,[1] an approximate equation governing the diffusion of gamma-rays through matter was derived. The gamma-ray energies for which the equation is valid extends from a few Mev up to energies (depending on the material) where the radiation of gamma-rays by the secondary electrons (photoelectrons, Compton recoils, and pairs) produced by the primary gamma-rays becomes important. In the second (II) paper of the series, the solution of the diffusion equation was considered in the energy range where the total cross section for gamma-rays was practically independent of energy. In all materials this latter energy range coincides practically with the energy range over which the diffusion equation itself is valid. However, in order to obtain a solution to the equation, even with this restriction, it was necessary to make a rather poor approximation to the Klein-Nishina formula; and this last approximation leads to rather large errors, especially for gamma-rays whose energy lies far below the energy of the incident gamma-rays. The present paper is directed towards refining the approximation somewhat, making certain corrections to the solution to improve its accuracy, and studying the magnitudes of the remaining errors. The notation used is the same as in I and II, and reference should be made to these papers for the meaning of symbols not sufficiently defined below.

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[1] L. L. Foldy, Phys. Rev. 81, 395 (1951), hereinafter referred to as I, and L. L. Foldy and R. K. Osborn, Phys. Rev. 81, 400 (1951), hereinafter referred to as II.

II. APPROXIMATIONS TO THE KLEIN-NISHINA FORMULA FOR WHICH THE DIFFUSION EQUATION CAN BE SOLVED

The equation governing the diffusion of gamma-rays derived in I is

$$\partial f(\sigma, \xi, \eta, \zeta)/\partial\zeta + \phi_T f(\sigma, \xi, \eta, \zeta) = (1/\pi)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(\sigma'/\sigma) f(\sigma', \xi', \eta', \zeta) d\xi' d\eta', \quad (1)$$

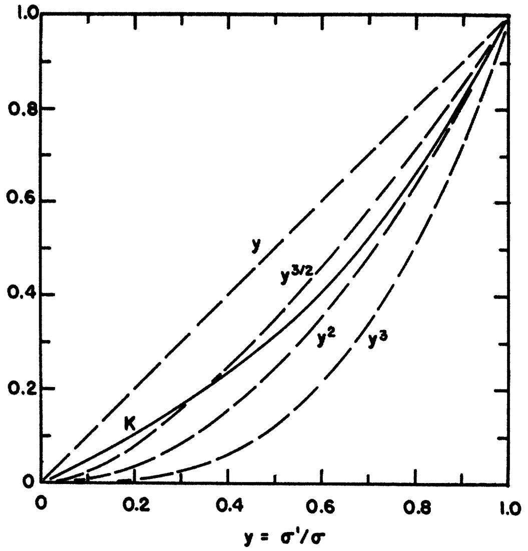

FIG. 1. The function $K(\sigma'/\sigma)$ together with several power law approximations to this function.

where f is the distribution function for the gamma rays in softness σ (reciprocal of the gamma-ray energy measured in units of the electron rest energy = gamma-ray wavelength measured in electron Compton wavelengths), and angle (as specified by the "rectangular" orientation variables ξ and η), at a depth ζ (dimensionless depth parameter) in the material. ϕ_T is the (constant) total gamma-ray cross section,

$$\sigma'=\sigma-\tfrac{1}{2}\{(\xi-\xi')^2+(\eta-\eta')^2\}, \tag{2}$$

and the kernel,

$$K(\sigma'/\sigma)=\tfrac{1}{2}(\sigma'/\sigma)[1+(\sigma'/\sigma)^2], \tag{3}$$

represents (apart from a constant dimensional factor) the Klein-Nishina cross section for scattering of a gamma-ray from a softness σ' to a softness σ.

We wish to note first that a solution of Eq. (1) can easily be obtained if the kernel K can be approximated by any power of σ'/σ, say $(\sigma'/\sigma)^n$. For, if one replaces K by this factor, then by the substitution,

$$f(\sigma,\xi,\eta,\zeta)=(\sigma_0/\sigma)^n g(\sigma,\xi,\eta,\zeta), \tag{4}$$

Eq. (1) can be reduced to the form

$$[\partial g(\sigma,\xi,\eta,\zeta)/\partial\zeta]+\phi_T g(\sigma,\xi,\eta,\zeta)$$
$$=(1/\pi)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\sigma',\xi',\eta',\zeta)d\xi'd\eta'. \tag{5}$$

This is the equation whose solution corresponding to monoenergetic gamma-rays of softness σ_0 incident normally on the surface of the material was found in II to be

$$g(\sigma,\xi,\eta,\zeta)=\Big[\sum_{m=0}^{\infty} g_m(\sigma,\xi,\eta,\zeta)\Big]\exp(-\phi_T\zeta), \tag{6}$$

where

$$\begin{aligned} g_0&=\delta(\sigma-\sigma_0)\delta(\xi)\delta(\eta),\\ g_1&=(\zeta/\pi)\delta[\sigma-\sigma_0-\tfrac{1}{2}(\xi^2+\eta^2)],\\ g_m&=\frac{(m-1)(2\zeta)^m}{2\pi(m!)^2}\left(\sigma-\sigma_0-\frac{\xi^2+\eta^2}{2m}\right)_*^{m-2},\quad m\geqslant 2.\end{aligned} \tag{7}$$

In the above, the asterisk subscript on the parentheses indicates that if the quantity contained in parentheses is negative, then the parentheses is to be replaced by zero. The distribution function f is then given simply by g multiplied by $(\sigma_0/\sigma)^n$. In II, the solution for the choice $n=1$ was examined.

In Fig. 1, we have plotted K as a function of (σ'/σ) together with several power law approximations ($n=1$, $\frac{3}{2}$, 2, and 3) to it. It can be seen that $n=1$ is not a very satisfactory approximation, being everywhere too high, but it is the only power law approximation behaving correctly for small σ'/σ. Since physically K determines the probability of a Compton scattering of a gamma-ray of softness σ' to a softness σ, this means that in this approximation we predict too many scattered gamma-rays at all energies, particularly at low energies. It will be noted further that the choice $n=2$ is a rather good approximation for $\sigma'/\sigma>0.5$; and, hence, the solution obtained with this choice will give quite accurate results for the distribution of gamma-rays in the range from half the incident energy to the incident energy ($\sigma_0<\sigma<2\sigma_0$), but it will predict far too few gamma-rays at the lower energies. On the other hand, the choice $n=\frac{3}{2}$ is a fair approximation over the range from $\sigma'/\sigma=0.2$ to $\sigma'/\sigma=1$ (better than 15 percent), and so might be relied on to give good results over the softness range (σ_0 to $5\sigma_0$). It represents probably the best power law approximation to the Klein-Nishina formula but will predict too many scattered gamma-rays at high energies at the expense of too few at low energies.

Now the fact that a change in the power by which the Klein-Nishina formula is approximated leads only to a change in the power of (σ_0/σ) by which g is multiplied to give the distribution function suggests that a still better approximation to the solution can be obtained by "interpolation" on the coefficient of g. The natural choice here is to take this coefficient to be just $K(\sigma_0/\sigma)$:

$$f(\sigma,\xi,\eta,\zeta)=\tfrac{1}{2}(\sigma_0/\sigma)[1+(\sigma_0/\sigma)^2]g(\sigma,\xi,\eta,\zeta), \tag{8}$$

since this decreases the number of high energy gamma-rays slightly and brings it into accord with what one obtains by a good approximation ($n=2$) for this energy region and at the same time increases the number of low energy gamma-rays substantially so as to give a number much closer to what one would anticipate on the basis of the results from the choice $n=1$. Hence, we would expect Eq. (8) to be superior to any solution obtained by a simple power law approximation to K.

Actually the approximate solution (8) has a very simple interpretation. The problem to which we are trying to find a solution corresponds to the diffusion of gamma-rays which are Compton scattered with a probability proportional to $K(\sigma'/\sigma)=\frac{1}{2}(\sigma'/\sigma)[1+(\sigma'/\sigma)^2]$. If we replace this problem by one in which the incident gamma-rays are divided into two equal groups, one group being scattered with a probability proportional to σ'/σ and the other with a probability proportional to $(\sigma'/\sigma)^3$, so that the average probability is given by K, then Eq. (8) would be the exact solution to this latter problem.

III. IMPROVEMENT OF THE SOLUTION

As we pointed out in II, the series (6) has the interpretation that the first term represents gamma-rays reaching the depth ζ unscattered, while the second, third, fourth, etc., terms represent, respectively, the gamma-rays reaching the depth ζ after having been scattered once, twice, three times, etc. Actually, the solution for a power law approximation to K given by Eq. (4) could have been obtained by an iteration procedure on the original diffusion equation. One is led to inquire why one can easily obtain a solution to the diffusion equation when K is approximated by a power law and yet a solution is difficult to obtain when the correct expression for K is used. By investigating the iteration method of solution the reason for this difference is found to be the following: Consider a gamma-ray scattered m times and as a result having its softness modified from σ_0 to σ. If $\sigma_1, \sigma_2, \cdots\sigma_{m-1}$ are the softnesses of the gamma ray after the first, second, $\cdots(m-1)$th, scatterings, then when K is approximated by $(\sigma'/\sigma)^n$, the probability that this sequence of scattering takes place is proportional to

$$(\sigma_0/\sigma_1)^n(\sigma_1/\sigma_2)^n\cdots(\sigma_{m-1}/\sigma)^n=(\sigma_0/\sigma)^n. \quad (9)$$

The fact that this expression is independent of the sequence of intermediate softnesses is what makes the simple solution possible, and the fact that it is independent of m (the number of scatterings) is what allows the factor $(\sigma_0/\sigma)^n$ to be factored out of the expression and appear as the coefficient of g in Eq. (4). If K had these same simple properties, then the solution of Eq. (1) could be written in the form (8).

Actually, K does not have these properties; but we can show that it does not deviate too greatly from having these properties. If we define the functions

$$\begin{aligned}K_m(\sigma_0/\sigma;\sigma_1,\sigma_2,\cdots\sigma_{m-1})&=K(\sigma_0/\sigma_1)K(\sigma_1/\sigma_2)\cdots K(\sigma_{m-1}/\sigma)\\&=K_{m-1}(\sigma_0/\sigma_{m-1})K(\sigma_{m-1}/\sigma),\end{aligned} \quad (10)$$

then we must show first that K_m is not very sensitive to variations of $\sigma_1, \sigma_2, \cdots\sigma_{m-1}$. In the appendix we give a proof that K_m is always bounded from above by K and from below by $(\sigma_0/\sigma)^2$ for any values of the intermediate softnesses.[2] Referring to Fig. 1 we see that this confines K_m to a relatively narrow range, particularly for $\sigma_0/\sigma>0.5$. Insofar as K_m may be approximated by K, Eq. (8) will be the solution to our problem.

[2] From this it follows by elementary arguments that $(\sigma_0/\sigma)^2g(\sigma,\xi,\eta,\zeta)\leqslant f(\sigma,\xi,\eta,\zeta)\leqslant\frac{1}{2}(\sigma_0/\sigma)[1+(\sigma_0/\sigma)^2]g(\sigma,\xi,\eta,\zeta)$.

Actually, however, it is possible to improve on the solution given by Eq. (8), since a tractable form can still be obtained even when the K_m are approximated by functions which are different for different m. Statistically, we would expect that a gamma-ray would reach the softness σ from an initial softness σ_0 in m scatterings far more frequently through those sequences of intermediate softnesses which are uniformly distributed between σ_0 and σ_m than by such sequences which include one or more large changes in softness and many small ones. This in turn would lead one to expect that if one had to pick an *effective* K_m for scattering independent of $\sigma_1, \sigma_2, \cdots\sigma_{m-1}$, the best choice would lie close to that K_m, which we shall denote as $\bar{K}_m(\sigma_0/\sigma)$, for which $\sigma_1, \sigma_2, \cdots\sigma_{m-1}$ are uniformly spaced between σ_0 and σ. We have plotted these for $m=1, 2, 3,$ and 4 in Fig. 2a. The exact expressions for $\bar{K}_m(\sigma_0/\sigma)$ are quite complicated, but one can approximate them closely by the series of functions,

$$W_m(\sigma_0/\sigma)=(\sigma_0/\sigma)\{(\sigma_0/\sigma)+(m+1)^{-1}[1-(\sigma_0/\sigma)]^2\}, \quad (11)$$

which have a simple analytical form. These have been plotted in Fig. 2b for comparison with the curves in Fig. 2a. W_m is slightly larger than $\bar{K}_m$ for small values of (σ_0/σ); but this is fortuitous, since we would expect the best effective value for K_m to lie above $\bar{K}_m$ in this region.

We shall now show that to within the error made in replacing $W_m(\sigma_0/\sigma')K(\sigma'/\sigma)$ by $W_{m+1}(\sigma_0/\sigma)$, the function

$$f(\sigma,\xi,\eta,\zeta)=\Big[\sum_{m=0}^{\infty}W_m(\sigma_0/\sigma)g_m(\sigma,\xi,\eta,\zeta)\Big]\exp(-\phi_T\zeta), \quad (12)$$

with the g_m given by Eq. (7), is a solution of Eq. (1). The proof rests on the fact that

$$\partial g_{m+1}(\sigma,\xi,\eta,\zeta)/\partial\zeta=(1/\pi)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g_m(\sigma',\xi',\eta',\zeta)d\xi'd\eta', \quad (13)$$

as can be seen by substituting Eq. (6) in Eq. (5). If one substitutes Eq. (12) in Eq. (1), one obtains easily

$$W_{m+1}(\sigma_0/\sigma)\partial g_{m+1}(\sigma,\xi,\eta,\zeta)\,\partial\zeta=(1/\pi)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K(\sigma'/\sigma)W_m(\sigma_0/\sigma')\times g_m(\sigma',\xi',\eta',\zeta)d\xi'd\eta',$$

which establishes our contention. On the basis of the discussion above, it is unlikely that the errors in the solution (12) are large at any depth and for any softness.

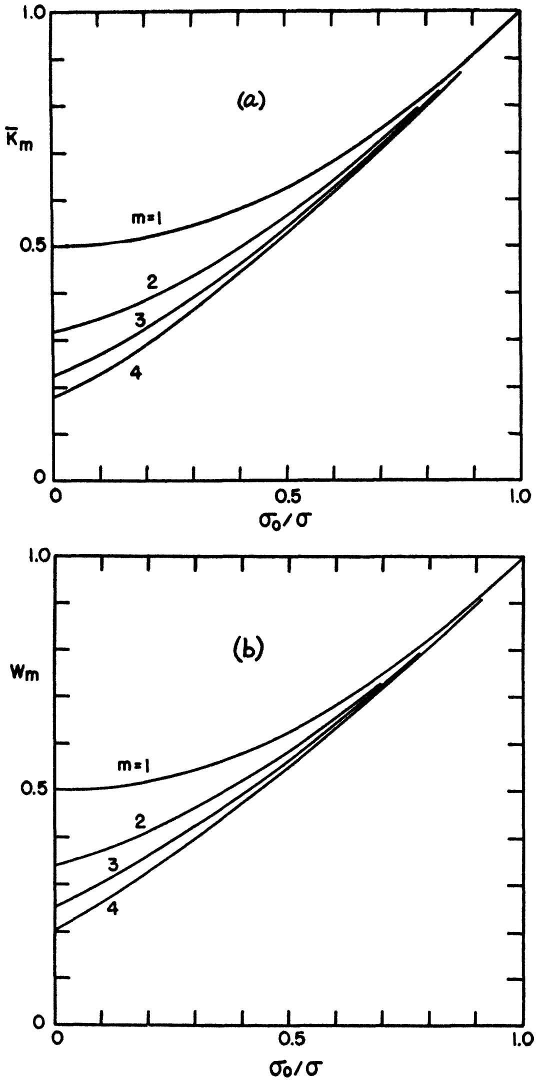

FIG. 2(a) The function $\bar{K}_m(\sigma_0/\sigma)$ for $m=1, 2, 3, 4$; (b) the functions $W_m(\sigma_0/\sigma)$ for $m=1, 2, 3, 4$, representing approximations to the corresponding functions $\bar{K}_m$.

Absolute, but rather wide, limits on the errors are provided by the inequality in reference 2.

IV. FURTHER REFINEMENTS IN THE SOLUTION

The best solution we have obtained so far for our diffusion problem is given by Eq. (12). In this section we shall make some small modifications of this solution to ameliorate the effects of some of the other approximations we have been forced to make.

(a) Improvement in the Angular Distribution

In the solution (12) each term for $m\geqslant 2$ cuts off at an angle $\vartheta=[2m(\sigma-\sigma_0)]^{\frac{1}{2}}$ because of the asterisk on the parentheses in g_m. One can easily show that the rigorous cut off in angle for an m-fold scattered gamma-ray of softness σ actually occurs at the angle ϑ fixed by the equation: $m\{1-\cos(\vartheta/m)\}=\sigma-\sigma_0$. Hence, we might expect an improvement in the angular distribution represented by Eq. (8) if the factor $(\sigma-\sigma_0-\vartheta^2/2m)_*$ is replaced by the factor $(\sigma-\sigma_0/m\{1-\cos(\vartheta/m)\})_*$.

(b) Correction of the Term for Single Scattering

One can easily show that the correct form for the term in Eq. (8) representing a single scattering of a gamma-ray when employing the rigorous Klein-Nishina formula should be

$$(2\pi)^{-1}\zeta(\sigma_0/\sigma)[1+(\sigma_0/\sigma)^2-(\sigma_0/\sigma)\sin^2\vartheta] \times\delta(\sigma-\sigma_0-\{1-\cos\vartheta\}).$$

Hence, we can employ this correct term in place of the result obtained above for the single-scattering term in the series (8).

(c) Correction of the Exponential Absorption Factor

It has been assumed in the work above that the total gamma-ray cross section is independent of softness in the range of energies in which we are working. Actually this cross section varies somewhat with energy. Since the exponential factor $\exp(-\phi_T\zeta)$ represents simply the over-all absorption of the gamma-rays in penetrating to the depth ζ and since a gamma-ray with softness σ at a depth ζ has generally filtered down to this softness through a series of intermediate softness values, some improvement of the solution would be anticipated if one replaced the constant ϕ_T in the exponential by the average value of ϕ_T over the softness range from σ_0 to σ. We shall represent this average value by $\bar{\phi}_T(\sigma_0, \sigma)$.

Making all of the above corrections, our "best" solution of the diffusion problem now takes the form

$$f(\sigma, \xi, \eta, \zeta)=\left[\sum_{m=0}^{\infty} f_m\right]\exp[-\bar{\phi}_T(\sigma_0, \sigma)\zeta],$$

$$f_0=\delta(\sigma-\sigma_0)\delta(\xi)\delta(\eta),$$

$$f_1=(2\pi)^{-1}\zeta(\sigma_0/\sigma)[1+(\sigma_0/\sigma)^2-(\sigma_0/\sigma)\sin^2\vartheta] \times\delta(\sigma-\sigma_0-\{1-\cos\vartheta\}),$$

$$f_m=\frac{\sigma_0}{\sigma}\left\{\frac{\sigma_0}{\sigma}+\frac{1}{m+1}\left(1-\frac{\sigma_0}{\sigma}\right)^2\right\}\frac{(m-1)(2\zeta)^m}{2\pi(m!)^2} \times(\sigma-\sigma_0-m\{1-\cos(\vartheta/m)\})_*^{m-2}, \quad m\geqslant 2. \quad (14)$$

V. SPECIAL CASES

In certain special cases, the solution given by Eq. (14) can be made to take simpler forms by the use of methods such as those employed in II. In particular, for the special cases where $\vartheta^2\ll 2(\sigma-\sigma_0)$ or where $\rho=[8\zeta(\sigma-\sigma_0)]^{\frac{1}{2}}\gg\vartheta^2/2(\sigma-\sigma_0)$, the series in Eq. (14)

may be summed to give the closed form:

$$
\begin{aligned}
f=&[\delta(\sigma-\sigma_0)\delta(\xi)\delta(\eta)+(\zeta/2\pi)(\sigma_0/\sigma)\\
&\times[1+(\sigma_0/\sigma)^2-(\sigma_0/\sigma)\sin^2\vartheta]\delta(\sigma-\sigma_0-\{1-\cos\vartheta\})\\
&+(4/\pi)\zeta^2(\sigma_0/\sigma)^2(\{[1-I_0(\rho)+\tfrac{1}{2}\rho I_1(\rho)]/\rho^4\}\\
&+\{(\sigma/\sigma_0)(1-\sigma_0/\sigma)^2[\rho-2I_1(\rho)+\rho I_2(\rho)]/\rho^5\})\\
&\times\exp[-\vartheta^2/2(\sigma-\sigma_0)]]\exp[-\bar{\phi}_T(\sigma_0,\sigma)\zeta]. \quad (15)
\end{aligned}
$$

To obtain the distribution function at any depth integrated over all angles for the gamma-rays, it was found necessary to omit the slight improvement in the angular distribution, resulting from the replacement of $\theta^2/2m$ by $m[1-\cos(\vartheta/m)]$ in order to carry out the integration. Once this is done, the series can be summed to give for the integral distribution function

$$
\begin{aligned}
F(\sigma,\zeta)=&\int\int f(\sigma,\xi,\eta,\zeta)d\xi d\eta\\
=&\{\delta(\sigma-\sigma_0)+[4\zeta(\sigma_0/\sigma)^2I_1(\rho)/\rho]\\
&+8\zeta(\sigma_0/\sigma)(1-\sigma_0/\sigma)^2I_2(\rho)/\rho^2\}\\
&\times\exp[-\bar{\phi}_T(\sigma_0,\sigma)\zeta]. \quad (16)
\end{aligned}
$$

For convenience in calculations the functions

$$[1-I_0(\rho)+\tfrac{1}{2}\rho I_1(\rho)]/\rho^4,\ [\rho-2I_1(\rho)+\rho I_2(\rho)]/\rho^5,$$

$I_1(\rho)/\rho$, and $I_2(\rho)/\rho^2$, occurring in the above formulas, have been plotted as a function of ρ in Fig. 3.

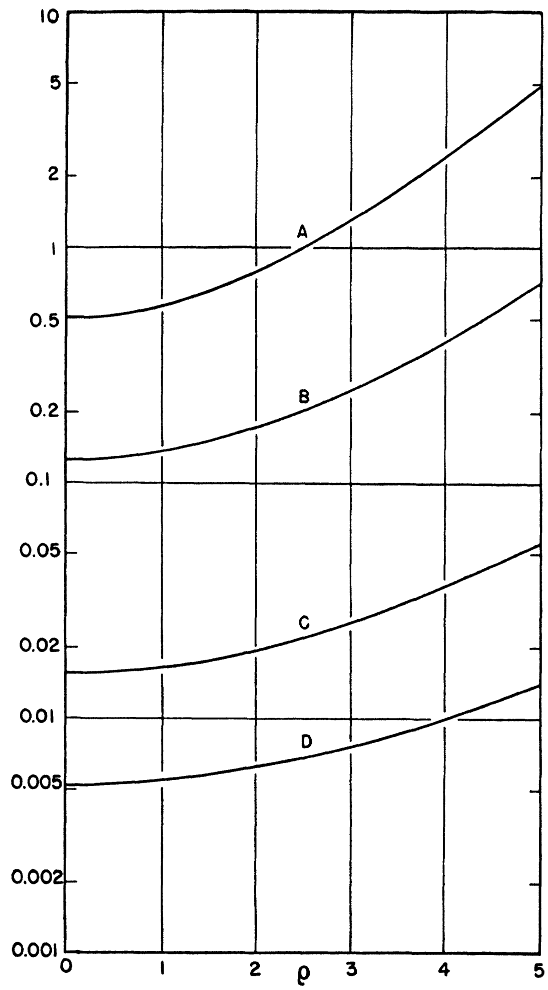

FIG. 3. Plots of the functions entering Eqs. (15) and (16). (A) The function $I_1(\rho)/\rho$; (B) the function $I_2(\rho)/\rho^2$; (C) the function $[1-I_0(\rho)+\frac{1}{2}\rho I_1(\rho)]/\rho^4$; (D) the function $[\rho-2I_1(\rho)+\rho I_2(\rho)]/\rho^5$.

APPENDIX

We wish to establish that

$$(\sigma_0/\sigma)^2\leqslant K_m(\sigma_0/\sigma;\sigma_1,\sigma_2,\cdots\sigma_{m-1})\leqslant K_1(\sigma_0/\sigma)\equiv K(\sigma_0/\sigma). \quad \text{(A-1)}$$

We shall use the method of mathematical induction based on the fact that

$$K(\sigma_0/\sigma)\geqslant(\sigma_0/\sigma)^2. \quad \text{(A-2)}$$

We have first that if

$$K_m(\sigma_0/\sigma)\geqslant(\sigma_0/\sigma)^2, \quad \text{(A-3)}$$

then

$$K_{m+1}(\sigma_0/\sigma)=K_m(\sigma_0/\sigma_m)K(\sigma_m/\sigma)\geqslant(\sigma_0/\sigma_m)^2(\sigma_m/\sigma)^2=(\sigma_0/\sigma)^2, \quad \text{(A-4)}$$

which combined with Eq. (A-2) is sufficient to establish the inferior limit in Eq. (A-1).

Secondly, we shall show that if

$$K_m(\sigma_0/\sigma)\leqslant K(\sigma_0/\sigma),$$

then

$$K_{m+1}(\sigma_0/\sigma)\leqslant K(\sigma_0/\sigma),$$

which combined with Eq. (A-2) is sufficient to establish the superior limit in Eq. (A-2). We have

$$K_{m+1}(\sigma_0/\sigma)=K_m(\sigma_0/\sigma_m)K(\sigma_m/\sigma)\leqslant K(\sigma_0/\sigma_m)K(\sigma_m/\sigma),$$

$$
\begin{aligned}
K(\sigma_0/\sigma_m)K(\sigma_m/\sigma)=&\tfrac{1}{2}(\sigma_0/\sigma_m)[1+(\sigma_0/\sigma_m)^2]\cdot\tfrac{1}{2}(\sigma_m/\sigma)[1+(\sigma_m/\sigma)^2]\\
=&\tfrac{1}{2}(\sigma_0/\sigma)[1+(\sigma_0/\sigma)^2]\\
&\{\tfrac{1}{2}[1+(\sigma_m/\sigma)^2][1+(\sigma_0/\sigma_m)^2]/[1+(\sigma_0/\sigma)^2]\\
=&K(\sigma_0/\sigma)\cdot\{\tfrac{1}{2}+\tfrac{1}{2}[(\sigma_m/\sigma)^2+(\sigma_0/\sigma_m)^2]/\\
&[1+(\sigma_0/\sigma)^2]\}\leqslant K(\sigma_0/\sigma).
\end{aligned}
$$

The last inequality follows from the fact that $(\sigma_m/\sigma)^2+(\sigma_0/\sigma_m)^2$ has its maximum value for σ_m equal to σ_0 or σ, this maximum value being just $1+(\sigma_0/\sigma)^2$.