The Theory of Quantized Fields. I*

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The conventional correspondence basis for quantum dynamics is here replaced by a self-contained quantum dynamical principle from which the equations of motion and the commutation relations can be deduced. The theory is developed in terms of the model supplied by localizable fields. A short review is first presented of the general quantum-mechanical scheme of operators and eigenvectors, in which emphasis is placed on the differential characterization of representatives and transformation functions by means of infinitesimal unitary transformations. The fundamental dynamical principle is stated as a variational equation for the transformation function connecting eigenvectors associated with different spacelike surfaces, which describes the temporal development of the system. The generator of the infinitesimal transformation is the variation of the action integral operator, the space-time volume integral of the invariant lagrange function operator. The invariance of the lagrange function preserves the form of the dynamical principle under coordinate transformations, with the exception of those transformations which include a reversal in the positive sense of time, where a separate discussion is necessary. It will be shown in Sec. III that the requirement of invariance under time reflection imposes a restriction upon the operator properties of fields, which is simply the connection between the spin and statistics of particles. For a given dynamical system, changes in the transformation function arise only from alterations of the eigenvectors associated with the two surfaces, as generated by operators constructed from field variables attached to those surfaces. This yields the operator principle of stationary action, from which the equations of motion are obtained. Commutation relations are derived from the generating operator associated with a given surface. In particular, canonical commutation relations are obtained for those field components that are not restricted by equations of constraint. The surface generating operator also leads to generalized Schrödinger equations for the representative of an arbitrary state. Action integral variations which correspond to changing the dynamical system are discussed briefly. A method for constructing the transformation function is described, in a form appropriate to an integral spin field, which involves solving Hamilton-Jacobi equations for ordered operators. In Sec. III, the exceptional nature of time reflection is indicated by the remark that the charge and the energy-momentum vector behave as a pseudoscalar and pseudovector, respectively, for time reflection transformations. This shows, incidentally, that positive and negative charge must occur symmetrically in a completely covariant theory. The contrast between the pseudo energy-momentum vector and the proper displacement vector then indicates that time reflection cannot be described within the unitary transformation framework. This appears most fundamentally in the basic dynamical principle. It is important to recognize here that the contributions to the lagrange function of half-integral spin fields behave like pseudoscalars with respect to time reflection. The non-unitary transformation required to represent time reflection is found to be the replacement of a state vector by its dual, or complex conjugate vector, together with the transposition of all operators. The fundamental dynamical principle is then invariant under time reflection if inverting the order of all operators in the lagrange function leaves an integral spin contribution unaltered, and reverses the sign of a half-integral spin contribution. This implies the essential commutativity, or anti-commutativity, of integral and half-integral field components, respectively, which is the connection between spin and statistics.

I. INTRODUCTION

DESPITE extensive developments in the concepts and techniques of the theory of quantized fields, quantitative success has been achieved thus far only in the restricted domain of quantum electrodynamics. Furthermore, the existence of divergences, whether concealed or explicit, serves to emphasize that the present quantum theory of fields must, in some respect, be incomplete. It is not our purpose to propose a solution of this basic problem, but rather to present a general theory of quantum field dynamics which unifies several independently developed procedures and which may provide a framework capable of admitting fundamentally new physical ideas.

Quantum mechanics involves two distinct sets of hypotheses—the general mathematical scheme of linear operators and state vectors with its associated probability interpretation and the commutation relations and equations of motion for specific dynamical systems. It is the latter aspect that we wish to develop, by substituting a single quantum dynamical principle for the conventional array of assumptions based on classical hamiltonian dynamics and the correspondence principle.[1] We shall find it useful, however, first to review briefly some aspects of the mathematical formalism that find repeated application in the construction of our theory.

The simultaneous eigenvectors of some complete set of commuting hermitian operators, $\Psi(\alpha')$, provide a description of the arbitrary state Ψ by means of the representative

$$(\alpha'|)=(\Psi(\alpha'),\Psi), \tag{1.1}$$

which has the interpretation of a probability amplitude. Two such representations, associated with different complete sets of commuting operators, are related by

$$(\alpha'|)=\int(\alpha'|\beta')d\beta'(\beta'|), \tag{1.2}$$

where $\int d\beta'$ indicates integration and summation over

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[1] Although our attention will be focused on field dynamics, the analogous development of particle quantum dynamics should be evident.

the totality of eigenvalues β', and

$$(\alpha'|\beta')=(\Psi(\alpha'),\Psi(\beta')) \tag{1.3}$$

is the transformation function. As a special example of Eq. (1.2), we have

$$(\alpha'|\gamma')=\int(\alpha'|\beta')d\beta'(\beta'|\gamma'), \tag{1.4}$$

the multiplicative composition law of transformation functions.

The set of commuting hermitian operators

$$\bar{\alpha}=U\alpha U^{-1}, \tag{1.5}$$

which is obtained from α with the aid of the arbitrary unitary operator U, has the property that its eigenvalues are identical with those of α, and that its eigenvectors are given by

$$\Psi(\bar{\alpha}')=U\Psi(\alpha'), \tag{1.6}$$

where $\bar{\alpha}'$ and α' are the same set of eigenvalues. Conversely, two sets of operators that possess the same eigenvalue spectrum are related by a unitary transformation. Note that the transformation function $(\bar{\alpha}'|\alpha'')$ may also be viewed as the matrix of U^{-1} in the original eigenvector system,

$$\begin{aligned}(\bar{\alpha}'|\alpha'')&=(U\Psi(\alpha'),\Psi(\alpha''))=(\Psi(\alpha'),U^{\dagger}\Psi(\alpha''))\\&=(\alpha'|U^{-1}|\alpha'').\end{aligned} \tag{1.7}$$

The unitary operator

$$U=1-(i/\hbar)F,\quad U^{-1}=1+(i/\hbar)F, \tag{1.8}$$

in which F is an infinitesimal hermitian operator, induces an infinitesimal transformation in the commuting set of operators,

$$\bar{\alpha}=U\alpha U^{-1}=\alpha-\delta\alpha, \tag{1.9}$$

where

$$i\hbar\delta\alpha=\alpha F-F\alpha=[\alpha,F]. \tag{1.10}$$

If the system is such that it is possible to obtain operators $\delta\alpha$ that commute with the complete set α, one can treat the $\delta\alpha$ as arbitrary, infinitesimal numbers, and $\Psi(\bar{\alpha}')$ provides an eigenvector of α with the eigenvalue set $\alpha'+\delta\alpha$. This evidently corresponds to the special circumstance of α having a continuous eigenvalue spectrum.

The concept of infinitesimal unitary transformation can be used to provide a differential characterization for the representative of a state, or for a transformation function. The change in the representative $(\alpha'|)$ when the commuting set of operators is altered by the unitary transformation generated by the infinitesimal hermitian operator F, is given by

$$\delta(\alpha'|)=((\alpha-\delta\alpha)'|)-(\alpha'|)=(\delta\Psi(\alpha'),\Psi), \tag{1.11}$$

where

$$\delta\Psi(\alpha')=U\Psi(\alpha')-\Psi(\alpha')=-(i/\hbar)F\Psi(\alpha'). \tag{1.12}$$

Therefore,

$$\delta(\alpha'|)=(i/\hbar)(\Psi(\alpha'),F\Psi)=(i/\hbar)(\alpha'|F|), \tag{1.13}$$

or

$$(\hbar/i)\delta(\alpha'|)=\int(\alpha'|F|\alpha'')d\alpha''(\alpha''|), \tag{1.14}$$

which is a differential equation for the representative $(\alpha'|)$. In a similar manner, we can characterize the transformation function $(\alpha'|\beta')$ by the effect of altering the two commuting sets α and β into $\alpha-\delta\alpha$ and $\beta-\delta\beta$, as induced by the two infinitesimal generating operators F_α, and F_β. Thus,

$$\begin{aligned}\delta(\alpha'|\beta')&=(\delta\Psi(\alpha'),\Psi(\beta'))+(\Psi(\alpha'),\delta\Psi(\beta'))\\&=(i/\hbar)(\alpha'|(F_\alpha-F_\beta)|\beta'),\end{aligned} \tag{1.15}$$

or

$$(\hbar/i)\delta(\alpha'|\beta')=\int(\alpha'|F_\alpha|\alpha'')d\alpha''(\alpha''|\beta')-\int(\alpha'|\beta'')d\beta''(\beta''|F_\beta|\beta'). \tag{1.16}$$

II. QUANTUM DYNAMICS OF LOCALIZABLE FIELDS

A localizable field is a dynamical system characterized by one or more operator functions of the space-time coordinates, $\phi^\alpha(x)$. Contained in this statement are the assumptions that the operators x_μ, representing position measurements, are commutative,

$$[x_\mu,x_\nu]=0, \tag{2.1}$$

and furthermore, that they commute with the field operators,

$$[x_\mu,\phi^\alpha]=0, \tag{2.2}$$

so that

$$(x|\phi^\alpha|x')=\delta(x-x')\phi^\alpha(x). \tag{2.3}$$

The difficulties associated with current field theories may be attributable to the implicit hypothesis of localizability. However, our development of quantum field dynamics will be confined to such fields. It remains to be seen whether other systems can be included within its scope.

The problem of constructing a complete set of commuting operators, that is, of simultaneously measurable physical quantities, necessarily involves specific properties of the fields. Nevertheless, as a general principle associated with relativistic requirements, we must expect such mutually commuting operators to be formed from field quantities at physically independent space-time points, that is, points which cannot be connected even by light signals. A continuous set of such points form a spacelike surface, which is a geometrical concept independent of the coordinate system. Therefore, a base vector system, $\Psi(\zeta',\sigma)$, will be specified by a spacelike surface σ and by the eigenvalues ζ' of a complete set of commuting operators constructed from field quantities attached to that surface. A change of representation will correspond, in general, to the introduction of another set of commuting operators on a different spacelike surface. Of particular importance is the transformation $\zeta_2, \sigma_2 \to \zeta_1, \sigma_1$, in which ζ_1 and ζ_2 are similarly constructed operator sets which possess the same eigenvalue spectrum and are therefore related by a unitary transformation [Eqs. (1.5) and (1.6)],

$$\zeta_1 = U_{12}\zeta_2 U_{12}^{-1}, \quad \Psi(\zeta_1', \sigma_1) = U_{12}\Psi(\zeta_2', \sigma_2), \quad \zeta_1' = \zeta_2', \tag{2.4}$$

so that [Eq. (1.7)]

$$(\zeta_1', \sigma_1 | \zeta_2'', \sigma_2) = (\zeta_2', \sigma_2 | U_{12}^{-1} | \zeta_2'', \sigma_2). \tag{2.5}$$

A description of the temporal development of a system is evidently accomplished by stating the relationship between eigenvectors associated with different spacelike surfaces, or, in other words, by exhibiting the transformation function (2.5). Accordingly, we may expect that the quantum dynamical laws will find their proper expression in terms of the transformation function. A differential formulation of this type will now be constructed.

The operator U_{12}^{-1} describes the development of the system from σ_2 to σ_1 and involves, not only the detailed dynamical characteristics of the system in this space-time region, but also the choice of commuting operators, ζ_1 and ζ_2, on the surfaces σ_1 and σ_2. Any infinitesimal change in the quantities on which the transformation function depends induces a corresponding alteration in U_{12}^{-1},

$$\delta(\zeta_1', \sigma_1 | \zeta_2'', \sigma_2) = (\zeta_2', \sigma_2 | \delta U_{12}^{-1} | \zeta_2'', \sigma_2). \tag{2.6}$$

Now it is a consequence of the unitary property that $iU_{12}\delta U_{12}^{-1}$ must be hermitian. Accordingly, we write

$$\delta U_{12}^{-1} = (i/\hbar) U_{12}^{-1} \delta W_{12}, \tag{2.7}$$

where δW_{12} is an infinitesimal hermitian operator, and obtain

$$\delta(\zeta_1', \sigma_1 | \zeta_2'', \sigma_2) = (i/\hbar)(\zeta_1', \sigma_1 | \delta W_{12} | \zeta_2'', \sigma_2). \tag{2.8}$$

The composition law of transformation functions [Eq. (1.4)],

$$(\zeta_1', \sigma_1 | \zeta_3''', \sigma_3) = \int (\zeta_1', \sigma_1 | \zeta_2'', \sigma_2) d\zeta'' (\zeta_2'', \sigma_2 | \zeta_3''', \sigma_3), \tag{2.9}$$

imposes a restriction on δW, the generating operator of infinitesimal transformations. Thus,

$$(\zeta_1', \sigma_1 | \delta W_{13} | \zeta_3''', \sigma_3) = \int (\zeta_1', \sigma_1 | \delta W_{12} | \zeta_2'', \sigma_2) d\zeta'' (\zeta_2'', \sigma_2 | \zeta_3''', \sigma_3) + \int (\zeta_1', \sigma_1 | \zeta_2'', \sigma_2) d\zeta'' (\zeta_2'', \sigma_2 | \delta W_{23} | \zeta_3''', \sigma_3), \tag{2.10}$$

or

$$\delta W_{13} = \delta W_{12} + \delta W_{23}; \tag{2.11}$$

the infinitesimal generating operators satisfy an additive law of composition.

Our basic assumption is that δW_{12} is obtained by variation of the quantities contained in a hermitian operator W_{12}, which must have the general form

$$W_{12} = (1/c) \int_{\sigma_2}^{\sigma_1} (dx) \mathcal{L}[x], \tag{2.12}$$

according to the additive requirement (2.11). Individual systems are described by stating $\mathcal{L}$ as an invariant hermitian function of the fields and their coordinate derivatives,

$$\mathcal{L}[x] = \mathcal{L}(\phi^\alpha(x), \phi_\mu{}^\alpha(x)), \quad \phi_\mu{}^\alpha(x) = \partial_\mu \phi^\alpha(x). \tag{2.13}$$

In conformity with their classical analogs, we shall call W and $\mathcal{L}$ the action integral and lagrange function operators, respectively. The invariance of the lagrange function, and therefore of the action integral, guarantees that our fundamental dynamical principle,

$$\delta(\zeta_1', \sigma_1 | \zeta_2'', \sigma_2) = (i/\hbar)(\zeta_1', \sigma_1 | \delta W_{12} | \zeta_2'', \sigma_2) = (i/\hbar c)(\zeta_1', \sigma_1 | \delta \int_{\sigma_2}^{\sigma_1} (dx) \mathcal{L} | \zeta_2'', \sigma_2), \tag{2.14}$$

is unaltered in form by a change in the coordinate system. An exception must be made, however, for those coordinate transformations that include a reversal in the positive sense of time, which require a separate discussion. We shall see that the requirement of invariance under time reflection imposes a general restriction upon the commutation properties of fields, which is simply the connection between the spin and statistics of elementary particles.

If the parameters of the system are not altered, the variation of the transformation function in Eq. (2.14) arises only from infinitesimal changes of ζ_1, σ_1 and ζ_2, σ_2. Such transformations may be characterized by infinitesimal generating operators, $F(\sigma_1)$ and $F(\sigma_2)$, which act on the eigenvectors $\Psi(\zeta_1', \sigma_1)$ and $\Psi(\zeta_2'', \sigma_2)$, and are therefore expressed in terms of operators associated with the surfaces σ_1 and σ_2, respectively. On referring to Eq. (1.15), we obtain for such variations,

$$\delta W_{12} = F(\sigma_1) - F(\sigma_2). \tag{2.15}$$

This is the operator principle of stationary action, for it states that the action integral operator is unaltered by infinitesimal variations of the field quantities in the interior of the region bounded by σ_1 and σ_2, being dependent only on operators attached to the boundary surfaces. The equations of motion for the field are contained in this principle.[2]

[2] In the following discussions, one should keep in mind that the lagrange functions of the simple systems usually considered are no more than quadratic in the components of individual fields.

The evaluation of δW_{12} involves adding the independent effects of changing the field components at each point by $\delta_0\phi^\alpha(x)$, and of altering the region of integration by a displacement δx_μ of the points on the boundary surfaces. Thus,

$$\delta W_{12}=(1/c)\int_{\sigma_2}^{\sigma_1}(dx)\delta_0\mathfrak{L}+(1/c)\left(\int_{\sigma_1}-\int_{\sigma_2}\right)d\sigma_\mu\delta x_\mu\mathfrak{L}, \quad (2.16)$$

where

$$\delta_0\mathfrak{L}=(\partial\mathfrak{L}/\partial\phi^\alpha)\delta_0\phi^\alpha+(\partial\mathfrak{L}/\partial\phi_\mu{}^\alpha)\partial_\mu\delta_0\phi^\alpha$$
$$=[(\partial\mathfrak{L}/\partial\phi^\alpha)-\partial_\mu(\partial\mathfrak{L}/\partial\phi_\mu{}^\alpha)]\delta_0\phi^\alpha+\partial_\mu[(\partial\mathfrak{L}/\partial\phi_\mu{}^\alpha)\delta_0\phi^\alpha]. \quad (2.17)$$

This expression for $\delta_0\mathfrak{L}$ is to be understood symbolically, since the order of the operators in $\mathfrak{L}$ must not be altered in the course of effecting the variation. Accordingly, the commutation properties of $\delta_0\phi^\alpha$ are involved in obtaining the consequences of the stationary requirement on the action integral. For simplicity, we shall introduce here the explicit assumption that the commutation properties of $\delta_0\phi^\alpha$ and the structure of the lagrange function must be so related that identical contributions are produced by terms that differ fundamentally only in the position of $\delta_0\phi^\alpha$. We may now infer the equations of motion

$$\partial_\mu(\partial\mathfrak{L}/\partial\phi_\mu{}^\alpha)=\partial\mathfrak{L}/\partial\phi^\alpha. \quad (2.18)$$

From the resulting form of δW_{12} we obtain the infinitesimal generating operator $F(\sigma)$, which acts on eigenvectors associated with the surface σ,

$$F(\sigma)=(1/c)\int_\sigma d\sigma_\mu[(\partial\mathfrak{L}/\partial\phi_\mu{}^\alpha)\delta_0\phi^\alpha+\mathfrak{L}\delta x_\mu]. \quad (2.19)$$

The total variation, $\delta\phi^\alpha(x)$, is composed additively of the variation $\delta_0\phi^\alpha(x)$ at the point x, and of the change in $\phi^\alpha(x)$ produced by moving from the point x on σ to $x+\delta x$ on $\sigma+\delta\sigma$. In evaluating the latter, we shall take into account that the field components $\phi^\alpha(x)$, although stated in terms of some fixed coordinate system, are most advantageously considered in relation to the local coordinate system provided by σ at the point x. Only such motions are contemplated that correspond to a local rigid displacement of the surface σ. This restriction is expressed by

$$\partial_\mu\delta x_\nu=-\partial_\nu\delta x_\mu, \quad (2.20)$$

being the condition that an infinitesimal space vector on σ be mapped into one of equal length on $\sigma+\delta\sigma$. The displacement induced change in $\phi^\alpha(x)$ may be obtained by an alteration in the coordinate system that reduces, in the neighborhood of x, to the equivalent local coordinate transformation. Thus, under the infinitesimal coordinate transformation

$$x_\mu'-x_\mu=-\delta x_\mu, \quad (2.21)$$

where

$$\delta x_\mu=\epsilon_\mu-\epsilon_{\mu\nu}x_\nu, \quad \epsilon_{\mu\nu}=-\epsilon_{\nu\mu}=\partial_\mu\delta x_\nu, \quad (2.22)$$

the field components suffer a linear transformation, as expressed by

$$\phi^{\alpha\prime}(x')-\phi^\alpha(x)=(i/\hbar)\tfrac{1}{2}\epsilon_{\mu\nu}S_{\mu\nu}{}^{\alpha\beta}\phi^\beta(x). \quad (2.23)$$

Therefore,

$$\phi^{\alpha\prime}(x)-\phi^\alpha(x)=\partial_\mu\phi^\alpha(x)\delta x_\mu+(i/\hbar)\tfrac{1}{2}\epsilon_{\mu\nu}S_{\mu\nu}{}^{\alpha\beta}\phi^\beta(x), \quad (2.24)$$

and

$$\delta\phi^\alpha(x)=\delta_0\phi^\alpha(x)+\phi_\mu{}^\alpha(x)\delta x_\mu+(i/\hbar)\tfrac{1}{2}\partial_\mu\delta x_\nu S_{\mu\nu}{}^{\alpha\beta}\phi^\beta(x). \quad (2.25)$$

With the introduction of the total variation, the infinitesimal generating operator $F(\sigma)$ assumes the form

$$F(\sigma)=\int_\sigma d\sigma_\mu[\Pi_\mu{}^\alpha\delta\phi^\alpha+(1/c)\mathfrak{L}\delta x_\mu-\Pi_\mu{}^\alpha\phi_\nu{}^\alpha\delta x_\nu-(i/2\hbar)\Pi_\mu{}^\alpha S_{\lambda\nu}{}^{\alpha\beta}\phi^\beta\partial_\lambda\delta x_\nu], \quad (2.26)$$

where

$$c\Pi_\mu{}^\alpha=\partial\mathfrak{L}/\partial\phi_\mu{}^\alpha. \quad (2.27)$$

To simplify the last term of Eq. (2.26), we define

$$f_{\mu\lambda\nu}=-f_{\lambda\mu\nu}=(i/2\hbar)[\Pi_\mu{}^\alpha S_{\lambda\nu}{}^{\alpha\beta}\phi^\beta+\Pi_\nu{}^\alpha S_{\lambda\mu}{}^{\alpha\beta}\phi^\beta+\Pi_\lambda{}^\alpha S_{\nu\mu}{}^{\alpha\beta}\phi^\beta], \quad (2.28)$$

and obtain

$$(i/2\hbar)\Pi_\mu{}^\alpha S_{\lambda\nu}{}^{\alpha\beta}\phi^\beta\partial_\lambda\delta x_\nu=f_{\mu\lambda\nu}\partial_\lambda\delta x_\nu=\partial_\lambda(f_{\mu\lambda\nu}\delta x_\nu)+\partial_\lambda f_{\lambda\mu\nu}\delta x_\nu, \quad (2.29)$$

since the last two terms of $f_{\mu\lambda\nu}$ are symmetrical in λ and ν, and therefore do not contribute to Eq. (2.29), in view of Eq. (2.20). We now remark that, in virtue of $f_{\mu\lambda\nu}=-f_{\lambda\mu\nu}$,

$$\int_\sigma d\sigma_\mu\partial_\lambda(f_{\mu\lambda\nu}\delta x_\nu)=0, \quad (2.30)$$

provided $f_{\mu\lambda\nu}\delta x_\nu$ effectively approaches zero, with sufficient rapidity, at infinitely remote points[3] on σ. Finally then,

$$F(\sigma)=\int_\sigma d\sigma_\mu[\Pi_\mu{}^\alpha\delta\phi^\alpha+(1/c)T_{\mu\nu}\delta x_\nu], \quad (2.31)$$

where

$$(1/c)T_{\mu\nu}=(1/c)\mathfrak{L}\delta_{\mu\nu}-\Pi_\mu{}^\alpha\phi_\nu{}^\alpha-\partial_\lambda f_{\lambda\mu\nu} \quad (2.32)$$

is the stress tensor operator. As we shall demonstrate, this tensor has the property of being symmetrical,

$$T_{\mu\nu}=T_{\nu\mu}, \quad (2.33)$$

[3] All such characterizations of a spatially closed system, in terms of an operator approaching zero at infinity, are to be understood as a restriction to states for which the matrix elements of the operator have this property.

as an expression of the conservation of angular momentum.

Conservation laws are associated with variations that leave the action integral unchanged, since

$$\delta W_{12}=F(\sigma_1)-F(\sigma_2)=0 \tag{2.34}$$

implies the constancy of the corresponding generating operator. The mechanical conservation laws for an isolated system are derived by considering a rigid displacement of the entire field, or equivalently, of the coordinate system, which is described by a common infinitesimal translation and rotation of the surfaces σ_1 and σ_2,

$$\delta x_\mu=\epsilon_\mu-\epsilon_{\mu\nu}x_\nu, \quad \epsilon_{\mu\nu}=-\epsilon_{\nu\mu}, \tag{2.35}$$

combined with the field variation $\delta\phi^\alpha=0$. The displacement generating operator is then given by

$$F_{\delta x}(\sigma)=\epsilon_\mu P_\mu(\sigma)+\tfrac{1}{2}\epsilon_{\mu\nu}J_{\mu\nu}(\sigma), \tag{2.36}$$

where

$$P_\nu(\sigma)=(1/c)\int_\sigma d\sigma_\mu T_{\mu\nu}, \tag{2.37}$$

and

$$J_{\mu\nu}(\sigma)=(1/c)\int d\sigma_\lambda M_{\lambda\mu\nu},$$
$$M_{\lambda\mu\nu}=x_\mu T_{\lambda\nu}-x_\nu T_{\lambda\mu}. \tag{2.38}$$

Accordingly,

$$P_\nu(\sigma_1)-P_\nu(\sigma_2)=0, \tag{2.39}$$

and

$$J_{\mu\nu}(\sigma_1)-J_{\mu\nu}(\sigma_2)=0, \tag{2.40}$$

which are the conservation laws for the energy-momentum vector, and the angular momentum tensor, respectively. Since the surfaces σ_1 and σ_2 are arbitrary, we infer the corresponding differential conservation laws,

$$\partial_\mu T_{\mu\nu}=0, \tag{2.41}$$

and

$$\partial_\lambda M_{\lambda\mu\nu}=0, \tag{2.42}$$

which, in conjunction, imply the symmetry of the stress tensor:

$$\partial_\lambda M_{\lambda\mu\nu}=T_{\mu\nu}-T_{\nu\mu}=0. \tag{2.43}$$

The conservation law of charge can be obtained from the required invariance of the hermitian lagrange function under constant phase transformations—the multiplication of mutually hermitian conjugate pairs of field components by $\exp(\pm i\gamma)$. We consider infinitesimal phase transformations and, for convenience, write

$$\gamma=(e/\hbar c)\delta\lambda. \tag{2.44}$$

Thus, we postulate the invariance of $\mathfrak{L}$ under the infinitesimal transformation

$$\delta\phi^\alpha=-(ie/\hbar c)\epsilon^\alpha\delta\lambda\phi^\alpha, \tag{2.45}$$

where ϵ^α is characteristic of the field component ϕ^α, and may assume the values 0, or ± 1. The associated generating operator is

$$F_{\delta\lambda}(\sigma)=-(ie/\hbar c)\int_\sigma d\sigma_\mu \Pi_\mu{}^\alpha\epsilon^\alpha\phi^\alpha\delta\lambda$$
$$=(1/c)Q(\sigma)\delta\lambda, \tag{2.46}$$

where

$$Q(\sigma)=(1/c)\int_\sigma d\sigma_\mu j_\mu \tag{2.47}$$

and

$$j_\mu=-(iec/\hbar)\Pi_\mu{}^\alpha\epsilon^\alpha\phi^\alpha. \tag{2.48}$$

The implied conservation law,

$$Q(\sigma_1)-Q(\sigma_2)=0, \tag{2.49}$$

is that of the total charge in the system.

It is important to notice the ambiguity in the lagrange function that is associated with given equations of motion. Thus, two lagrange functions that are related by

$$\bar{\mathfrak{L}}(\phi^\alpha, \phi_\mu{}^\alpha)=\mathfrak{L}(\phi^\alpha, \phi_\mu{}^\alpha)+c\partial_\nu f_\nu(\phi^\alpha, \phi_\mu{}^\alpha) \tag{2.50}$$

provide action integral operators that differ by surface integrals:

$$\bar{W}_{12}=W_{12}+\left(\int_{\sigma_1}-\int_{\sigma_2}\right)d\sigma_\nu f_\nu. \tag{2.51}$$

Therefore, the principle of stationary action for $\bar{W}_{12}$ is automatically satisfied by the equations of motion deduced from W_{12}, and

$$\delta\bar{W}_{12}=\bar{F}(\sigma_1)-\bar{F}(\sigma_2), \tag{2.52}$$

where

$$\bar{F}=F+\delta w, \quad w=\int d\sigma_\nu f_\nu. \tag{2.53}$$

Hence, augmenting a lagrange function by the divergence of an arbitrary vector does not affect the equations of motion, but modifies the infinitesimal generating operator associated with a given surface σ. However, this ambiguity of the lagrange function corresponds precisely to the possibility of subjecting the commuting set of operators on σ to an arbitrary unitary transformation.

We verify this statement by specializing the general transformation theory to unitary transformations on a given surface. Let us introduce $\bar{\zeta}$, a new set of commuting operators on σ, which are obtained from ζ by a unitary transformation,

$$\Psi(\bar{\zeta}', \sigma)=\mathfrak{U}\Psi(\zeta', \sigma), \tag{2.54}$$

where $\mathfrak{U}$ is characterized by an infinitesimal hermitian generating operator δw, according to

$$\delta\mathfrak{U}^{-1}=(i/\hbar)\mathfrak{U}^{-1}\delta w. \tag{2.55}$$

As the analog of Eq. (2.8) we have, therefore,

$$\delta(\bar{\zeta}', \sigma|\zeta'', \sigma)=(i/\hbar)(\bar{\zeta}', \sigma|\delta w|\zeta'', \sigma); \tag{2.56}$$

but

$$\delta w = \bar{F} - F, \qquad (2.57)$$

where $\bar{F}$ and F are, respectively, the operators generating infinitesimal transformations of $\bar{\zeta}$ and ζ. This is just of the form (2.53); and conversely, by employing a particular w we obtain from Eq. (2.56) a differential equation to determine the transformation function that defines the new representation.

The commutation relations of our theory are implicit in the significance of F as an infinitesimal generating operator. We shall consider first those transformations that do not alter the surface σ, so that $\delta x_\nu = 0$. It is convenient to write

$$d\sigma_\mu = n_\mu d\sigma, \qquad (2.58)$$

where n_μ is a unit timelike vector and $d\sigma$ is the numerical measure of the surface element. To avoid irrelevant geometrical complications in the following discussion, we shall henceforth restrict σ to be a plane surface, so that n_μ is constant on σ. Note, incidentally, that coordinate derivatives can be decomposed into components normal and tangential to σ,

$$\partial_\mu = n_\mu \partial_n + \partial_{t\mu},$$
$$\partial_n = -n_\mu \partial_\mu, \quad \partial_{t\mu} = (\delta_{\mu\nu} + n_\mu n_\nu)\partial_\nu, \qquad (2.59)$$

and that the equations of motion read

$$\partial_n \Pi^\alpha = (1/c)(\partial \mathcal{L}/\partial \phi^\alpha) - \partial_{t\mu} \Pi_\mu{}^\alpha. \qquad (2.60)$$

We have here introduced the notation

$$\Pi^\alpha = n_\mu \Pi_\mu{}^\alpha \qquad (2.61)$$

for a quantity which, more precisely, should be written $\Pi^\alpha(x, \sigma)$.

The generating operator F now becomes

$$F_{\delta\phi} = \int d\sigma \Pi^\alpha \delta\phi^\alpha. \qquad (2.62)$$

Another significant form, associated with a different base vector system, is obtained from Eq. (2.53) with

$$f_\nu = -\Pi_\nu{}^\alpha \phi^\alpha. \qquad (2.63)$$

Indeed, we have

$$\delta \int d\sigma_\nu f_\nu = -\delta \int d\sigma \Pi^\alpha \phi^\alpha$$
$$= -\int d\sigma (\Pi^\alpha \delta\phi^\alpha + \delta\Pi^\alpha \phi^\alpha), \qquad (2.64)$$

so that

$$\bar{F} = F_{\delta\Pi} = -\int d\sigma \delta\Pi^\alpha \phi^\alpha. \qquad (2.65)$$

It should be emphasized again that these operator expressions are symbolic in the sense that the actual positions in which $\delta\phi^\alpha$ and $\delta\Pi^\alpha$ appear depend upon the structure of the lagrange function.

To obtain the proper interpretation of $F_{\delta\phi}$ or $F_{\delta\Pi}$, it is necessary to recognize that some of the Π^α can be identically equal to zero. This expresses the possibility that derivatives in timelike directions of some of the ϕ^α may not occur in the lagrange function. Accordingly, we shall divide the quantities ϕ^α and Π^α into two sets: ϕ^a and Π^a, called the canonical variables, and ϕ^A, Π^A, termed the constraint variables, in which the second set is characterized by

$$\Pi^A \equiv 0. \qquad (2.66)$$

The name ascribed to the ϕ^A refers to the fact that, for these quantities, the equations of motion (2.60) degenerate into equations of constraint,

$$(1/c)(\partial \mathcal{L}/\partial \phi^A) = \partial_{t\mu} \Pi_\mu{}^A, \qquad (2.67)$$

that is, relations among the variables on σ. The nature of these relations can be made more apparent by exploiting the requirement that Eq. (2.66) be independent of the coordinate system. We shall later show that the implied restriction on the structure of $\mathcal{L}$ is expressed by

$$n_\mu \Pi_\nu{}^A - n_\nu \Pi_\mu{}^A = (i/\hbar) \Pi^a S_{\mu\nu}{}^{aA}. \qquad (2.68)$$

On multiplication with n_ν, we obtain from this equation that

$$\Pi_\mu{}^A = (i/\hbar) \Pi^a S_{\mu\nu}{}^{aA} n_\nu, \qquad (2.69)$$

which enables the constraint equations to be written

$$(1/c)(\partial \mathcal{L}/\partial \phi^A) = (i/\hbar) \partial_\mu \Pi^a S_{\mu\nu}{}^{aA} n_\nu. \qquad (2.70)$$

We shall now assume that it is possible to solve the left side of Eq. (2.70) for ϕ^A, thus exhibiting explicitly the constraint variables as functions of the canonical variables. This excludes systems for which the ϕ^A are fundamentally ambiguous in consequence of the existence of gauge transformations. The latter situation will be discussed subsequently in terms of the familiar example provided by the electromagnetic field.

It is evident from these considerations, and from the structure of the generating operators,

$$F_{\delta\phi} = \int d\sigma \Pi^a \delta\phi^a,$$
$$F_{\delta\Pi} = -\int d\sigma \delta\Pi^a \phi^a, \qquad (2.71)$$

that only the canonical variables are dynamically independent on σ. Accordingly, $F_{\delta\phi}$ is to be interpreted as the generator of that infinitesimal transformation of the commuting operator set ζ on σ which is produced by changing ϕ^a into $\phi^a - \delta\phi^a$. Similarly, $F_{\delta\Pi}$ is regarded as generating the infinitesimal transformation of ζ in which Π^a is replaced by $\Pi^a - \delta\Pi^a$. Thus, ϕ^a and Π^a are special examples of a set of independent field coordinates, and the most general possibility is implicit in the transformation (2.53). Associated with any such set of operators is the conjugate set appearing in F, as Π^a is conjugate to ϕ^a, and $-\phi^a$ to Π^a.

We shall now examine the change in the matrix of G, an arbitrary function of field variables on σ, which is produced by the infinitesimal transformation generated by $F_{\delta\phi}$, say. Thus, we have

$$\begin{aligned}\delta(\zeta', \sigma|G|\zeta'', \sigma) &= (\Psi(\zeta', \sigma), G\delta\Psi(\zeta'', \sigma)) \\ &\quad + (\delta\Psi(\zeta', \sigma), G\Psi(\zeta'', \sigma)) \\ &= -(i/\hbar)(\zeta', \sigma|[G, F_{\delta\phi}]|\zeta'', \sigma). \quad (2.72)\end{aligned}$$

On the other hand, we have

$$\delta(\zeta', \sigma|G|\zeta'', \sigma) = (\zeta', \sigma|\delta_\phi G|\zeta'', \sigma), \quad (2.73)$$

where $\delta_\phi G$ means the change in G produced on increasing ϕ^a by $\delta\phi^a$. This simply expresses the fact that replacing ϕ^a by $\phi^a - \delta\phi^a$ in both G and ζ leaves the relation between them, and therefore the matrix, unaltered. We thereby infer the commutation relation,

$$[G, F_{\delta\phi}] = i\hbar\delta_\phi G, \quad (2.74)$$

with its evident generalization in regard to the field coordinates, including, in particular,

$$[G, F_{\delta\Pi}] = i\hbar\delta_\Pi G. \quad (2.75)$$

Of special importance are the results obtained from Eqs. (2.74) and (2.75) with $G = \phi^a$ and Π^a:

$$\begin{aligned}&\left[\phi^a(x), \int_\sigma d\sigma'\Pi^b(x')\delta\phi^b(x')\right] = i\hbar\delta\phi^a(x), \\ &\left[\Pi^a(x), \int_\sigma d\sigma'\Pi^b(x')\delta\phi^b(x')\right] = 0,\end{aligned} \quad (2.76)$$

and

$$\begin{aligned}&\left[\int_\sigma d\sigma'\delta\Pi^b(x')\phi^b(x'), \Pi^a(x)\right] = i\hbar\delta\Pi^a(x), \\ &\left[\int_\sigma d\sigma'\delta\Pi^b(x')\phi^b(x'), \phi^a(x)\right] = 0,\end{aligned} \quad (2.77)$$

in which we have invoked the dynamical independence of the ϕ^a and the Π^a.

To extract explicit commutation relations among the ϕ^a and Π^a we must know the operator properties of $\delta\phi^a$ and $\delta\Pi^a$. The requirement that the formalism be invariant with respect to time reflection supplies the desired information. It will be shown in Sec. III that $\delta\phi^b(x')$ and $\delta\Pi^b(x')$ commute with all field quantities $\phi^a(x)$ and $\Pi^a(x)$, on σ, except when both a and b designate components of fields that possess half-integral spin, in which event they anti-commute. Accordingly, the commutation relations of Eqs. (2.76) and (2.77) become

$$\begin{aligned}&\int_\sigma d\sigma'[\phi^a(x), \Pi^b(x')]_\pm\delta\phi^b(x') = i\hbar\delta\phi^a(x), \\ &\int_\sigma d\sigma'[\Pi^a(x), \Pi^b(x')]_\pm\delta\phi^b(x') = 0, \\ &\int_\sigma d\sigma'\delta\Pi^b(x')[\phi^b(x'), \Pi^a(x)]_\pm = i\hbar\delta\Pi^a(x), \\ &\int_\sigma d\sigma'\delta\Pi^b(x')[\phi^b(x'), \phi^a(x)]_\pm = 0,\end{aligned} \quad (2.78)$$

where

$$[A, B]_- = AB - BA \quad (2.79)$$

and

$$[A, B]_+ = AB + BA. \quad (2.80)$$

Since the $\delta\phi^a$ and $\delta\Pi^a$ are quite arbitrary, we have derived the fundamental commutation relations,

$$\begin{aligned}&[\phi^a(x), \Pi^b(x')]_\pm = i\hbar\delta_{ab}\delta_\sigma(x-x'), \\ &[\phi^a(x), \phi^b(x')]_\pm = [\Pi^a(x), \Pi^b(x')]_\pm = 0. \quad (2.81)\end{aligned}$$

Here $\delta_\sigma(x-x')$ denotes the three-dimensional delta-function, which is defined by

$$\int_\sigma d\sigma'\delta_\sigma(x-x')f(x') = f(x), \quad (2.82)$$

where $f(x)$ is an arbitrary function. The commutation properties of the ϕ^A can then be obtained from their explicit expression in terms of the canonical variables. Thus, according to Eq. (2.70),

$$(1/c)[\partial\mathcal{L}/\partial\phi^A(x), \phi^a(x')]_\pm = \partial_\mu\delta_\sigma(x-x')S_{\mu\nu}{}^{aA}n_\nu, \quad (2.83)$$

and

$$[\partial\mathcal{L}/\partial\phi^A(x), \Pi^a(x')]_\pm = 0. \quad (2.84)$$

In the requirement that commutators be employed, for components of an integral spin field, and anti-commutators for components of a half-integral spin field, we have the connection between the spin and statistics of particles. We shall note here that the commutation properties of a Bose-Einstein system, that is, an integral spin field, can be represented by means of differential operators. According to Eq. (1.13), a suitable representative of an arbitrary state obeys

$$\begin{aligned}\delta(\zeta', \sigma|) &= (i/\hbar)(\zeta', \sigma|F_{\delta\phi}|) \\ &= (i/\hbar)\left(\zeta', \sigma\left|\int_\sigma d\sigma\Pi^a\delta\phi^a\right|\right), \quad (2.85)\end{aligned}$$

in which σ is not altered. Now the characteristic property of an integral spin field is that $\delta\phi^a$ commutes with all dynamical variables and can therefore be treated as a number. The representation involved in

Eq. (2.85) is evidently that which is labeled by the continuous eigenvalues of the $\phi^a(x)$ at all points of σ. In terms of the notation

$$\delta(\phi',\sigma|)=\int_\sigma d\sigma\delta\phi^{a\prime}(x)(\delta/\delta\phi^{a\prime}(x))(\phi',\sigma|), \quad (2.86)$$

we obtain

$$(\hbar/i)(\delta/\delta\phi^{a\prime}(x))(\phi',\sigma|)=(\phi',\sigma|\Pi^a(x)|), \quad (2.87)$$

and similarly,

$$i\hbar(\delta/\delta\Pi^{a\prime}(x))(\Pi',\sigma|)=(\Pi',\sigma|\phi^a(x)|). \quad (2.88)$$

We shall make further application of the general commutation relations (2.74) and (2.75) by successively placing $G=P_\nu$, $J_{\mu\nu}$, and Q. According to Eq. (2.30), the last term of Eq. (2.32) makes no contribution to P_ν, so that

$$P_\nu=-\int d\sigma\Pi^a\partial_\nu\phi^a+(1/c)\int d\sigma_\nu\mathfrak{L}. \quad (2.89)$$

In the evaluation of δP_ν we encounter

$$(1/c)\int d\sigma_\nu\delta\mathfrak{L}=\int d\sigma_\nu\partial_\mu(\Pi_\mu{}^\alpha\delta\phi^\alpha)=\int d\sigma_\mu\partial_\nu(\Pi_\mu{}^\alpha\delta\phi^\alpha)$$
$$=\int d\sigma\partial_\nu(\Pi^a\delta\phi^a), \quad (2.90)$$

whence

$$\delta P_\nu=\int d\sigma(\partial_\nu\Pi^a\delta\phi^a-\delta\Pi^a\partial_\nu\phi^a). \quad (2.91)$$

The rearrangements of Eq. (2.90) have involved Eq. (2.17), as simplified by the equations of motion, and the assumption that the system is spatially closed. We thereby obtain

$$[\phi^a(x),P_\nu]=(\hbar/i)\partial_\nu\phi^a(x),$$
$$[\Pi^a(x),P_\nu]=(\hbar/i)\partial_\nu\Pi^a(x), \quad (2.92)$$

in virtue of the commutativity of P_ν with $\delta\phi^a$ and $\delta\Pi^a$, which is a consequence of the fact that half-integral spin field components must appear paired in the vector P_ν. Incidentally, a commutator $[F^{(1)},F^{(2)}]$, which has been evaluated by considering the effect on $F^{(2)}$ of the transformation generated by $F^{(1)}$, can equally well be viewed from the reverse standpoint. Thus, the relations (2.92) also exhibit P_ν in the role of the translation generator.

The angular momentum tensor $J_{\mu\nu}$ is easily brought into a form analogous to (2.89),

$$J_{\mu\nu}=-\int d\sigma\Pi^a[(x_\mu\partial_\nu-x_\nu\partial_\mu)\phi^a+(i/\hbar)S_{\mu\nu}{}^{a\beta}\phi^\beta]$$
$$+(1/c)\int(d\sigma_\nu x_\mu\mathfrak{L}-d\sigma_\mu x_\nu\mathfrak{L}). \quad (2.93)$$

The contribution of the second term to $\delta J_{\mu\nu}$ is evaluated as follows,

$$\delta(1/c)\int(d\sigma_\nu x_\mu\mathfrak{L}-d\sigma_\mu x_\nu\mathfrak{L})$$
$$=\int[d\sigma_\nu x_\mu\partial_\lambda(\Pi_\lambda{}^\alpha\delta\phi^\alpha)-d\sigma_\mu x_\nu\partial_\lambda(\Pi_\lambda{}^\alpha\delta\phi^\alpha)]$$
$$=\int d\sigma_\lambda(x_\mu\partial_\nu-x_\nu\partial_\mu)(\Pi_\lambda{}^\alpha\delta\phi^\alpha)$$
$$+\int(d\sigma_\mu\Pi_\nu{}^\alpha-d\sigma_\nu\Pi_\mu{}^\alpha)\delta\phi^\alpha$$
$$=\int d\sigma[(x_\mu\partial_\nu-x_\nu\partial_\mu)(\Pi^a\delta\phi^a)$$
$$+(n_\mu\Pi_\nu{}^\alpha-n_\nu\Pi_\mu{}^\alpha)\delta\phi^\alpha]. \quad (2.94)$$

Therefore,

$$\delta J_{\mu\nu}=-\int d\sigma\delta\Pi^a[(x_\mu\partial_\nu-x_\nu\partial_\mu)\phi^a+(i/\hbar)S_{\mu\nu}{}^{a\beta}\phi^\beta]$$
$$+\int d\sigma[(x_\mu\partial_\nu-x_\nu\partial_\mu)\Pi^\alpha-(i/\hbar)\Pi^bS_{\mu\nu}{}^{b\alpha}$$
$$+n_\mu\Pi_\nu{}^\alpha-n_\nu\Pi_\mu{}^\alpha]\delta\phi^\alpha. \quad (2.95)$$

We have thus derived the commutation relations

$$[\phi^a,J_{\mu\nu}]=(x_\mu(\hbar/i)\partial_\nu-x_\nu(\hbar/i)\partial_\mu)\phi^a+S_{\mu\nu}{}^{a\beta}\phi^\beta,$$
$$[\Pi^a,J_{\mu\nu}]=(x_\mu(\hbar/i)\partial_\nu-x_\nu(\hbar/i)\partial_\mu)\Pi^a-\Pi^bS_{\mu\nu}{}^{ba}$$
$$+(\hbar/i)(n_\mu\Pi_\nu{}^a-n_\nu\Pi_\mu{}^a), \quad (2.96)$$
$$0=-\Pi^aS_{\mu\nu}{}^{aA}+(\hbar/i)(n_\mu\Pi_\nu{}^A-n_\nu\Pi_\mu{}^A),$$

which exhibit $J_{\mu\nu}$ in the role of the rotation generator and illustrate the formation of $J_{\mu\nu}$ as the superposition of orbital and spin angular momenta. The third equation of Eq. (2.96), the statement that $\Pi^A\equiv0$ is a property independent of the coordinate system, has already been employed in Eq. (2.68).

According to Eqs. (2.47) and (2.48), the charge operator is given by

$$Q=-(ie/\hbar)\int d\sigma\Pi^a\epsilon^a\phi^a. \quad (2.97)$$

Therefore, we have

$$\delta Q=-(ie/\hbar)\int d\sigma(\delta\Pi^a\epsilon^a\phi^a+\Pi^a\epsilon^a\delta\phi^a), \quad (2.98)$$

from which we obtain the commutation relations

$$[\phi^a,Q]=e\epsilon^a\phi^a,\quad [\Pi^a,Q]=-e\epsilon^a\Pi^a. \quad (2.99)$$

These indicate the significance of e as the elementary charge, and exhibit Q in the role of the phase transformation generator. Note, however, that the derivation of Eq. (2.99) from the latter viewpoint is not restricted to the canonical variables, although nothing new is obtained thereby.

The general infinitesimal operator (2.31) describes the transformation from the commuting operator set ζ, σ to $\zeta-\delta\zeta$, $\sigma+\delta\sigma$, as indicated by

$$\Psi((\zeta-\delta\zeta)', \sigma+\delta\sigma)=[1-(i/\hbar)F]\Psi(\zeta', \sigma). \quad (2.100)$$

The operator F is additively composed of two parts,

$$F=F_{\delta\phi}+F_{\delta x}, \quad (2.101)$$

where $F_{\delta\phi}$ induces, via $\delta\phi^a$, a change in the commuting operator set defined in relation to the local coordinate system provided by a fixed σ,

$$\Psi((\zeta-\delta\zeta)', \sigma)=[1-(i/\hbar)F_{\delta\phi}]\Psi(\zeta', \sigma), \quad (2.102)$$

while

$$F_{\delta x}=(1/c)\int d\sigma_\mu T_{\mu\nu}\delta x_\nu \quad (2.103)$$

generates the change in σ described by δx_ν, for a fixed set of commuting operators defined relative to σ,

$$\Psi(\zeta', \sigma+\delta\sigma)=[1-(i/\hbar)F_{\delta x}]\Psi(\zeta', \sigma). \quad (2.104)$$

Consistent with our restriction to plane surfaces, we consider only rigid displacements of σ, for which the generating operator has already been given in Eq. (2.36).

Differential equations that describe the change in the representative of an arbitrary state, as produced by rigid displacements, are inferred from

$$\delta_x(\zeta', \sigma|)=(\delta_x\Psi(\zeta', \sigma), \Psi)=(i/\hbar)(\zeta', \sigma|F_{\delta x}|). \quad (2.105)$$

In terms of the notation

$$\delta_x(\zeta', \sigma|)=\epsilon_\mu\delta_\mu(\zeta', \sigma|)+\tfrac{1}{2}\epsilon_{\mu\nu}\delta_{\mu\nu}(\zeta', \sigma|), \quad (2.106)$$

we obtain generalized Schrödinger equations[4] for translations,

$$(\hbar/i)\delta_\mu(\zeta', \sigma|)=(\zeta', \sigma|P_\mu(\sigma)|)$$
$$=\int(\zeta', \sigma|P_\mu(\sigma)|\zeta'', \sigma)d\zeta''(\zeta'', \sigma|), \quad (2.107)$$

and rotations,

$$(\hbar/i)\delta_{\mu\nu}(\zeta', \sigma|)=(\zeta', \sigma|J_{\mu\nu}(\sigma)|)$$
$$=\int(\zeta', \sigma|J_{\mu\nu}(\sigma)|\zeta'', \sigma)d\zeta''(\zeta'', \sigma|). \quad (2.108)$$

An operator $G(\sigma)$, which is constructed from field quantities on σ, has a matrix $(\zeta', \sigma|G(\sigma)|\zeta'', \sigma)$ that is independent of σ, since the relation between $G(\sigma)$ and the ζ on σ is unchanged by an alteration of the surface. The components of $P_\mu(\sigma)$ referred to axes based on σ are of this nature; and, consequently, the matrix of $P_\mu(\sigma)$ in Eq. (2.107) involves the orientation of σ relative to the coordinate system, but is otherwise independent of σ. Commutation relations between P_μ, $J_{\mu\nu}$ and $G(\sigma)$ follow from this property of the $G(\sigma)$ matrix. Thus, we have

$$0=\delta_x G(\sigma)-(i/\hbar)[G(\sigma), F_{\delta x}], \quad (2.109)$$

whence

$$[G(\sigma), P_\mu]=(\hbar/i)\delta_\mu G(\sigma), \quad (2.110)$$

and

$$[G(\sigma), J_{\mu\nu}]=(\hbar/i)\delta_{\mu\nu}G(\sigma). \quad (2.111)$$

As the first of several illustrations of these commutation relations, we choose $G(\sigma)=\phi^\alpha(x)$. According to Eq. (2.25), we have

$$[\phi^\alpha(x), P_\mu]=(\hbar/i)\partial_\mu\phi^\alpha(x), \quad (2.112)$$

and

$$[\phi^\alpha(x), J_{\mu\nu}]=(x_\mu(\hbar/i)\partial_\nu-x_\nu(\hbar/i)\partial_\mu)\phi^\alpha(x)+S_{\mu\nu}{}^{\alpha\beta}\phi^\beta(x), \quad (2.113)$$

which is in agreement with Eqs. (2.92) and (2.96), but without the restriction of the latter to the components ϕ^a. A particularly simple example is provided by $G(\sigma)=Q$, the total charge. Since this operator is independent of σ, we have

$$[Q, P_\mu]=[Q, J_{\mu\nu}]=0, \quad (2.114)$$

which state, inversely, that P_μ and $J_{\mu\nu}$ are unaffected by phase transformations. The effect of a displacement of σ on the quantity $G(\sigma)=P_\lambda e_\lambda(\sigma)$, where $e_\lambda(\sigma)$ is an arbitrary vector that is rigidly attached to σ, comes entirely from the rotation of the vector $e_\lambda(\sigma)$,

$$\delta_x(P_\lambda e_\lambda(\sigma))=-\epsilon_{\mu\nu}P_\mu e_\nu(\sigma). \quad (2.115)$$

Therefore, we have

$$[P_\mu, P_\nu]=0, \quad (2.116)$$

and

$$[P_\lambda, J_{\mu\nu}]=i\hbar(\delta_{\nu\lambda}P_\mu-\delta_{\mu\lambda}P_\nu). \quad (2.117)$$

Our last example, $G(\sigma)=J_{\lambda\kappa}e_\lambda{}^{(1)}(\sigma)e_\kappa{}^{(2)}(\sigma)$, where both $e_\lambda{}^{(1)}(\sigma)$ and $e_\kappa{}^{(2)}(\sigma)$ are arbitrary vectors rigidly attached to σ, is actually an extension of the type of operator under consideration, since

$$J_{\lambda\kappa}e_\lambda{}^{(1)}e_\kappa{}^{(2)}=(1/c)\int d\sigma_\mu[x_\lambda e_\lambda{}^{(1)}T_{\mu\kappa}e_\kappa{}^{(2)}-x_\kappa e_\kappa{}^{(2)}T_{\mu\lambda}e_\lambda{}^{(1)}] \quad (2.118)$$

involves space-time coordinates, in addition to field variables. The necessary revision of Eq. (2.109) is

$$\delta_x G(\sigma)=(i/\hbar)[G(\sigma), F_{\delta x}]+\partial_x G(\sigma), \quad (2.119)$$

where $\partial_x G(\sigma)$ denotes the displacement induced change in $G(\sigma)$, associated with the explicit appearance of space-time coordinates. In the example provided by

[4] Note that these Schrödinger equations have been obtained from the Heisenberg picture, in which the arbitrary state vector is fixed. P. A. M. Dirac, *The Principles of Quantum Mechanics* (The Clarendon Press, Oxford, 1947), third edition, Sec. 32.

Eq. (2.118), $\partial_x G(\sigma)$ arises from the translation (but not rotation) of σ,

$$\partial_x(J_{\lambda\kappa}e_\lambda^{(1)}e_\kappa^{(2)})=(\epsilon_\lambda P_\kappa-\epsilon_\kappa P_\lambda)e_\lambda^{(1)}e_\kappa^{(2)}. \quad (2.120)$$

On combining this with

$$\delta_x(J_{\lambda\kappa}e_\lambda^{(1)}e_\kappa^{(2)})=-\epsilon_{\mu\nu}J_{\mu\kappa}e_\nu^{(1)}e_\kappa^{(2)}-\epsilon_{\mu\nu}J_{\lambda\mu}e_\lambda^{(1)}e_\nu^{(2)}, \quad (2.121)$$

we again obtain Eq. (2.117), and

$$[J_{\lambda\kappa}, J_{\mu\nu}]=i\hbar(\delta_{\lambda\nu}J_{\mu\kappa}+\delta_{\lambda\mu}J_{\kappa\nu}+\delta_{\kappa\nu}J_{\lambda\mu}+\delta_{\kappa\mu}J_{\nu\lambda}). \quad (2.122)$$

We may remark, as an example of a general procedure for constructing representations of operator commutation properties, that the identity

$$[[\phi^\alpha, J_{\lambda\kappa}], J_{\mu\nu}]-[[\phi^\alpha, J_{\mu\nu}], J_{\lambda\kappa}] =[\phi^\alpha, [J_{\lambda\kappa}, J_{\mu\nu}]] \quad (2.123)$$

leads to analogous commutation relations for the representatives of orbital and spin angular momentum in Eq. (2.113).

In a final comment concerning commutation relations, we observe that the commutators of generating operators are of significance in connection with integrability conditions for the infinitesimal transformations generated by these operators.[5] If $F^{(1)}$ and $F^{(2)}$ are two such generators of infinitesimal transformations, we have

$$\begin{aligned}\delta^{(1)}\Psi(\zeta',\sigma)&=\Psi((\zeta-\delta^{(1)}\zeta)',\sigma+\delta^{(1)}\sigma)-\Psi(\zeta',\sigma)\\&=-(i/\hbar)F^{(1)}\Psi(\zeta',\sigma),\\ \delta^{(2)}\Psi(\zeta',\sigma)&=\Psi((\zeta-\delta^{(2)}\zeta)',\sigma+\delta^{(2)}\sigma)-\Psi(\zeta',\sigma)\\&=-(i/\hbar)F^{(2)}\Psi(\zeta',\sigma).\end{aligned} \quad (2.124)$$

Now, the difference between the results of the two ways in which these transformations can be successively applied may be regarded as the effect of a third, related transformation,

$$\begin{aligned}(\delta^{(1)}\delta^{(2)}-\delta^{(2)}\delta^{(1)})\Psi(\zeta',\sigma)&=\Psi((\zeta+(\delta^{(1)}\delta^{(2)}-\delta^{(2)}\delta^{(1)})\zeta)',\sigma+(\delta^{(1)}\delta^{(2)}-\delta^{(2)}\delta^{(1)})\sigma)-\Psi(\zeta',\sigma)\\&=-(i/\hbar)F^{[12]}\Psi(\zeta',\sigma).\end{aligned} \quad (2.125)$$

Therefore,

$$[F^{(1)}, F^{(2)}]=i\hbar F^{[12]} \quad (2.126)$$

is a condition necessary to the integrability of Eq. (2.124). A simple illustration of this viewpoint is provided by rigid displacements:

$$\begin{aligned}\delta^{(1,2)}x_\mu&=\epsilon_\mu^{(1,2)}-\epsilon_{\mu\nu}^{(1,2)}x_\nu,\\ F_{\delta x^{(1,2)}}&=\epsilon_\mu^{(1,2)}P_\mu+\tfrac{1}{2}\epsilon_{\mu\nu}^{(1,2)}J_{\mu\nu},\end{aligned} \quad (2.127)$$

since

$$\begin{aligned}(\delta^{(1)}\delta^{(2)}-\delta^{(2)}\delta^{(1)})x_\mu&=-\epsilon_{\mu\nu}^{(1)}\epsilon_\nu^{(2)}+\epsilon_{\mu\nu}^{(2)}\epsilon_\nu^{(1)}-(-\epsilon_{\mu\lambda}^{(1)}\epsilon_{\lambda\nu}^{(2)}+\epsilon_{\mu\lambda}^{(2)}\epsilon_{\lambda\nu}^{(1)})x_\nu\\&=\epsilon_\mu^{[12]}-\epsilon_{\mu\nu}^{[12]}x_\nu\end{aligned} \quad (2.128)$$

is another rigid displacement. The ensuing commutation relations are just Eqs. (2.116), (2.117), and (2.122).

In our discussions of the variational principle (2.14), we have dealt with the properties of a given dynamical system. The principle is also applicable, however, to variations in which the system is altered, as characterized by a change in the structure of the lagrange function. For a variation of this type, we have

$$\delta(\zeta_1',\sigma_1|\zeta_2'',\sigma_2)=(i/\hbar c)\int_{\sigma_2}^{\sigma_1}(dx)(\zeta_1',\sigma_1|\delta\mathfrak{L}[x]|\zeta_2'',\sigma_2), \quad (2.129)$$

or

$$\delta(\zeta_1',\sigma_1|\zeta_2'',\sigma_2)=(i/\hbar c)\int_{\sigma_2}^{\sigma_1}(dx)\int(\zeta_1',\sigma_1|\zeta^a,\sigma)d\zeta^a(\zeta^a,\sigma|\delta\mathfrak{L}[x]|\zeta^b,\sigma)d\zeta^b(\zeta^b,\sigma|\zeta_2'',\sigma_2), \quad (2.130)$$

where the surface σ contains the point x. If two independent variations of this nature are applied successively we obtain

$$\begin{aligned}\delta^{(2)}\delta^{(1)}(\zeta_1',\sigma_1|\zeta_2'',\sigma_2)&=(i/\hbar c)\int_{\sigma_2}^{\sigma_1}(dx)\Big[\int\delta^{(2)}(\zeta_1',\sigma_1|\zeta^a,\sigma)d\zeta^a(\zeta^a,\sigma|\delta^{(1)}\mathfrak{L}[x]|\zeta_2'',\sigma_2)\\&\qquad+\int(\zeta_1',\sigma_1|\delta^{(1)}\mathfrak{L}[x]|\zeta^b,\sigma)d\zeta^b\delta^{(2)}(\zeta^b,\sigma|\delta_2'',\sigma_2)\Big]\\&=(i/\hbar c)^2\int_{\sigma_2}^{\sigma_1}(dx)\Big[\int_{\sigma}^{\sigma_1}(dx')(\zeta_1',\sigma_1|\delta^{(2)}\mathfrak{L}[x']\delta^{(1)}\mathfrak{L}[x]|\zeta_2'',\sigma_2)\\&\qquad+\int_{\sigma_2}^{\sigma}(dx')(\zeta_1',\sigma_1|\delta^{(1)}\mathfrak{L}[x]\delta^{(2)}\mathfrak{L}[x']|\zeta_2'',\sigma_2)\Big].\end{aligned} \quad (2.131)$$

We shall introduce here a notation for chronologically ordered operators,

$$(A(x)B(x'))_+=\begin{matrix}A(x)B(x'), & x_0>x_0'\\ B(x')A(x), & x_0'>x_0,\end{matrix} \quad (2.132)$$

[5] See, for example, H. Weyl, *The Theory of Group and Quantum Mechanics* (E. P. Dutton and Company, Inc., New York, 1931), p. 177.

which is an invariant concept provided that the operators involved commute when $x-x'$ is a spacelike interval and that the positive sense of time is preserved. Thus we may write Eq. (2.131) more compactly as

$$\delta^{(1)}\delta^{(2)}(\zeta_1', \sigma_1|\zeta_2'', \sigma_2)=\delta^{(2)}\delta^{(1)}(\zeta_1', \sigma_1|\zeta_2'', \sigma_2) = (i/\hbar c)^2\int_{\sigma_2}^{\sigma_1}(dx)\int_{\sigma_2}^{\sigma_1}(dx')(\zeta_1', \sigma_1|(\delta^{(1)}\mathfrak{L}[x]\delta^{(2)}\mathfrak{L}[x'])_+|\zeta_2'', \sigma_2). \quad (2.133)$$

These results will find frequent application in later work.

We shall conclude this section by indicating, in connection with a Bose-Einstein field, a method for constructing the transformation function $(\zeta_1', \sigma_1|\zeta_2'', \sigma_2)$, which has as its classical analog the Hamilton-Jacobi theory of field mechanics. The actual motion of the system is implicit in the form assumed by the variation of the action integral,

$$\delta W_{12}=\left[\int_\sigma d\sigma\Pi^a\delta\phi^a+\epsilon_\mu P_\mu(\sigma)+\tfrac{1}{2}\epsilon_{\mu\nu}J_{\mu\nu}(\sigma)\right]_{\sigma_2}^{\sigma_1}, \quad (2.134)$$

in which we continue the restriction to plane spacelike surfaces. It is implied by Eq. (2.134) that W_{12} can be exhibited as a function of σ_1, σ_2, and of the ϕ^a on these surfaces, and therefore that the Π^a, P_μ and $J_{\mu\nu}$ associated with each surface can also be so exhibited. With the aid of commutation relations between the ϕ^a on σ_1 and on σ_2, it will be possible to order the operators in Eq. (2.134) so that the ϕ^a on σ_1 everywhere stand to the left of the ϕ^a on σ_2. The differential expression, thus ordered, shall be denoted by $\delta\mathcal{W}(\phi_1, \sigma_1; \phi_2, \sigma_2)$, from which we obtain differential equations connecting the various ordered operators,

$$\begin{aligned}(\delta/\delta\phi^a(x_1))\mathcal{W}&=\Pi^a(x_1), \quad (\delta/\delta\phi^a(x_2))\mathcal{W}=-\Pi^a(x_2),\\ \delta_\mu^{(1)}\mathcal{W}&=P_\mu(\sigma_1), \quad \delta_\mu^{(2)}\mathcal{W}=-P_\mu(\sigma_2), \quad (2.135)\\ \delta_{\mu\nu}^{(1)}\mathcal{W}&=J_{\mu\nu}(\sigma_1), \quad \delta_{\mu\nu}^{(2)}\mathcal{W}=-J_{\mu\nu}(\sigma_2),\end{aligned}$$

where x_1 and x_2 are arbitrary points on σ_1 and σ_2, respectively. In conjunction with the commutation relations (2.81), these Hamilton-Jacobi operator equations serve to determine the ordered operator $\mathcal{W}(\phi_1, \sigma_1; \phi_2, \sigma_2)$, to within an additive constant.

It is important to recognize that $\mathcal{W}\neq W_{12}$, and indeed, that $\mathcal{W}$ is a non-hermitian operator.[6] This is a consequence of the noncommutativity of the ϕ^a on σ_1 and on σ_2 in a manner which depends upon the location of these surfaces. Thus, if the operator W_{12} is first ordered and then varied, the result will differ from what is obtained by ordering δW_{12}. We now turn to the differential characterization of the transformation function labelled by eigenvalues of ϕ^a on σ_1 and σ_2,

$$\delta(\phi', \sigma_1|\phi'', \sigma_2) = (i/\hbar)(\phi', \sigma_1|\delta\mathcal{W}(\phi_1, \sigma_1; \phi_2, \sigma_2)|\phi'', \sigma_2), \quad (2.136)$$

and observe that, in virtue of the ordering in $\delta\mathcal{W}$, the operators ϕ^a on σ_1 and on σ_2 act directly on their respective eigenvectors and can be replaced by the associated eigenvalues:

$$\delta(\phi', \sigma_1|\phi'', \sigma_2) = (i/\hbar)\delta\mathcal{W}(\phi', \sigma_1; \phi'', \sigma_2)(\phi', \sigma_1|\phi'', \sigma_2). \quad (2.137)$$

The transformation function is thereby obtained as[7]

$$(\phi', \sigma_1|\phi'', \sigma_2)=\exp[(i/\hbar)\mathcal{W}(\phi', \sigma_1; \phi'', \sigma_2)], \quad (2.138)$$

where the constant of integration, which is additively contained in $\mathcal{W}$, can be determined from the condition

$$\lim_{\sigma_1\to\sigma_2}(\phi', \sigma_1|\phi'', \sigma_2)=\delta(\phi'-\phi''). \quad (2.139)$$

III. TIME REFLECTION

The general physical requirement of invariance with respect to coordinate transformations applies not only to translations and rotations of the coordinate system, but also to reflections of the coordinate axes. Among

[6] The elementary example of a one-dimensional free particle will suffice to illustrate this. The Hamilton-Jacobi equations for the construction of $\mathcal{W}(x(t_1), x(t_2), t)$, $t=t_1-t_2$, are

$$(\partial/\partial x(t_1))\mathcal{W}=-(\partial/\partial x(t_2))\mathcal{W}=p, \quad -(\partial/\partial t)\mathcal{W}=p^2/2m.$$

According to the solution of the equations of motion,

$$x(t_1)-x(t_2)=(t/m)p,$$

we have

$$[x(t_1), x(t_2)]=-i\hbar t/m,$$

whence

$$\begin{aligned}-(\partial/\partial t)\mathcal{W}&=(m/2t^2)(x(t_1)-x(t_2))^2\\&=(m/2t^2)[x^2(t_1)-2x(t_1)x(t_2)+x^2(t_2)]-i\hbar/2t.\end{aligned}$$

The solution of the Hamilton-Jacobi operator equations is

$$\mathcal{W}=(m/2t)[x^2(t_1)-2x(t_1)x(t_2)+x^2(t_2)]+\tfrac{1}{2}i\hbar\log(At),$$

which should be compared with the hermitian action integral

$$\begin{aligned}W_{12}=\tfrac{1}{2}mv^2t&=(m/2t)(x(t_1)-x(t_2))^2\\&=(m/2t)[x^2(t_1)-2x(t_1)x(t_2)+x^2(t_2)]-\tfrac{1}{2}i\hbar.\end{aligned}$$

Incidentally, the analog of Eq. (2.138) is

$$\begin{aligned}(x', t_1|x'', t_2)&=\exp[(i/\hbar)\mathcal{W}(x', x'', t)]\\&=(At)^{-\frac{1}{2}}\exp[(im/2\hbar t)(x'-x'')^2],\end{aligned}$$

where the constant A is determined to be

$$A=2\pi i\hbar/m$$

from the analog of Eq. (2.139),

$$\lim_{t\to 0}(x', t_1|x'', t_2)=\delta(x'-x'').$$

[7] The exponential form of Eq. (2.138) is familiar as a basis for establishing a correspondence connection with classical Hamilton-Jacobi particle mechanics. Dirac employed this form in a discussion of unitary transformations and recognized, in part, that the Hamilton-Jacobi equations are rigorous as relations among ordered operators (see the end of the section quoted in reference 4). In Feynman's version of quantum mechanics [R. P. Feynman, Revs. Modern Phys. **20**, 367 (1948)], the exponential form is employed for infinitesimal time intervals, with the real part of $\mathcal{W}$ defined as the classical action integral.

the latter transformations, time reflection has a singular position. Its special nature can be indicated by the transformation properties of some integrated physical quantities. Thus, the expectation value of the energy-momentum vector,

$$\langle P_\nu \rangle = (1/c)\int_\sigma d\sigma_\mu \langle T_{\mu\nu}\rangle, \tag{3.1}$$

is actually a pseudovector with respect to time reflection. With the plane surface σ chosen perpendicular to the time axis, the components of $\langle P_\nu\rangle$ are obtained as three-dimensional volume integrals,

$$\begin{aligned}\langle P_0\rangle &= (1/c)\int d\sigma \langle T_{00}\rangle,\\ \langle P_k\rangle &= (1/c)\int d\sigma\langle T_{0k}\rangle, \quad k=1, 2, 3,\end{aligned} \tag{3.2}$$

and the time reflection $x_0 \rightarrow -x_0$, $x_k \rightarrow x_k$ induces $\langle P_0\rangle \rightarrow \langle P_0\rangle$, $\langle P_k\rangle \rightarrow -\langle P_k\rangle$, according to the transformation properties of tensors. This differs in sign from a proper vector transformation. In particular, the energy does not reverse sign under time reflection. More generally, this property of $\langle P_\nu\rangle$ is obtained from the pseudovector character of $d\sigma_\mu$, which expresses the pseudoscalar nature of a four-dimensional volume element with respect to time reflection. Similarly, the expectation value of the charge

$$\langle Q\rangle = (1/c)\int d\sigma_\mu \langle j_\mu\rangle = (1/c)\int d\sigma\langle j_0\rangle \tag{3.3}$$

behaves as a pseudoscalar under time reflection. Hence, this transformation interchanges positive and negative charge, and both signs must occur symmetrically in a covariant theory. Indeed, for some purposes the requirement of charge symmetry can be substituted for the more incisive demand of invariance under time reflection.

The significant implication of these properties is that time reflection cannot be included within the general framework of unitary transformations. Thus, on referring to the Schrödinger equation for translations (2.107), or the analogous operator equation (2.110), we encounter a contradiction between the transformation properties of the proper vector translation operator δ_μ and of the pseudovector P_μ. This difficulty appears most fundamentally in our basic variational principle (2.14). With $\mathcal{L}$ behaving as a scalar and (dx) as a pseudoscalar, reflection of the time axis introduces a minus sign on the right side of this equation. However, it is important to notice that the scalar nature of $\mathcal{L}$ cannot be maintained for that part of the lagrange function which describes half-integral spin fields. Indeed, such contributions to $\mathcal{L}$ behave like pseudoscalars with respect to time reflection.[8] If we were to consider only such a half-integral spin field, the basic dynamical equation would preserve its structure under time reversal, but at the expense of violating the general transformation properties of all physical quantities; charge would remain unaltered, and energy would reverse sign under time reflection. The latter difficulty simply indicates that, on inclusion of the contributions of integral spin fields, the various parts of $\mathcal{L}$ would transform differently, thus emphasizing again the general failure of Eq. (2.14) to admit time reflection as a unitary transformation.

To aid in investigating the extended class of transformations that is required to include time reflection we shall introduce some notational developments. The scalar product of two vectors, Ψ_a and Ψ_b, can be written

$$(a|b) = \Psi_a{}^*\Psi_b = \Psi_b\Psi_a{}^*, \tag{3.4}$$

thereby being regarded as the invariant combination of a vector Ψ_b with the dual, complex conjugate vector $\Psi_a{}^*$ We allow operators to act both on the left and on the right of vectors, Ψ and Ψ^*. Thus, an operator associated with A, the transposed operator A^T, is defined by[9]

$$A\Psi = \Psi A^T, \quad \Psi^* A = A^T\Psi^*, \tag{3.5}$$

or by

$$(a|A|b) = \Psi_a{}^* A\Psi_b = \Psi_b A^T \Psi_a{}^*. \tag{3.6}$$

We also define the associated complex conjugate operator A^*,

$$(A\Psi)^* = A^*\Psi^*. \tag{3.7}$$

The connection with the hermitian conjugate operator $A^\dagger$ is obtained from the definition of the latter,

$$(A\Psi)^* = \Psi^* A^\dagger, \tag{3.8}$$

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[8] The fundamental invariant of a spin $\frac{1}{2}$ field is $\bar{\psi}\psi = \psi^\dagger\gamma_0\psi$. The transformation that represents time reflection, $\psi' = R\psi$, can be obtained from its equivalence with a rotation through the angle π in the (45) plane; $R = \exp[i\pi\frac{1}{2}\sigma_{45}] = i\sigma_{45}$. Accordingly,

$$\bar{\psi}'\psi' = \psi^\dagger R^{-1}\gamma_0 R\psi = -\bar{\psi}\psi,$$

which indicates the pseudoscalar character of the spin $\frac{1}{2}$ field lagrange function, with respect to time reflection. The corresponding behavior of fields with other spin values can be obtained from the observation that a spinor of rank n contains fields of spin $\frac{1}{2}n$, $\frac{1}{2}n-1$, $\cdots$. The basic invariant and time reflection operator for a spinor of rank n are

$$\bar{\psi}\psi = \psi^\dagger \prod_{k=1}^{n}\gamma_0{}^{(k)}\psi,$$

and

$$R = \exp\left[i\pi\tfrac{1}{2}\sum_{k=1}^{n}\sigma_{45}{}^{(k)}\right] = \prod_{k=1}^{n} i\sigma_{45}{}^{(k)}.$$

Therefore,

$$\bar{\psi}'\psi' = \psi^\dagger R^{-1}\prod_{k=1}^{n}\gamma_0{}^{(k)}R\psi = (-1)^n\bar{\psi}\psi,$$

which shows the pseudoscalar nature of the lagrange function for all half-integral spin fields.

[9] Note how the familiar property of transposition, $(AB)^T = B^TA^T$, follows from this definition: $AB\Psi = A(\Psi B^T) = \Psi B^T A^T$.

namely,

$$A^\dagger = A^{*T}. \tag{3.9}$$

Conventional quantum mechanics contemplates transformations only within the Ψ vector space, and contragradient transformations within the dual Ψ^* space. We shall now consider transformations that interchange the two spaces, as in

$$\Psi_a \rightarrow \Psi_{\bar{a}} = \Psi_a^*. \tag{3.10}$$

The effect of Eq. (3.10) is indicated by

$$(a|b) = \Psi_a^* \Psi_b = \Psi_{\bar{a}} \Psi_{\bar{b}}^* = (\bar{b}|\bar{a}), \tag{3.11}$$

and

$$(a|A|b) = \Psi_a^* A \Psi_b = \Psi_{\bar{a}} A \Psi_b^* = (\bar{b}|A^T|\bar{a}). \tag{3.12}$$

More generally, if

$$\Psi_{\bar{a}}^* = R\Psi_a, \tag{3.13}$$

where R is a unitary operator, we have

$$(a|b) = (\bar{b}|\bar{a}), \quad (a|A|b) = (\bar{b}|\bar{A}|\bar{a}), \tag{3.14}$$

in which

$$\bar{A} = (RAR^{-1})^T. \tag{3.15}$$

Now, we have

$$\overline{AB} = (RABR^{-1})^T = (RBR^{-1})^T(RAR^{-1})^T = \bar{B}\bar{A}, \tag{3.16}$$

and therefore

$$(a|[A,B]|b) = -(\bar{b}|[\bar{A},\bar{B}]|\bar{a}). \tag{3.17}$$

We have here precisely the sign change that is required to preserve the structure of equations like Eq. (2.110) under time reflection.

We now examine whether it is possible to satisfy the requirement of invariance under time reflection by means of transformations of the type (3.13). When we introduce the coordinate transformation

$$\bar{x}_0 = -x_0, \quad \bar{x}_k = x_k, \quad k = 1, 2, 3, \tag{3.18}$$

in conjunction with the eigenvector transformation

$$\Psi^*(\bar{\zeta}', \sigma) = R\Psi(\zeta', \sigma), \tag{3.19}$$

the fundamental dynamical equation (2.14) becomes

$$\delta(\bar{\zeta}_2'', \sigma_2|\bar{\zeta}_1', \sigma_1) = (i/\hbar c)(\bar{\zeta}_2'', \sigma_2|\delta\int_{\sigma_1}^{\sigma_2}(d\bar{x})\bar{\mathfrak{L}}|\bar{\zeta}_1', \sigma_1), \tag{3.20}$$

where

$$\bar{\mathfrak{L}} = (R\mathfrak{L}R^{-1})^T = \mathfrak{L}^T((R\phi^\alpha R^{-1})^T, \pm\bar{\partial}_\mu(R\phi^\alpha R^{-1})^T). \tag{3.21}$$

In the last statement, the $\pm$ sign indicates the effect of the coordinate transformation (3.18) on the components of the gradient vector, while the notation $\mathfrak{L}^T(\ )$ symbolizes the reversal in the order of all factors induced by the operation of transposition. The operator R will now be chosen to produce that linear transformation of the ϕ^α,

$$R\phi^\alpha R^{-1} = R^{\alpha\beta}\phi^\beta, \tag{3.22}$$

which compensates the effect of the gradient vector transformation. Thus. we have

$$\bar{\mathfrak{L}} = (\pm)\mathfrak{L}^T(\phi^{\alpha T}, \bar{\partial}_\mu\phi^{\alpha T}), \tag{3.23}$$

where the $(\pm)$ sign here refers to the fact that the structure of the lagrange function, for half-integral spin fields, can be maintained only at the expense of a change in sign. We now see that if

$$\bar{\mathfrak{L}} = \mathfrak{L}(\phi^{\alpha T}, \bar{\partial}_\mu\phi^{\alpha T}), \tag{3.24}$$

the form of our fundamental dynamical equation will have been preserved under time reflection, since Eq. (3.20) will then differ from Eq. (2.14) only in the substitution of $\phi^{\alpha T}$ for ϕ^α as the appropriate field variable, and in the interchange of σ_1 and σ_2, which simply reflects the reversed temporal sense in which the dynamical development of the system is to be traced.

Invariance under time reflection thus requires that inverting the order of all factors in the lagrange function leave a scalar term unchanged, and reverse the sign of a pseudoscalar term. This can be satisfied, of course, by an explicit symmetrization or antisymmetrization of the various terms in $\mathfrak{L}$. When the lagrange function, thus arranged, is employed in the principle of stationary action, the variations $\delta_0\phi^\alpha$ will likewise be disposed in a symmetrical or antisymmetrical manner. We must now recall that the equations of motion (2.18), which do not depend explicitly on the nature of the field commutation properties, have been obtained by postulating the equality of terms in $\delta_0\mathfrak{L}$ that differ basically only in the location of $\delta_0\phi^\alpha$. Since such terms appear with the same sign in scalar components of $\mathfrak{L}$, and with opposite signs in pseudoscalar components, we deduce a corresponding commutativity, or anticommutativity, between $\delta_0\phi^\alpha$ and the other operators in the individual terms of $\delta_0\mathfrak{L}$.

The information concerning commutation properties that has thus been obtained is restricted to operators at common space-time points, since this is the nature of the terms in $\mathfrak{L}$. Commutation relations between field quantities located at distinct points of a space-like surface are implied by the general compatibility requirement for physical quantities attached to points with a spacelike interval. Components of integral spin fields, and bilinear combinations of the components of half-integral spin fields, are the basic physicial quantities to which this compatibility condition applies. By considering the general possibilities of coupling between the various fields, we may draw from these two expressions of relativistic invariance the consequence that the variations $\delta\phi^b(x')$, and therefore the conjugate variations $\delta\Pi^b(x')$, commute or anticommute with $\phi^a(x)$, $\Pi^a(x)$ for all x and x' on a given σ, where the relation of anticommutativity holds when both a and b refer to components of half-integral spin fields. The consistency of this statement with the general commutation relations that have already been deduced from it is easily verified. By subjecting the canonical variables in Eq. (2.81) to independent variations, we obtain

$$[\phi^a(x), \delta\phi^b(x')]_\pm = [\Pi^a(x), \delta\phi^b(x')]_\pm = 0,$$
$$[\phi^a(x), \delta\Pi^b(x')]_\pm = [\Pi^a(x), \delta\Pi^b(x')]_\pm = 0, \quad (3.25)$$

which is valid for all x, x' on σ. In addition, Eq. (2.81) properly states that all physical quantities commute at distinct points of σ.

We conclude that the connection between the spin and statistics of particles is implicit in the requirement of invariance under coordinate transformations.[10]

[10] The discussion of the spin and statistics connection by W. Pauli [Phys. Rev. 58, 716 (1940)] is somewhat more negative in character, although based on closely related physical requirements. Thus, Pauli remarks that Bose-Einstein quantization of a half-integral spin field implies an energy that possesses no lower bound, and that Fermi-Dirac quantization of an integral spin field leads to an algebraic contradiction with the commutativity of physical quantities located at points with a spacelike interval. Another postulate which has been employed, that of charge symmetry [W. Pauli and F. J. Belinfante, Physica 7, 177 (1940)], suffices to determine the nature of the commutation relations for sufficiently simple systems. As we have noticed, it is a consequence of time reflection invariance. The comments of Feynman on vacuum polarization and statistics [Phys. Rev. 76, 749 (1949)] appear to be an illustration of the charge symmetry requirement, since a contradiction is established when the charge symmetrical concept of the vacuum is applied to a Bose-Einstein spin ½ field, or to a Fermi-Dirac spin 0 field.

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Diffusion of High Energy Gamma-Rays through Matter. III. Refinement of the Solution of the Diffusion Equation*

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In Part I of the present series of papers an approximate equation was derived governing the diffusion of high energy gamma-rays through matter. In Part II an approximate solution of this diffusion equation was obtained in the energy region where the total gamma-ray cross section was substantially independent of energy. In the present paper, by consideration of the methods employed in obtaining the solution in II, a refinement of the solution is carried out which reduces the errors introduced by the approximations made both in the energy distribution and the angular distribution of the multiply-scattered gamma-rays. The solution is also modified to take into partial account the effect of small variations of the total gamma-ray cross section with energy. An upper and lower bound on the solution is obtained when the cross section is independent of energy.

I. INTRODUCTION

In the first (I) of the present series of papers,[1] an approximate equation governing the diffusion of gamma-rays through matter was derived. The gamma-ray energies for which the equation is valid extends from a few Mev up to energies (depending on the material) where the radiation of gamma-rays by the secondary electrons (photoelectrons, Compton recoils, and pairs) produced by the primary gamma-rays becomes important. In the second (II) paper of the series, the solution of the diffusion equation was considered in the energy range where the total cross section for gamma-rays was practically independent of energy. In all materials this latter energy range coincides practically with the energy range over which the diffusion equation itself is valid. However, in order to obtain a solution to the equation, even with this restriction, it was necessary to make a rather poor approximation to the Klein-Nishina formula; and this last approximation leads to rather large errors, especially for gamma-rays whose energy lies far below the energy of the incident gamma-rays. The present paper is directed towards refining the approximation somewhat, making certain corrections to the solution to improve its accuracy, and studying the magnitudes of the remaining errors. The notation used is the same as in I and II, and reference should be made to these papers for the meaning of symbols not sufficiently defined below.

II. APPROXIMATIONS TO THE KLEIN-NISHINA FORMULA FOR WHICH THE DIFFUSION EQUATION CAN BE SOLVED

The equation governing the diffusion of gamma-rays derived in I is

$$\partial f(\sigma, \xi, \eta, \zeta)/\partial\zeta + \phi_T f(\sigma, \xi, \eta, \zeta) = (1/\pi)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(\sigma'/\sigma) f(\sigma', \xi', \eta', \zeta) d\xi' d\eta', \quad (1)$$

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[1] L. L. Foldy, Phys. Rev. 81, 395 (1951), hereinafter referred to as I, and L. L. Foldy and R. K. Osborn, Phys. Rev. 81, 400 (1951), hereinafter referred to as II.