On the Scattering of a Particle by a Static Potential

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The behavior of the solutions of scattering integral equations is studied as a function of the potential strength λ. From an analysis of the second Born approximation for a Yukawa potential it seems indicated that the Born expansion for a nuclear potential has no useful domain of applicability. The convergence of the Born expansion is discussed. It is shown that the Fredholm theory of integral equations enables one to express the solutions as a quotient of infinite power series in λ which still converge when the Born expansion breaks down. Only in exceptional cases can this method be used for obtaining rapid numerical estimates.

I. INTRODUCTION

THE energy range over which experimental data on nucleon-nucleon scattering are available has in recent years been considerably extended. It seems worthwhile, therefore, to inquire how suitable the various theoretical methods for analyzing these data are, especially in the domain of higher energies. We shall here consider the scattering theory in terms of a conventional potential picture, an assumption which may well turn out to be inadequate.

The general situation in the low energy region (up to $\sim$10 Mev, say) is quite satisfactory. A phase shift analysis is not too involved owing to the fact that the problem is essentially one of S-states only. Moreover, variational methods have proved to be quite powerful for the codification of the low energy experimental material.[1]

For higher energies the variational approach has, to our knowledge, not been applied. The phase shift method still seems the most reliable one so far devised, but it is well known to become an increasingly cumbersome tool.

For rapid estimates in this region one often takes recourse to the first Born approximation. But apart from general statements that this will be more trustworthy the higher the energy, or the weaker the potential strength, little is known about the convergence of the Born expansion (for its definition see Sec. IIa). It is, of course, illuminating to have information on the higher approximations in this expansion. Few attempts in this direction are known to us. We may mention here the work of Wu[2] on the Born approximation for the scattering by a gaussian potential and that of Källén[3] who calculated in second Born approximation the S, P, D-phase shifts for a central Yukawa potential. In Sec. IIb of this paper we will present a case where the second Born approximation is actually very amenable to evaluation, *viz.*, that of the Yukawa potential. It is particularly convenient for the discussion that the results for this case are presentable in closed form. It will be shown that the Born expansion for a central Yukawa potential fitted to describe the low energy interactions is quite unreliable, even at energies of $\sim$100 Mev and more. Indeed, bearing in mind that at not very much higher energies the potential picture is bound to break down anyway, it would seem justified to state that there is no really useful domain of applicability, whatsoever, for the Born expansion for a nuclear potential.

Apart from the investigation of some of the lower order Born approximations it would seem not without interest to study the general behavior of the solutions of scattering integral equations as a function of the potential strength λ. To this subject[3a] Secs. III and IV of the present paper are devoted. In Sec. III we treat the scattering in S-states only. It is easy to find the radius λ_0 of convergence of the Born expansion for the example of the Hulthén potential. More generally, it is then shown that also for values of λ lying outside the convergence domain the solutions of the problem satisfying the required boundary conditions can be obtained by invoking convergent power series in λ. Indeed, it suffices that $\int_0^\infty r|V(r)|dr$ is finite to express these solutions (and thus also the scattering matrix) as a quotient of two such series, each of which converges for arbitrarily large λ. This method, which essentially is based on Fredholm's theory of integral equations, constitutes a departure from the usual Born expansion. But for $\lambda<\lambda_0$ this new representation can, of course, be reduced to the Born series. However, the calculation of a number of terms sufficient for a good numerical estimate is possible only in exceptional cases. The exponential potential is one such instance, and in Sec. III the rapidity of convergence and a comparison with the Born expansion is illustrated with the help of this example.

In Sec. IV the general problem is discussed where one does not confine oneself to S-states, in fact, where no use of a development in spherical harmonics is made. Here too one can show the convergence of a quotient representation of the solutions under similar conditions

[1] J. Schwinger, hectographed notes on nuclear physics, Harvard (1947); J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949); Revs. Modern Phys. **22**, 77 (1950).

[2] T. Y. Wu, Phys. Rev. **73**, 934 (1948).

[3] G. Källén, Ark. Fysik **2**, 33 (1950).

[3a] See also T. Kato, Prog. Theor. Phys. 4, 514 (1949); J. Phys. Soc. Japan 4, 334 (1949); Prog. Theor. Phys. **5**, 207 (1950). This author investigates the dependence of eigenvalues and eigenvectors on λ.

as for S-states. These requirements are actually fulfilled for the case of the Yukawa potential already mentioned. However, the convergence is too slow for the energies of interest to make rapid estimates by means of this method. It would seem that the best one can hope for is still a good variational guess (see also Sec. IV).

II. THE SCATTERING INTEGRAL EQUATION; THE BORN EXPANSION

(a) General Formalism

In this section we will first collect some preliminary formulas for the scattering of a particle in a central field of force and will then discuss the results of the first and second Born approximation for the case of the Yukawa potential. The details of this calculation will be found in Appendix I.

The Schroedinger equation for the neutron-proton system referring to the center of mass is[4]

$$[\Delta+k^2-\lambda V(r)]\psi=0, \quad (1)$$

where the customary dimensionless units have been introduced: length is measured in units r_0, the characteristic length of the potential, and

$$k=(ME_0r_0^2/2\hbar^2)^{\frac{1}{2}},$$

where E_0 is the energy of the incident particle in the laboratory system. λ is the potential strength, defined by the usual conditions of normalization of $V(r)$ at small distances. Whatever spin and exchange dependence the interaction may have is implicitly contained in λ.

We look for the solution of Eq. (1) which is regular and which at large distances behaves like a plane wave plus an outgoing spherical wave; i.e., we want to solve the integral equation

$$\psi(\mathbf{x})=\psi_0(\mathbf{x})+\lambda\int K(\mathbf{x},\mathbf{y})\psi(\mathbf{y})d\mathbf{y}, \qquad \psi_0(\mathbf{x})=\exp(i\mathbf{kx}), \quad (2)$$

$$K(\mathbf{x},\mathbf{y})=-\exp(ik|\mathbf{x}-\mathbf{y}|)V(y)/4\pi|\mathbf{x}-\mathbf{y}| = -G(\mathbf{x},\mathbf{y})V(y). \quad (3)$$

The Green's function G which we have picked guarantees that the required boundary conditions are fulfilled.

Asymptotically $\psi(\mathbf{x})$ as given by Eq. (2) is of the following general form

$$\psi(\mathbf{x})\sim\psi_0(\mathbf{x})+|\mathbf{x}|^{-1}\exp(ik|\mathbf{x}|)\cdot f(\theta), \quad (4)$$

where $f(\theta)$ is the amplitude for the scattering over an angle θ. The differential cross section is given by

$$d\sigma/d\Omega=r_0^2|f(\theta)|^2, \quad (5)$$

where, in the presence of spin-dependent forces, the appropriate averages over spin states have to be performed.

The Born expansion is defined as the representation of $\psi(\mathbf{x})$ in Eq. (2), and thus of $f(\theta)$ (or more generally of the scattering matrix), and of $d\sigma/d\Omega$ as a power series in λ. This amounts to iterating Eq. (2), which gives

$$\psi(\mathbf{x})=\psi_0(\mathbf{x})+\sum_{n=1}^{\infty}\lambda^n\int K_n(\mathbf{x},\mathbf{y})\psi_0(\mathbf{y})d\mathbf{y}, \quad (6)$$

in which

$$K_1=K,\quad K_n(\mathbf{x},\mathbf{y})=\int K(\mathbf{x},\mathbf{z})K_{n-1}(\mathbf{z},\mathbf{y})d\mathbf{z},\quad n>1. \quad (7)$$

It has to be verified, however, under what conditions this procedure is consistent, i.e., whether the necessary convergence requirements for the series occurring on the right-hand side of Eq. (7) are fulfilled. This question will be discussed in the following sections.

The expansion of $f(\theta)$ corresponding to that of ψ given by Eq. (6) is

$$f(\theta)=-\frac{1}{4\pi}\sum_{n=1}^{\infty}\lambda^n\int \exp(-i\mathbf{k}'\mathbf{x}_1)V(x_1)K_{n-1}(\mathbf{x}_1,\mathbf{x}_2)\times\exp(i\mathbf{kx}_2)d\mathbf{x}_1d\mathbf{x}_2, \qquad K_0(\mathbf{x}_1,\mathbf{x}_2)=\delta(\mathbf{x}_1-\mathbf{x}_2). \quad (8)$$

Here θ is the angle between $\mathbf{k}$ and $\mathbf{k}'$. The vector $\mathbf{k}'$ has the same direction as $\mathbf{x}$, and $|\mathbf{k}'|=|\mathbf{k}|$.

It is often convenient to express the right-hand side of Eq. (8) as an integral over momentum space. To achieve this put

$$(\mathbf{p}|V|\mathbf{q})=\int d\mathbf{x}V(x)\exp(-i(\mathbf{p}-\mathbf{q})\mathbf{x}), \quad (9)$$

$$K_n(\mathbf{p},\mathbf{q})=(1/8\pi^3)\int d\mathbf{x}d\mathbf{y}K_n(\mathbf{x},\mathbf{y})\exp(-i\mathbf{px}+i\mathbf{qy}). \quad (10)$$

Then

$$f(\theta)=-\frac{1}{4\pi}\sum_{n=1}^{\infty}\lambda^n\int d\mathbf{p}(\mathbf{k}'|V|\mathbf{p})K_{n-1}(\mathbf{p},\mathbf{k}). \quad (11)$$

We note the following useful formulas:

$$K_{n+1}(\mathbf{p},\mathbf{q})=\int K_n(\mathbf{p},\mathbf{t})K(\mathbf{t},\mathbf{q})d\mathbf{t}, \quad (12)$$

$$\int K_n(\mathbf{x},\mathbf{x})d\mathbf{x}=\int K_n(\mathbf{p},\mathbf{p})d\mathbf{p}, \quad (13)$$

$$K(\mathbf{p},\mathbf{q})=-(\mathbf{p}|V|\mathbf{q})/8\pi^3(p^2-k^2-i\epsilon),\quad \epsilon>0, \quad (14)$$

where in Eq. (14) k^2 has been given a small positive imaginary part $i\epsilon$ which serves to make integrations over $\mathbf{p}$ well defined and in accordance with the condition that our kernel corresponds to outgoing waves only. In all final results one should let ϵ tend to zero.

[4] It is, of course, irrelevant for the subsequent discussion of some general properties of the solutions of Eq. (1) that we consider here the particular physical system of two interacting nucleons.

For the purpose of the discussion given in the next section we subject Eq. (2) to an expansion in spherical harmonics. Thus, G is written as follows[5]

$$G(\mathbf{x},\mathbf{x}')=\tfrac{1}{4}\sum_{l=0}^{\infty}(l+\tfrac{1}{2})^{\frac{1}{2}}G_l(r,r')P_l(\vartheta),$$

where ϑ denotes the angle between $\mathbf{x}$ and $\mathbf{x}'$ and $r=|\mathbf{x}|$, $r'=|\mathbf{x}'|$. $P_l(\vartheta)$ are the normalized Legendre polynomials, while

$$G_l(r,r')=(rr')^{-\frac{1}{2}}J_{l+\frac{1}{2}}(kr)[(-)^lJ_{-l-\frac{1}{2}}(kr')+iJ_{l+\frac{1}{2}}(kr')],\quad r'>r. \tag{15}$$

For $r'<r$, $G_l(r,r')$ is defined by the right-hand side of Eq. (15), where now r and r' are interchanged. Put

$$\psi(\mathbf{x})=\textstyle\sum_l P_l(\theta)\varphi_l(r),\quad \exp(i\mathbf{kx})=\sum_l f_l(r)P_l(\theta).$$

Using the well-known expansion of a plane wave with respect to Legendre polynomials and applying the addition theorem for spherical harmonics, one gets

$$\varphi_l(r)=\{\pi(2l+1)/kr\}^{\frac{1}{2}}i^lJ_{l+\frac{1}{2}}(kr)-\tfrac{1}{2}\lambda\pi\int_0^\infty G_l(r,r')V(r')\varphi_l(r')r'^2dr'$$

For S-states ($l=0$) one therefore has, with $-ikr\varphi_0(r)2^{\frac{1}{2}}=\varphi(r)$:

$$\varphi(r)=e^{-ikr}-e^{ikr}+\lambda\int_0^\infty g(r,r')V(r')\varphi(r')dr', \tag{16a}$$

$$g(r,r')=-(1/2ik)\{\exp[ik(r+r')]-\exp(ik|r-r'|)\}.$$

The asymptotic form of $\varphi(r)$ is

$$\varphi(r)\sim e^{-ikr}-S_0(\lambda,k)e^{ikr}. \tag{16b}$$

$S_0(\lambda,k)$, the eigenvalue of the scattering matrix corresponding to angular momentum zero and wave number k is given by

$$S_0(\lambda,k)=1+(\lambda/k)\int_0^\infty \sin krV(r)\varphi(r)dr. \tag{16c}$$

In accordance with the definition introduced above, the Born expansion of $\varphi(r)$ and of $S_0(\lambda,k)$ is obtained by developing these quantities in a power series in λ.

(b) Example: The Yukawa Potential

Disregarding for the moment the question whether Eq. (7) has any meaning at all, we will next investigate an example in which the terms $\sim\lambda$ and λ^2 of Eq. (6) are calculated explicitly; or, more precisely, we shall compute the contributions $\sim\lambda^2$ and λ^3 to Eq. (5).

The case to be considered is that of the n-p-scattering by a spin and exchange dependent central Yukawa potential with constants fitted so as to account for the deuteron binding energy and the epithermal scattering. We are well aware of the academic character of a purely central force for the n-p system, but then our present investigation is one of method. We thus have

$$\begin{aligned}V(x)&=e^{-x}/x,\\ \lambda_{\mathrm{tr}}&=(a_{\mathrm{tr}}P_x+b_{\mathrm{tr}})B_{\mathrm{tr}},\\ \lambda_{\mathrm{s}}&=(a_{\mathrm{s}}P_x+b_{\mathrm{s}})B_{\mathrm{s}}.\end{aligned} \tag{17}$$

"tr" and "s" refer to triplet and singlet states, respectively. P_x is the space exchange operator. The constants a, b determine the exchange type of the forces and are normalized so that

$$a_{\mathrm{s}}+b_{\mathrm{s}}=1,\quad a_{\mathrm{tr}}+b_{\mathrm{tr}}=-3.$$

We will consider two cases of exchange type for which numerical results by means of phase shift analysis are available in the literature, for energies which are of interest for the present purposes. These are first the "even" theory, proposed by Serber, for which

$$a_{\mathrm{s}}=b_{\mathrm{s}}=\tfrac{1}{2};\quad a_{\mathrm{tr}}=b_{\mathrm{tr}}=-\tfrac{3}{2}\quad\text{(even theory)}; \tag{18}$$

secondly, the "symmetrical" theory, for which

$$a_{\mathrm{s}}=2,\quad b_{\mathrm{s}}=-1;\quad a_{\mathrm{tr}}=-2,\quad b_{\mathrm{tr}}=-1\quad\text{(symmetrical theory)}. \tag{19}$$

The constants B depend on what range is chosen for the Yukawa potential. We shall use the following set of values:

$$r_0=1.18\times10^{-13}\text{ cm},\quad B_{\mathrm{s}}=-1.58,\quad B_{\mathrm{tr}}=+0.76. \tag{20}$$

For the time being we shall drop the subscripts "tr" and "s" and turn to the evaluation of the differential cross section up to the order indicated above. In the result so obtained we then must perform the spin averaging. Thus we have in our approximation, using Eqs. (5), (11), and (14):

$$\frac{d\sigma}{d\Omega}=\left(\frac{r_0}{4\pi}\right)^2\left|(\mathbf{k}'|\lambda V|\mathbf{k})-\frac{1}{8\pi^3}\int d\mathbf{p}(\mathbf{k}'|\lambda V|\mathbf{p})\frac{1}{p^2-k^2-i\epsilon}(\mathbf{p}|\lambda V|\mathbf{k})\right|^2.$$

According to Eq. (17)

$$\begin{aligned}(\mathbf{k}'|V|\mathbf{k})&=4\pi/(|\mathbf{k}-\mathbf{k}'|^2+1),\\ (\mathbf{k}'|P_xV|\mathbf{k})&=4\pi/(|\mathbf{k}+\mathbf{k}'|^2+1).\end{aligned}$$

Inserting this, one readily obtains

$$\frac{d\sigma}{d\Omega}=B^2r_0^2\left|\frac{a}{2\eta(1+\xi)+1}+\frac{b}{2\eta(1-\xi)+1}-\frac{B(a^2+b^2)}{2\pi^2}\Phi(\xi)-\frac{Bab}{\pi^2}\Phi(-\xi)\right|^2,\quad \eta=k^2,\quad \xi=\cos\theta. \tag{21}$$

[5] See G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, London, 1944), second edition, p. 366, Eqs. (9) and (10).

Here

$$\Phi(\xi)=\int d\mathbf{p}\frac{1}{|\mathbf{p}-\mathbf{k}'|^2+1}\cdot\frac{1}{p^2-\eta-i\epsilon}\cdot\frac{1}{|\mathbf{p}-\mathbf{k}|^2+1}. \quad (22)$$

To be consistent we may retain only the terms $\sim B^2$ and B^3 in Eq. (21). Evidently, we then need only the real part $\mathrm{Re}\Phi(\xi)$ of $\Phi(\xi)$. Hence,

$$d\sigma/d\Omega=d\sigma_2/d\Omega+d\sigma_3/d\Omega,$$

where

$$\frac{d\sigma_2}{d\Omega}=B^2r_0^2\left[\frac{a}{2\eta(1+\xi)+1}+\frac{b}{2\eta(1-\xi)+1}\right]^2, \quad (23)$$

$$\frac{d\sigma_3}{d\Omega}=-\frac{B^3r_0^2}{\pi^2}\left[\frac{a}{2\eta(1+\xi)+1}+\frac{b}{2\eta(1-\xi)+1}\right] \times[(a^2+b^2)\mathrm{Re}\Phi(\xi)+2ab\mathrm{Re}\Phi(-\xi)]. \quad (24)$$

It will be shown in Appendix I that

$$\mathrm{Re}\Phi(\xi)=\frac{\sqrt{2}\pi^2}{\{\eta(1-\xi)(2\eta^2+4\eta+1-2\eta^2\xi)\}^{\frac{1}{2}}} \times\tan^{-1}\left\{\frac{\eta(1-\xi)}{2(2\eta^2+4\eta+1-2\eta^2\xi)}\right\}^{\frac{1}{2}}. \quad (25)$$

Thus Eqs. (23)-(25) give us the differential cross section. By carefully performing the limiting transition $r_0\rightarrow 0$ which leads to the coulomb case, one will verify that the right-hand side of Eq. (24) vanishes. This is in accordance with the well-known circumstance that the approximation $\sim\lambda^2$ happens to give the rigorous answer for the coulomb potential.

The total cross section σ is obtained by integration over ξ. Putting again $\sigma=\sigma_2+\sigma_3$, we get

$$\sigma_2=4\pi B^2r_0^2\left[\frac{a^2+b^2}{1+4\eta}+\frac{ab}{2\eta(1+2\eta)}\ln(1+4\eta)\right].$$

For the purpose of integrating over angles, the expression (25) can be considerably simplified, provided we restrict ourselves to sufficiently high energies. It is then not hard to see that the $\tan^{-1}$ can be replaced by its argument. We have verified that in this way one obtains an error which is only $\sim$2% at 90 Mev and which gets progressively smaller for still higher energies. Proceeding in this way, one obtains

$$\sigma_3=-\frac{\pi B^3r_0^2}{2\eta^3}\cdot\frac{1}{\beta^2-\gamma^2}\left[\{\beta(a+b)^3-\gamma(a-b)^3\}\ln\frac{\gamma+1}{\gamma-1} -\{\gamma(a+b)^3-\beta(a-b)^3\}\ln\frac{\beta+1}{\beta-1}\right], \quad (26a)$$

where the following abbreviations have been used:

$$\beta=(2\eta^2+4\eta+1)/2\eta^2,\quad \gamma=(2\eta+1)/2\eta. \quad (26b)$$

We now apply these results to the two cases mentioned above.

1. *Even Theory*

Using Eq. (18), approximating the expression (25) in the way mentioned above, and performing the spin averaging, one obtains

$$\frac{d\sigma_2}{d\Omega}=\frac{r_0^2}{16}(B_s^2+27B_{tr}^2)\left[\frac{1}{2\eta(1+\xi)+1}+\frac{1}{2\eta(1-\xi)+1}\right]^2,$$

$$\frac{d\sigma_3}{d\Omega}=\frac{r_0^2}{16\eta^3}(|B_s|^3+81B_{tr}^3)\cdot\frac{\beta\gamma}{(\beta^2-\xi^2)(\gamma^2-\xi^2)},$$

$$\sigma_2=\frac{\pi r_0^2}{8}(B_s^2+27B_{tr}^2)\left[\frac{4}{1+4\eta}+\frac{1}{\eta(1+2\eta)}\ln(1+4\eta)\right],$$

$$\sigma_3=\frac{\pi r_0^2(|B_s|^3+81B_{tr}^3)}{4\eta(3\eta+1)(4\eta^2+5\eta+1)}\cdot \times[(4\eta^2+5\eta+1)-2\eta(2\eta+1)\ln(2\eta+1)].$$

Using Eq. (20) we compute the various quantities for $E_0=90$ Mev, corresponding to $\eta=1.5$. This energy is chosen so as to enable a comparison with the values given by Christian and Hart,[6] which were obtained by numerical evaluation of phase shifts. According to these authors

$$\sigma=90\text{ mb},\quad R=d\sigma(180°)/d\sigma(90°)=3.25,$$

whereas

$$\sigma_2=88\text{ mb},\ R_2=d\sigma_2(180°)/d\sigma_2(90°)=20.5/3.9=5.2,$$

$$\sigma_3=52\text{ mb},\ R_3=\frac{d\sigma_2(180°)+d\sigma_3(180°)}{d\sigma_2(180°)+d\sigma_3(90°)}=\frac{20.5+8.1}{3.9+3.0}=4.1.$$

Thus, the first Born approximation seems to give a good approximation to σ; but evidently this is misleading, as follows from the comparison of R_2 and R. Going to the next approximation gives a better picture of the angular distribution, but now one is far off again with respect to the total cross section. Clearly, the Born expansion is still quite useless at this energy, and the situation does not get much better for larger E_0, as is shown by Table I. Thus, even at 270 Mev (where for that matter the potential picture is getting to be quite far fetched) σ_3 still gives a correction of more than 20 percent to σ_2. One must therefore conclude that the Born expansion is nowhere quantitatively reliable.

2. *Symmetrical Theory*

In view of the experimental evidence this seems a quite academic case. We are mentioning it only in order

[6] R. S. Christian and E. W. Hart, Phys. Rev. **77**, 441 (1949), see especially Table III. We are indebted to Dr. Christian for supplying us with further unpublished numerical results.

TABLE I. Born approximation for Yukawa potential.

E_0(Mev)	σ_2(mb)	σ_3(mb)	R_2	R_3
120	66	31	7.7	6.1
150	51	18	10.7	8.5
180	42	14	14	11.5
210	36	11	18	15.0
240	31	8	22	18.5
270	27	6	29	24.6

to have another means of comparison, *viz.*, with the results of Chew and Goldberger.[7] Also working with a Yukawa potential and with the constants (20), they found

$$\sigma = 140 \text{ mb}, \quad R = 6.81 \text{ for } E_0 = 80 \text{ Mev}.$$

Using Eq. (19) we obtain

$$\sigma_2 = 133 \text{ mb}, \quad R_2 = 56.9/4.7 = 12.1;$$
$$\sigma_3 = 46 \text{ mb}, \quad R_3 = (56.9-3.7)/(4.7+3.9) = 6.2.$$

Note that $d\sigma_3(180°)$ is negative, which is not unexpected for an interaction of which a big part is exchange force. Qualitatively the same situation obtains as for the even theory.

Rather than pursue the investigation to higher terms in the expansion—the calculation of which is very much more complicated—it would seem preferable at this stage to inquire into the general validity of power series expansions in λ for scattering problems. To this subject the next two sections are devoted.

III. λ-DEPENDENCE OF THE SOLUTIONS; *S*-STATES

In this section we will discuss the general structure of the solution $\varphi(\lambda; k, r)$ of the integral equation (16a) for *S*-states. This will lead us to conclusions about the validity of the Born expansion. For reasons of simplicity we will assume that the potential $\lambda V(r)$ contains no exchange operators unless otherwise stated. Our results, however, will be general. Further, we will assume that $\int_0^\infty r|V(r)|dr$ is finite, which condition could be replaced by more general ones.

$\varphi(\lambda; k, r)$ is a solution of the differential equation

$$\varphi'' + k^2\varphi - \lambda V\varphi = 0 \qquad (27)$$

and is completely characterized by the boundary conditions (16b), and $\varphi(\lambda; k, 0) = 0$.

Since, in the first place, we are interested in the series expansion of $\varphi(\lambda; k, r)$ and $S_0(\lambda; k)$ in powers of λ, we have to investigate the analytical character of these quantities as functions of λ. It is then convenient to build up the solution φ from such other solutions of Eq. (27) for which a power series expansion in λ is always possible. It is easy to find functions of this kind. We have, in fact, only to look for solutions of Eq. (27), which are defined by *initial values which are independent of* λ. Since we can solve Eq. (27) under the given initial conditions for any complex value of λ, it is intuitively clear that for given k and r such solutions are entire functions of λ. This is the assertion of a theorem by H. Poincaré.[8] A convenient choice (although not the only one) for such a solution is defined by

$$\lim_{r\to\infty} e^{ikr} f(\lambda; k, r) = 1; \qquad (28)$$

and $f(\lambda; k, r)$ is seen to satisfy the integral equation:

$$f(\lambda; k, r) = e^{-ikr} + (\lambda/k)\int_r^\infty \sin k(r'-r)V(r')f(\lambda; k\ r')dr'. \qquad (29)$$

The power series in λ, which one gets by successive iteration of Eq. (29), converges[9] for any λ. Of course, $f(\lambda; k, r)$ will not be zero for $r=0$. However, the solution $\varphi(\lambda; k, r)$ which does vanish at the origin, can easily be expressed in terms of f:

$$\varphi(\lambda; kr) = [1/f(\lambda; -k)][f(\lambda; -k)f(\lambda; k, r) - f(\lambda; k)f(\lambda; -k, r)], \qquad (30)$$

where

$$f(\lambda; k) \equiv f(\lambda; k, 0) \qquad (31)$$

and [see Eq. (16b)]

$$S_0(\lambda; k) = f(\lambda; k)/f(\lambda; -k). \qquad (32)$$

Thus, $\varphi(\lambda; k, r)$ and $S_0(\lambda; k)$ are meromorphic functions in λ with poles at the zero's of $f(\lambda; -k)$, which we denote by $\lambda_1, \lambda_2 \cdots \lambda_n \cdots$. If λ_1 is the zero with the smallest absolute value, then the Born expansion will certainly diverge for $|\lambda| > |\lambda_1|$ and converge for $|\lambda| < |\lambda_1|$. We will not try here to give estimates for λ_1 or $|\lambda_1|$. However, it is easy to see that: (1) for the Born expansion to converge it suffices that

$$|\lambda| \int_0^\infty r|V(r)|dr < 1.$$

This is an immediate consequence of (16a) and

$$|e^{ik(r+r')} - e^{ik|r-r'|}| \le 2|k|r'.$$

(2) For real, nonvanishing k the λ_n cannot be real, for the Wronskian of $f(\lambda; k, r)$ and $f(\lambda; -k, r)$ is independent of r, and from (28) it follows that

$$f(\lambda; k, r)[\partial f(\lambda; -k, r)/\partial r] - [\partial f(\lambda; k, r)/\partial r]f(\lambda; -k, r) = 2ik. \qquad (33)$$

[7] G. F. Chew and M. L. Goldberger, Phys. Rev. **73**, 1409 (1948). For a correction see F. Rohrlich and J. Eisenstein, Phys. Rev. **75**, 705 (1949), footnote on p. 711.

[8] H. Poincaré, Acta Math. 4, 215 (1884); see, e.g., *Enzyklopädie der Mathematischen Wissenschaft*, Vol. 2, part 2, p. 501. Compare for this and the following discussion, R. Jost, Helv. Phys. Acta **22**, 256 (1947); V. Bargmann, Phys. Rev. **75**, 301 (1949); Revs. Modern Phys. **21**, 488 (1949). A solution with this property was used by M. Born, Z. Physik **38**, 803 (1926) [p. 810] for the discussion of the one-dimensional scattering. The statement made in Born's paper, p. 816, concerning the convergence of the three-dimensional Born expansion seems to us erroneous, however, (see Sec. IV of this paper). Instead of using the solution (28), one may for the present purpose equally well use a solution $U(\lambda; k, r)$ specified by $U(\lambda; k, 0) = 0$, $U'(\lambda; k, 0) = 1$. This solution was used by C. E. Fröberg, Ark. Mat. Astron. Fysik **34 A**, No. 28 (1948). However, Eq. (4) for the determination of the phase shift is incorrect.

[9] V. Bargmann, Revs. Modern Phys. **21**, 488 (1949), Appendix.

Furthermore, $f(\lambda; -k, r) = f^*(\lambda^*; k, r)$. If, therefore, λ_n would be real, Eq. (33) would yield a contradiction for $r=0$, since the left-hand side would vanish.

The following example of a $V(r)$, for which the λ_n are simple functions of n and k, may serve to illustrate the general situation:[10]

$$V(r) = e^{-r}/(1-e^{-r}). \tag{34}$$

Here

$$f(\lambda; k, r) = e^{-ikr}F(i[k+(k^2+\lambda^2)^{\frac{1}{2}}], i[k-(k^2+\lambda^2)^{\frac{1}{2}}]; 1+2ik; e^{-r}), \tag{35}$$

where $F(\alpha, \beta; \gamma; z)$ is the hypergeometric function, and

$$f(\lambda; k) = \prod_{n=1}^{\infty}\left(1+\frac{\lambda}{n(n+2ik)}\right). \tag{36}$$

The zero's λ_n of $f(\lambda; -k)$ lie at the points

$$\lambda_n = -n^2+2ikn, \tag{37}$$

and the Born expansion converges for

$$|\lambda| < (1+4k^2)^{\frac{1}{2}}. \tag{38}$$

It is desirable to relate the form (30) for the solution $\varphi(\lambda; k, r)$ directly to the original integral equation (16a). This is indeed necessary for the extension of the present discussion to the more general Eq. (2). The way to do this is the application of Fredholm's[11] theory to the integral equation (16a).

Putting

$$g(r, r')V(r') = K(r, r'), \tag{39}$$

we get

$$\varphi(\lambda; k, r) = (e^{-ikr}-e^{ikr}) + [\lambda/\Delta(\lambda; k)]\int_0^\infty \Delta(\lambda; k; r, r')(e^{-ikr'}-e^{+ikr'})dr', \tag{40}$$

with

$$\Delta(\lambda; k) = 1+\sum_{n=1}^{\infty}\frac{(-\lambda)^n}{n!}\int_0^\infty dr_1\cdots\int_0^\infty dr_n \begin{vmatrix} K(r_1, r_1) & K(r_1, r_2) & \cdots & K(r_1, r_n) \\ \vdots & & & \\ K(r_n, r_1) & K(r_n, r_2) & \cdots & K(r_n, r_n) \end{vmatrix} \tag{41}$$

and

$$\Delta(\lambda; k; r, r') = K(r, r')+\sum_{n=1}^{\infty}\frac{(-\lambda)^n}{n!}\int_0^\infty dr_1\cdots\int_0^\infty dr_n \begin{vmatrix} K(r, r') & K(r, r_1) & \cdots & K(r, r_n) \\ K(r_1, r') & K(r_1, r_1) & \cdots & K(r_1, r_n) \\ \vdots & & & \\ K(r_n, r') & K(r_n, r_1) & \cdots & K(r_n, r_n) \end{vmatrix}. \tag{42}$$

The series implied in Eq. (40) converge absolutely for any λ. Again $\varphi(\lambda; k, r)$ appears as the quotient of two entire functions, and we will prove in Appendix II that Eqs. (40) and (30) are actually identical in the sense that

$$\Delta(\lambda; k) \equiv f(\lambda; -k). \tag{43}$$

The zero's of $\Delta(\lambda; k)$, i.e., λ_n, now appear as eigenvalues of the kernel $K(r, r')$; i.e., for $\lambda = \lambda_n$ there exists a solution of Eq. (27) for which

$$\psi(\lambda_n; k, 0) = 0 \quad\text{and}\quad \psi(\lambda_n; k, r) \sim e^{ikr}. \tag{44}$$

The fact that the imaginary part of λ_n does not vanish means that a stationary solution of the Schroedinger equation with the property (44) exists only if the potential $\lambda_n V(r)$ acts as a source (see Sec. IV).

We conclude this section with an example which shows how rapid a convergence one can expect for $f(\lambda, k)$, or rather the phase shift η, as a power series in λ for nuclear potentials. We take

$$V = e^{-r}.$$

Then

$$f(\lambda; k, r) = \exp\{-ir\ln(-\lambda)\}\Gamma(2ik+1) \times J_{2ik}[2(-\lambda)^{\frac{1}{2}}\exp(-r/2)],$$

$$f(\lambda; k) = \sum_0^\infty \lambda^n \alpha_n(k),$$

$$\alpha_0 = 1,$$

$$\alpha_n(k) = [n!(2ik+1)(2ik+2)\cdots(2ik+n)]^{-1}.$$

We further use $r_0 = 0.75\times10^{-13}$ cm, $\lambda = 2.13$. For these values Eq. (27) can be considered as the radial equation for ^3S-states, with the binding energy of the deuteron appropriately fitted. We have calculated η for $E_0 = 40$ Mev ($k = 0.514$). The results are collected in Table II. The first column indicates the highest power of λ which has been included. η_F is the phase shift computed by the Fredholm method from $f(\lambda; k)$. For comparison we also give η_B, the phase shift calculated

[10] L. Hulthén, Ark. Mat. Astron. Fysik **28A**, No. 5 (1942); **29B**, No. 1 (1942).

[11] See, e. g., A. B. Whittaker and C. D. Watson, *Modern Analysis* (Cambridge University Press, London, 1940), fourth edition, Chap. XI.

TABLE II. Phase shifts for exponential potential $E_0=40$ Mev.

Approx.	η_F	η_B
1	1.602	2.613
2	1.156	0.392
3	1.259	3.157
4	1.258	0.559

in the corresponding four Born approximations. The results are self-explanatory.

IV. THE λ-DEPENDENCE OF THE SOLUTIONS; GENERAL CASE

We now go back to the discussion of Eqs. (2) and (3). Following the Fredholm procedure, already outlined in the previous section, we will show that for all finite λ the solution $\psi(\lambda;\mathbf{k},\mathbf{x})$ can again be represented as the quotient of two convergent power series in λ.

However, we cannot without further ado apply the general formulas of the Fredholm theory [see Eqs. (40)–(42)]. The reason for this is that $K(\mathbf{x},\mathbf{x})$ becomes infinite, owing to the singularity of the Green's function, so that the Fredholm determinants do not exist. It is, however, easy to circumvent this obstacle. Obviously, the formulas of Fredholm still give solutions (proper convergence assumed), if we replace all those elements along the diagonal of the determinants which are of the form $K(\mathbf{x},\mathbf{x})$ by zero.[12] We will see in Appendix III that the following conditions, imposed[13] on $V(|\mathbf{x}|)$, are sufficient to insure the convergence of this procedure:

$$|V(|\mathbf{x}|)|\leq|\mathbf{x}|^{-1}F(|\mathbf{x}|), \tag{45}$$

where the function $F(r)$ satisfies the inequalities

$$\int F(r)dr\leq M, \tag{46}$$

$$F(r)\leq M'r^{-1}. \tag{47}$$

Equation (47) is a condition of smoothness on the potential prohibiting, e.g., the occurrence of high "peaks." (Since the conditions (46) and (47) look very reasonable from a physical standpoint, we have not tried to replace them by weaker ones.)

Thus,

$$\psi(\lambda;\mathbf{k},\mathbf{x})=\exp(i\mathbf{kx})+\lambda\int[\Delta(\lambda;\mathbf{x},\mathbf{x}')/\Delta(\lambda)]\exp(i\mathbf{kx}')d\mathbf{x}', \tag{48}$$

$$\Delta(\lambda;\mathbf{x},\mathbf{x}')=K(\mathbf{x},\mathbf{x}')+\sum_{n=1}^{\infty}\frac{(-\lambda)^n}{n!}\int d\mathbf{x}_1\cdots\int d\mathbf{x}_n\begin{vmatrix}K(\mathbf{x},\mathbf{x}') & K(\mathbf{x},\mathbf{x}_1) & K(\mathbf{x},\mathbf{x}_2) & \cdots & K(\mathbf{x},\mathbf{x}_n)\\ K(\mathbf{x}_1,\mathbf{x}') & 0 & K(\mathbf{x}_1,\mathbf{x}_2) & \cdots & K(\mathbf{x}_1,\mathbf{x}_n)\\ K(\mathbf{x}_2,\mathbf{x}') & K(\mathbf{x}_2,\mathbf{x}_1) & 0 & \cdots & K(\mathbf{x}_2,\mathbf{x}_n)\\ \vdots & & & & \\ K(\mathbf{x}_n,\mathbf{x}') & K(\mathbf{x}_n,\mathbf{x}_1) & K(\mathbf{x}_n,\mathbf{x}_2) & \cdots & 0\end{vmatrix}, \tag{49}$$

$$\Delta(\lambda)=1+\sum_{n=1}^{\infty}\frac{(-\lambda)^n}{n!}\int d\mathbf{x}_1\cdots\int d\mathbf{x}_n\begin{vmatrix}0 & K(\mathbf{x}_1,\mathbf{x}_2) & \cdots & K(\mathbf{x}_1,\mathbf{x}_n)\\ K(\mathbf{x}_2,\mathbf{x}_1) & 0 & \cdots & K(\mathbf{x}_2,\mathbf{x}_n)\\ & & & \\ K(\mathbf{x}_n,\mathbf{x}_1) & K(\mathbf{x}_n,\mathbf{x}_2) & \cdots & 0\end{vmatrix}. \tag{50}$$

So we conclude that $\psi(\lambda;\mathbf{k},\mathbf{x})$ is again a meromorphic function of λ with poles at the zeros λ_n of $\Delta(\lambda)$, and the same is true for the function $f(\theta)$ defined in Eq. (4). The zero λ_1 with the smallest absolute value gives the radius of convergence for the Born expansion (6). Since it is easy to prove[14] that the Born expansion converges for $\lambda M<1$, it follows that $|\lambda_1|\geqq M^{-1}$, for any value of k.

The λ_n are eigenvalues of the kernel $K(\mathbf{x},\mathbf{y})$, and this means that the homogeneous equation

$$\psi_n(\mathbf{k},\mathbf{x})=\lambda_n\int K(\mathbf{x},\mathbf{x}')\psi_n(\mathbf{k},\mathbf{x}')d\mathbf{x}' \tag{51}$$

has at least one nontrivial solution with the asymptotic behavior

$$\psi_n(\mathbf{k},\mathbf{x})\sim|\mathbf{x}|^{-1}f_n(\theta)\exp(ik|\mathbf{x}|). \tag{52}$$

From (51) and (52) we can conclude that the imaginary part of λ_n for $\mathbf{k}\neq0$ cannot vanish. In fact:

$$i\frac{\partial}{\partial x_\nu}\left(\psi_n^*\frac{\partial\psi_n}{\partial x_\nu}-\psi_n\frac{\partial\psi_n^*}{\partial x_\nu}\right)=i(\lambda_n-\lambda_n^*)\psi^*V\psi \tag{53}$$

integrating (53) over a large sphere and using (52) yields[15]

$$k\cdot\int|f(\theta)|^2d\Omega=-i(\lambda_n^*-\lambda_n)\int\psi_n^*V\psi_n dx, \tag{54}$$

[12] This device is due to D. Hilbert, *Grundzüge einer allgemeinen Theorie der Integralgleichungen* (Leipzig, 1912), p. 30. H. Poincaré, *Oeuvres* (Paris, 1934), Vol. III, p. 560; Acta Math. 33, 57 (1901). It consists in multiplying formally numerator and denominator of the resolvent by $\exp[\lambda\int K(\mathbf{x},\mathbf{x})d\mathbf{x}]$.

[13] The generalization to noncentral potentials presents no difficulties.

[14] This follows from

$$\left|\int K_{n+1}(\mathbf{x},\mathbf{y})\exp(i\mathbf{ky})dy\right|\leq M\left|\int K_n(\mathbf{x},\mathbf{y})\exp(i\mathbf{ky})dy\right|.$$

[15] For positive imaginary k the eigenvalues of $K(\mathbf{x},\mathbf{y})$ are real and give the potential strengths necessary for a bound state with an energy given by k.

which proves the point. Equation (53) shows that $i(\lambda_n-\lambda_n^*)\psi^*V\psi$ acts as a source distribution allowing a stationary outgoing current distribution asymptotically described by Eq. (52).

Breaking off the summation in Eqs. (49) and (50) at a certain $n=n_0$, we are allowed to rearrange the terms and thus see that the solution $\psi(\lambda;\mathbf{k},\mathbf{x})$ can always be represented to an arbitrary high accuracy by a polynomial in the iterated kernels K_n of the form

$$\psi_{n_0}(\lambda;\mathbf{k},\mathbf{x})=\exp(i\mathbf{kx})+\sum_{n=1}^{n_0}a_n\int K_n(\mathbf{x},\mathbf{x}')\exp(i\mathbf{kx}')d\mathbf{x}', \quad (55)$$

with the a_n composed of the integrals $\int K_m(\mathbf{x},\mathbf{x})d\mathbf{x}$. In the same sense, $f(\theta)$ can be approached by a linear combination of Born approximations $f_n^{(B)}(\theta)$:

$$f_{n0}(\theta)=\frac{1}{4\pi}\sum_1^{n_0}a_nf_n^{(B)}(\theta), \quad (56)$$

[compare Eq. (8)] in which, according to Eqs. (48)–(50), the a_n are rational functions of λ, but, of course, not simply λ^n.

From the foregoing considerations it follows that the right-hand side of Eq. (55) with unknown parameters a_n is in principle well suited as Ansatz for a variational procedure. But this clearly does not save labor as compared with the evaluation of an adequate number of terms in the Fredholm expansion itself. However, it certainly is unwarranted to employ a partial Born expansion, i.e., Eq. (55) with $a_n=\lambda^n$, for such purposes whenever the potential strength gets too large.

Despite the formal appeal of the Fredholm procedure, it is not suited for numerical work, because of its slow convergence. We will briefly exemplify this by considering again the Yukawa potential and going up to the second power in numerator and denominator of Eq. (48). Up to this approximation the numerator is identical in the second Born approximation for $f(\theta)$. In the foregoing, we have tacitly assumed that λV did not involve exchange operators, but the extension to their being present is straightforward.

Using Eqs. (17) and (13) we find in our approximation for the denominator $\Delta(\lambda)$ in Eq. (48) (for both singlet and triplet states)

$$\Delta(\lambda)=1-\tfrac{1}{2}\lambda^2[(a^2+b^2)D_1+2abD_2], \quad (57)$$

$$D_1=(1/4\pi^4)\int d\mathbf{p}d\mathbf{t}[(p^2-\eta-i\epsilon)\times(t^2-\eta-i\epsilon)(|\mathbf{p}-\mathbf{t}|^2+1)^2]^{-1}, \quad (58)$$

$$D_2=(1/4\pi^4)\int d\mathbf{p}d\mathbf{t}[(p^2-\eta-i\epsilon)(t^2-\eta-i\epsilon)\times(|\mathbf{p}-\mathbf{t}|^2+1)(|\mathbf{p}+\mathbf{t}|^2+1)]^{-1}. \quad (59)$$

D_1 and D_2 are evaluated in Appendix III. Their respective real parts are

$$\mathrm{Re}D_1=[2(4\eta+1)]^{-1} \quad (60)$$

$$\mathrm{Re}D_2=-(1/2\eta+1)\cdot\ln[2(\eta+1)/(4\eta+1)]^{\frac{1}{2}}. \quad (61)$$

The right-hand side of both Eq. (60) and Eq. (61) are positive for $\eta=1.5$ corresponding to the energy $E_0=90$ Mev used in Sec. II. Therefore, the cross section increases over that obtained in the second Born approximation, which itself was already too large.

In conclusion, we remark that the analysis of the legitimacy of power series expansions put forward in this paper is to a large extent specific for the simple problem of scattering by a static potential. The situation is much more intricate for the Born expansions of relativistic field theories. In fact, the kernels encountered there are of a much more singular nature, so that one certainly cannot hope to apply the simple Fredholm theory to this case. This is, of course, merely a reflection on the deep-lying differences in mathematical structure between the fairly simple differential equations of the nonrelativistic Schroedinger theory and the basic field-theoretical equations. In fact, in those cases where the development of the scattering matrix in powers of the coupling parameter fails, we are confronted with a situation where one does not even know what to mean by a solution. It seems to us that a clarification of this point is needed before one can attempt actual evaluations in problems where the coupling constants involved are large.

One of us (R.J.) would like to express his gratitude to the Institute for Advanced Study for a grant-in-aid.

APPENDIX

(I) Born Approximation for the Yukawa Potential

We shall here compute $\Phi(\xi)$ defined by Eq. (22). Introducing auxiliary variables x_1, x_2, x_3 whose ranges are confined to $x_i\geq 0$, we may write[16]

$$\Phi(\xi)=2\int d\mathbf{p}\int dx_1dx_2dx_3\frac{\delta(x_1+x_2+x_3-1)}{[\{(\mathbf{p}-\mathbf{k}')^2+1\}x_1+(p^2-\eta-i\epsilon)x_2+\{(\mathbf{p}-\mathbf{k})^2+1\}x_3]^3}.$$

Transforming from $\mathbf{p}$ to the new variable

$$\mathbf{P}=\mathbf{p}-\mathbf{k}'x_1-\mathbf{k}x_3,$$

[16] See R. P. Feynman, Phys. Rev. **76**, 769 (1949).

one gets upon performing the **P**-integration

$$\Phi(\xi)=\tfrac{1}{2}\pi^2\int\frac{\delta(x_1+x_4+x_3-1)dx_1dx_2dx_3}{[\eta\{x_1(1-x_1)+x_3(1-x_3)-x_2-2\xi x_1x_3\}+x_1+x_3-i\epsilon x_2]^{5/2}}.$$

Integrate over x_2 and put $x_1+x_3=u$, $x_1-x_3=v$:

$$\Phi(\xi)=\tfrac{1}{4}\pi^2\int_0^1 du\int_{-u}^{u}dv[u-\eta\{1-2u+\tfrac{1}{2}(u^2+v^2)+\tfrac{1}{2}(u^2-v^2)\xi\}-i\epsilon(1-u)]^{-3/2}.$$

Owing to the imaginary term in the square bracket, this expression is never zero for $-u\leq v\leq u$. Hence, the v-integration is well-defined and yields

$$\Phi(\xi)=\tfrac{1}{2}\pi^2\int_0^1 udub^{-1}(u)\{b(u)-au^2\}^{-1/2},\quad a=\tfrac{1}{2}\eta(1-\xi),\quad b=u-\eta\{1-2u+\tfrac{1}{2}u^2(1+\xi)\}-i\epsilon(1-u).\tag{I.1}$$

The integrand has branch points at u_+^0 and u_-^0, where

$$u_\pm^0=(1/2\eta)[2\eta+1\pm(4\eta+1)^{1/2}+i\epsilon\{1\pm(4\eta+1)^{1/2}\}],$$

and poles at $u_\pm$:

$$u_\pm=[1/\eta(1+\xi)][2\eta+1\pm\chi(\eta)+i\epsilon\{1\pm(\eta(1-\xi)+1)\chi^{-1}(\eta)\}],\quad \chi(\eta)=\{(2\eta+1)^2-2\eta^2(1+\xi)\}^{1/2}.$$

The position of these points in the complex u plane is sketched in Fig. 1.

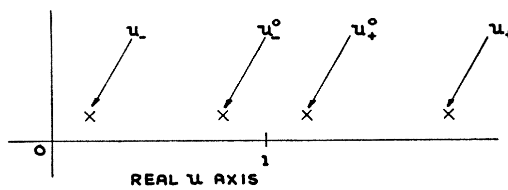

FIG. 1. Position of poles and branch points of the integrand in Eq. (I.1).

It is easy to find the indefinite integral over u. In substituting the limits one has to take care in the choice of branch. At $u=1$ the limit value is real. Since

$$\mathrm{Im}\{b(u)-au^2\}\leq 0,\quad 0\leq u\leq 1,$$

it follows from the position of u_-^0 that

$$\{b(u)-au^2\}^{1/2}_{u=0}=\{-\eta(1+i\epsilon)\}^{1/2}$$
$$=(-\eta)^{1/2}(1+\tfrac{1}{2}i\epsilon\eta^{-1})=-i\eta^{1/2}(1+\tfrac{1}{2}i\epsilon\eta^{-1}).$$

One may now let $\epsilon\rightarrow 0$, and one finds after some calculation

$$\Phi(\xi)=\frac{i\pi^2}{\eta(1+\xi)(u_+-u_-)}\ln\frac{u_+(1-u_-)I_-N_+}{u_-(1-u_+)I_+N_-},$$
$$I_\pm=(2\eta+1)u_\pm-2\eta-u_\pm\eta\{2(1-\xi)\}^{1/2},$$
$$N_\pm=1+u_\pm-iu_\pm\{2(1-\xi)\}^{1/2}.$$

The real and imaginary part of Φ are now readily found. We will only consider $\mathrm{Re}\Phi(\xi)$. As (see Fig. 1)

$$u_+(1-u_-)I_-/u_-(1-u_+)I_+>0,$$

we can for this purpose replace the ln in Φ by $\ln N_+/N_-$.

Upon taking the imaginary part of the latter quantity we get

$$\mathrm{Re}\Phi(\xi)=\frac{\pi^2}{\{2\eta(1-\xi)(2\eta^2+4\eta+1-2\eta^2\xi)\}^{1/2}}\cdot\tan^{-1}\left[\frac{2\{2\eta(1-\xi)(2\eta^2+4\eta+1-2\eta^2\xi)\}^{1/2}}{4\eta^2(1-\xi)+\eta(7+\xi)+2}\right].$$

(II) Proof that $\Delta(\lambda;\,k)=f(\lambda;\,-k)$

Using $2\tan^{-1}a=\tan^{-1}[2a/(1-a^2)]$ we are led to the expression quoted in Eq. (25). It is convenient to discuss for this proof the function $g(\lambda;k,r)=\exp(ikr)f(\lambda;k,r)$ rather than $f(\lambda;k,r)$ of Eq. (28). One verifies that g satisfies

$$g(\lambda;-k,r)=1+(\lambda/2ik)\int_r^\infty[\exp(2ik\{r'-r\})-1]V(r')g(\lambda;-k,r')dr'.$$

Now we get for $f(\lambda;-k)$ the expansion:

$$f(\lambda;-k)=g(\lambda;-k,0)=1+\sum_{n=1}^\infty\left(\frac{\lambda}{2ik}\right)^n\int_0^\infty dr_1\int_{r_1}^\infty dr_2\cdots\int_{r_{n-1}}^\infty dr_n(\exp 2ikr_1-1)$$
$$\times(\exp 2ik\{r_2-r_1\}-1)\cdots(\exp 2ik\{r_n-r_{n-1}\}-1)V(r_1)V(r_2)\cdots V(r_n).$$

On the other hand,

$$\Delta(\lambda;k)=1+\sum_{n=1}^\infty(n!)^{-1}\left(\frac{\lambda}{2ik}\right)^n\int_0^\infty dr_1\cdots\int_0^\infty dr_n\,\mathrm{Det}_{\mu\nu}\|\exp(ik\{r_\mu+r_\nu\})-\exp(ik|r_\mu-r_\nu|)\|V(r_1)V(r_2)\cdots V(r_n).$$

The integrand of the last integral is symmetric in all the variables; therefore,

$$\Delta(\lambda;k)=1+\sum_{n=1}^\infty\left(\frac{\lambda}{2ik}\right)^n\int_0^\infty dr_1\int_{r_1}^\infty dr_2\cdots\int_{r_{n-1}}^\infty dr_n\,\mathrm{Det}_{\mu\nu}\|\exp(ik\{r_\mu+r_\nu\})-\exp(ik|r_\mu-r_\nu|)\|V(r_1)V(r_2)\cdots V(r_n).$$

Writing $\xi_\mu=\exp(ikr_\mu)$, the determinant reads (remembering $r_1\leq r_2\leq r_3\cdots\leq r_n$)

$$D_n=\begin{vmatrix}(\xi_1^2-1) & \xi_2(\xi_1-1/\xi_1) & \xi_3(\xi_1-1/\xi_1) & \cdots & \xi_{n-1}(\xi_1-1/\xi_1) & \xi_n(\xi_1-1/\xi_1)\\ \xi_2(\xi_1-1/\xi_1) & (\xi_2^2-1) & \xi_3(\xi_2-1/\xi_2) & \cdots & \xi_{n-1}(\xi_2-1/\xi_2) & \xi_n(\xi_2-1/\xi_2)\\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \xi_{n-1}(\xi_1-1/\xi_1) & \xi_{n-1}(\xi_2-1/\xi_2) & \xi_{n-1}(\xi_3-1/\xi_3) & \cdots & (\xi_{n-1}^2-1) & \xi_n(\xi_{n-1}-1/\xi_{n-1})\\ \xi_n(\xi_1-1/\xi_1) & \xi_n(\xi_2-1/\xi_2) & \xi_n(\xi_3-1/\xi_3) & \cdots & \xi_n(\xi_{n-1}-1/\xi_{n-1}) & (\xi_n^2-1)\end{vmatrix}$$

This determinant is obviously a linear function of ξ_n^2 and it vanishes for $\xi_n=\xi_{n-1}$, so

$$D_n=[(\xi_n^2/\xi_{n-1}^2)-1]F(\xi_1\cdots\xi_{n-1}).$$

By putting $\xi_n=0$, one sees that $F(\xi_1\cdots\xi_{n-1})=D_{n-1}$. Therefore,

$$D_n=[(\xi_n^2/\xi_{n-1}^2)-1]D_{n-1}, \quad n=2, 3\cdots;$$

but $D_1=\xi_1^2-1$, and thus, inserting the definition of ξ_μ, one gets

$$D_n=(\exp 2ikr_1-1)(\exp 2ik\{r_2-r_1\}-1)\cdots(\exp 2ik\{r_n-r_{n-1}\}-1)$$

which completes the proof.

(III) The Convergence Proof for the Expansions (48)

As a preliminary step we will prove that the Fredholm formulas apply for the kernel of the iterated integral equation (2):

$$\psi(\lambda;\mathbf{k},\mathbf{x})=\exp(i\mathbf{kx})+\lambda\int K(\mathbf{x},\mathbf{x}')\exp(i\mathbf{kx}')d\mathbf{x}'+\lambda^2\int K_2(\mathbf{x},\mathbf{x}')\psi(\lambda;\mathbf{k},\mathbf{x}')d\mathbf{x}', \tag{III.1}$$

i.e., that

$$\psi(\lambda;\mathbf{k},\mathbf{x})=F_0(\mathbf{k},\mathbf{x})+\lambda^2\int H_2(\lambda^2;\mathbf{x},\mathbf{x}')F_0(\mathbf{k},\mathbf{x}')d\mathbf{x}', \tag{III.2}$$

where

$$F_0(\mathbf{k},\mathbf{x})=\exp(i\mathbf{kx})+\lambda\int K(\mathbf{x},\mathbf{x}')\exp(i\mathbf{kx}')d\mathbf{x}' \tag{III.3}$$

and

$$H_2(\lambda^2;\mathbf{x},\mathbf{x}')=\Delta_2(\lambda^2;\mathbf{x},\mathbf{x}')/\Delta_2(\lambda^2), \tag{III.4}$$

with

$$\Delta_2(\lambda^2;\mathbf{x},\mathbf{x}')=K_2(\mathbf{x},\mathbf{x}')+\sum_1^\infty\frac{(-\lambda^2)^n}{n!}\int d\mathbf{x}_1\cdots\int d\mathbf{x}_n\begin{vmatrix}K_2(\mathbf{x},\mathbf{x}') & K_2(\mathbf{x},\mathbf{x}_1) & \cdots & K_2(\mathbf{x},\mathbf{x}_n)\\ K_2(\mathbf{x}_1,\mathbf{x}') & K_2(\mathbf{x}_1,\mathbf{x}_1) & \cdots & K_2(\mathbf{x}_1,\mathbf{x}_n)\\ \cdot & \cdot & & \cdot\\ \cdot & \cdot & & \cdot\\ \cdot & \cdot & & \cdot\\ K_2(\mathbf{x}_n,\mathbf{x}') & K_2(\mathbf{x}_n,\mathbf{x}_1) & \cdots & K_2(\mathbf{x}_n,\mathbf{x}_n)\end{vmatrix} \tag{III.5}$$

and

$$\Delta_2(\lambda^2)=1+\sum_1^\infty\frac{(-\lambda^2)^n}{n!}\int d\mathbf{x}_1\cdots\int d\mathbf{x}_n\begin{vmatrix}K_2(\mathbf{x}_1,\mathbf{x}_1) & \cdots & K_2(\mathbf{x}_1,\mathbf{x}_n)\\ K_2(\mathbf{x}_n,\mathbf{x}_1) & \cdots & K_2(\mathbf{x}_n,\mathbf{x}_n)\end{vmatrix}. \tag{III.6}$$

As assumptions for the potential $V(x)$, we need Eqs. (46), (47). We first prove the convergence of the series for $\Delta(\lambda^2)$. For this we note

$$\begin{aligned}K_2(\mathbf{x},\mathbf{x}')&=\frac{1}{16\pi^2}\int dx_1\frac{\exp(ik|\mathbf{x}-\mathbf{x}_1|)}{|\mathbf{x}-\mathbf{x}_1|}V(|\mathbf{x}_1|)\frac{\exp(ik|\mathbf{x}_1-\mathbf{x}'|)}{|\mathbf{x}_1-\mathbf{x}'|}V(|\mathbf{x}'|)\\&=A(\mathbf{x},\mathbf{x}')(4\pi|\mathbf{x}'|)^{-1}V(|\mathbf{x}'|),\end{aligned} \tag{III.7}$$

and we are going to show that, using (46) and (47),

$$|A(\mathbf{x},\mathbf{x}')|\leq N<\infty. \tag{III.8}$$

Introducing the definition of $F(|\mathbf{x}|)$ from Eq. (45), we have

$$|A(\mathbf{x},\mathbf{x}')|\leq(1/4\pi)|\mathbf{x}'|\int d\mathbf{x}_1F(|\mathbf{x}_1|)/(|\mathbf{x}_1|\cdot|\mathbf{x}-\mathbf{x}_1|\cdot|\mathbf{x}_1-\mathbf{x}'|)=B(\mathbf{x},\mathbf{x}'). \tag{III.9}$$

The integral on the right-hand side is obviously only a function of $|\mathbf{x}|=R$, $|\mathbf{x}'|=Q$, and $\measuredangle(\mathbf{x},\mathbf{x}')=\Phi$, and we claim that for given R and Q it takes its maximum value for $\Phi=0$. This is intuitively clear from symmetry reasons, and it can be proved as follows:

$$B(x,x')=\frac{1}{4\pi}Q\int_0^\infty rdrF(r)\int_0^\pi\sin\vartheta d\vartheta\int_0^{2\pi}d\varphi\frac{1}{[R^2+r^2-2R\sin\vartheta\cos(\varphi-\frac{1}{2}\Phi)]^{\frac{1}{2}}[Q^2+r^2-2Qr\sin\vartheta\cos(\varphi+\frac{1}{2}\Phi)]^{\frac{1}{2}}}. \tag{III.10}$$

By use of the standard methods for evaluating elliptic integrals[17] one gets for the integral over φ:

$$[2/(a-b\cos\Phi)]K\{[(a'-b\cos\Phi)/(a-b\cos\Phi)]^{\frac{1}{2}}\}, \tag{III.11}$$

where $a\geq a'\geq b\geq 0$ are still functions of R, Q, r, ϑ.

Differentiating Eq. (III.11) with respect to $\cos\Phi$ yields[18]

$$[b/2^{\frac{1}{2}}(a-b\cos\Phi)^{\frac{1}{2}}]^{\frac{1}{2}}D\{[(a'-b\cos\Phi)/(a-b\cos\Phi)]^{\frac{1}{2}}\}\geq 0.$$

Therefore, we can restrict ourselves to the case $\Phi=0$, for which the integral $B(\mathbf{x},\mathbf{x}')$ can be evaluated simply by using polar coordinates with a polar axis along the common direction of the vectors $\mathbf{x}$ and $\mathbf{x}'$. The result is

$$B(\mathbf{x},\mathbf{x}')\leq\frac{1}{2}\left(\frac{Q}{R}\right)^{\frac{1}{2}}\left[\int_0^R drF(r)\log\left|\frac{r+(QR)^{\frac{1}{2}}}{r-(QR)^{\frac{1}{2}}}\right|+\int_Q^\infty drF(r)\log\left|\frac{r+(QR)^{\frac{1}{2}}}{r-(QR)^{\frac{1}{2}}}\right|+\log\frac{Q^{\frac{1}{2}}+R^{\frac{1}{2}}}{Q^{\frac{1}{2}}-R^{\frac{1}{2}}}\cdot\int_R^Q drF(r)\right], \tag{III.12}$$

where we assumed $Q\geq R$ which, for the estimate of B, is the less favorable case. Noting that the three logarithms are equal for $r=Q$, or $r=R$, we get the first estimate:

$$B(\mathbf{x},\mathbf{x}')\leq\tfrac{1}{2}(Q/R)^{\frac{1}{2}}\cdot M\cdot\log\{[(Q/R)^{\frac{1}{2}}+1]/[(Q/R)^{\frac{1}{2}}-1]\} \tag{III.13}$$

(where $M=\int_0^\infty drF(r)$) and, remembering the assumption $F(r)\leq M'/r$, a second estimate

$$B(\mathbf{x},\mathbf{x}')\leq\frac{1}{2}\left(\frac{Q}{R}\right)^{\frac{1}{2}}M'\int_0^\infty\frac{dr}{r}\log\left|\frac{r+(QR)^{\frac{1}{2}}}{r-(QR)^{\frac{1}{2}}}\right|=\frac{1}{2}\left(\frac{Q}{R}\right)^{\frac{1}{2}}M'\int_0^\infty\frac{dr}{r}\log\left|\frac{r+1}{r-1}\right|=\frac{\pi^2}{4}\left(\frac{Q}{R}\right)^{\frac{1}{2}}M'. \tag{III.14}$$

[17] A. B. Frank and C. D. Mises, *Differential und Integralgleichungender Physik* (1930), second edition, Vol. 1, p. 172. For the definitions of the complete elliptic integrals $K(k)$ and $D(k)$ see Jahnke-Emde, *Tables of Functions* (Dover Publications, New York, 1943), p. 73.

[18] Jahnke-Emde, reference 17, p. 76 and p. 73.

Therefore,

$$B(\mathbf{x},\mathbf{x}')\leq\frac{1}{2}\left(\frac{Q}{R}\right)^{\frac{1}{2}}\operatorname{Min}\left(M\log\frac{(Q/R)^{\frac{1}{2}}+1}{(Q/R)^{\frac{1}{2}}-1};\frac{\pi^2}{2}M'\right)\leq\frac{\pi^2}{4}M'\coth\frac{\pi^2}{4}\frac{M'}{M}=N. \qquad \text{(III.15)}$$

and *a fortiori* Eq. (III.8).

The completion of the convergence proof now goes along standard lines. First we estimate the determinant of Eq. (III.6):

$$\left|\operatorname*{Det}_{1\leq i,k\leq n}\|K_2(\mathbf{x}_i,\mathbf{x}_k)\|\right|=\left|\operatorname{Det}\|A(\mathbf{x}_i,\mathbf{x}_k)\|\cdot\frac{V(\mathbf{x}_1)}{4\pi|\mathbf{x}_1|}\frac{V(\mathbf{x}_2)}{4\pi|\mathbf{x}_2|}\cdots\frac{V(\mathbf{x}_n)}{4\pi|\mathbf{x}_n|}\right|.$$

By Hadamard's lemma[19] and Eq. (45) this is

$$\left|\operatorname*{Det}_{1\leq i,k\leq n}\|K_2(\mathbf{x}_i,\mathbf{x}_k)\|\right|\leqq N^n n^{n/2}\frac{F(|\mathbf{x}_1|)}{4\pi|\mathbf{x}_1|^2}\frac{F(|\mathbf{x}_2|)}{4\pi|\mathbf{x}_2|^2}\cdots\frac{F(|\mathbf{x}_n|)}{4\pi|\mathbf{x}_n|^2},$$

and because of Eq. (46)

$$\left|\int d\mathbf{x}_1\cdots\int d\mathbf{x}_n\operatorname{Det}\|K_2(\mathbf{x}_i,\mathbf{x}_k)\|\right|\leq M^nN^nn^{n/2}, \qquad \text{(III.16)}$$

which assures the convergence of Eq. (III.6) for any finite λ. To prove the convergence of

$$\int\Delta_2(\lambda^2,\mathbf{x},\mathbf{x}')F_0(\mathbf{k},\mathbf{x}')d\mathbf{x}', \qquad \text{(III.17)}$$

we first note that

$$|F_0(\mathbf{k},\mathbf{x})|\leq1+\frac{|\lambda|}{4\pi}\int d\mathbf{x}'\left|\frac{V(\mathbf{x}')}{\mathbf{x}-\mathbf{x}'}\right|\leq1+\frac{|\lambda|}{4\pi}\int d\mathbf{x}'\frac{F(|\mathbf{x}'|)}{|\mathbf{x}'||\mathbf{x}-\mathbf{x}'|}\leqq1+|\lambda|M$$

and therefore the determinant in Eq. (III.5) is less than

$$M^{n+1}N^{n+1}n^{\frac{1}{2}(n+1)}(1+|\lambda|M),$$

which makes Eq. (III.17) convergent for any finite λ.

The representation (III.2–6) of the solution for the scattering integral equation (2) has the disadvantage that the denominator $\Delta_2(\lambda^2)$ of the resolvent $H_2(\lambda^2;x,x')$ vanishes for $\pm\lambda_i$, where λ_i are the eigenvalues of the original integral equation. It is therefore possible to factor an entire function in the numerator and denominator of Eq. (III.4). This fact was proved in general by H. Poincaré.[20]

Since we know the convergence of our series, we can immediately use his results, which lead to the prescriptions of Sec. IV, Eqs. (49), (50), where numerator and denominator have no zero's in common.

(IV) The Fredholm Method for the Yukawa Potential

In this section we calculate the integrals D_1 and D_2 given by Eqs. (58), (59). D_1 is easily found by using the configuration space representation in which

$$D_1=(1/16\pi^2)\int d\mathbf{x}d\mathbf{y}\exp(2ik|\mathbf{x}-\mathbf{y}|)\exp(-x-y)/(|\mathbf{x}-\mathbf{y}|^2xy).$$

This integral is elementary, and its outcome is given in Eq. (60). D_2 is treated by introducing again auxiliary integrations over variables $x_i\geq0$, $i=1,\cdots,4$:

$$D_2=\frac{3!}{4\pi^4}\int d\mathbf{p}d\mathbf{q}\int dx_i\frac{\delta(x_1+x_2+x_3+x_4-1)}{[x_1(p^2-\eta)+x_2(q^2-\eta)+x_3\{(\mathbf{p}-\mathbf{q})^2+1\}+x_4\{(\mathbf{p}+\mathbf{q})^2+1\}-i\epsilon(x_1+x_2)]^4}.$$

The expression in square brackets can be written as the sum of a symmetrical quadratic expression $\Sigma a_{ik}y_iy_k$, where $(y_1,y_2,y_3)=\mathbf{p}$ and $(y_4,y_5,y_6)=\mathbf{q}$, and a constant term. We transform the six dimensional quadratic expression on principal axes:

$$\Sigma a_{ik}y_iy_k=\Sigma\lambda_iP_i^2.$$

One easily sees that

$$\Pi_i\lambda_i=d^3=[(x_1+x_3+x_4)(x_2+x_3+x_4)-(x_3-x_4)^2]^3.$$

Introducing $\lambda_i^{\frac{1}{2}}P_i$ as new variables and putting $\Sigma P_i^2=P^2$ we get

$$D_2=(3!O_6/4\pi^4)\int dx_i\frac{\delta(\Sigma x_i-1)}{d^{\frac{3}{2}}}\int\frac{P^5dP}{(P^2+A)^4},$$

where $O_6=\pi^3$ is the surface of a 6-dimensional unit sphere and

$$A=x_3+x_4-\eta(x_1+x_2)-i\epsilon(x_1+x_2).$$

Hence,

$$D_2=(1/4\pi)\int dx_i\delta(\Sigma x_i-1)d^{-\frac{3}{2}}A^{-1}.$$

We introduce

$$x_1+x_2=x,\quad x_3+x_4=v_1,\quad -x\leq y\leq x,$$
$$x_1-x_2=y,\quad x_3-x_4=v_2,\quad -v_1\leq v_2\leq v_1.$$

One can then integrate over v_2 and thereupon over v_1, noting that the argument of the δ-function now reads $x+v_1-1$. The result is

$$\begin{aligned}D_2&=\frac{1}{\pi}\int_0^1\frac{dx(1-x)}{(2-x)\{1-(\eta+1)x-i\epsilon x\}}\int_{-x}^{x}\frac{dy}{(2-x+y)(4x-3x^2-y^2)^{\frac{1}{2}}}\\&=\frac{1}{\pi}\int_0^1\frac{dx}{(2-x)\{1-(\eta+1)x-i\epsilon x\}}\tan^{-1}\left[\frac{(x-x^2)^{\frac{1}{2}}}{2-x}\right]\\&=-\frac{i(\eta+1)}{2\eta+1}\tan^{-1}\frac{\eta^{\frac{1}{2}}}{2\eta+1}+\frac{1}{\pi}\mathrm{P}\int_0^1\frac{dx}{\{1-(\eta+1)x\}(2-x)}\tan^{-1}\left[\frac{(x-x^2)^{\frac{1}{2}}}{2-x}\right],\end{aligned}$$

[19] See reference 11, p. 212.
[20] H. Poincaré, reference 12.

where P indicates that the principal value has to be taken with respect to the point $x=(\eta+1)^{-1}$. It is readily seen that the principal value integral can be written as

$$-(i/4\pi)\oint\frac{dx}{\{1-(\eta+1)x\}\{2-x\}}\ln\frac{2-x+i(x-x^2)^{\frac{1}{2}}}{2-x-i(x-x^2)^{\frac{1}{2}}} \quad \text{(IV.1)}$$

taken along the contour I of Fig. 2 which encircles the branch points $x=0$, 1 of $(x-x^2)$ in the negative sense (the plane is cut along the segment $0\leq x\leq 1$). The contributions of the two infinitesimal semi-circles near $x=(\eta+1)^{-1}$ cancel each other; indeed, the value of the logarithm in any given point of the path above the cut is equal and opposite in sign in the corresponding point below it.

We next deform I to the contour II, coming from $x=\infty$, encircling the pole of the integrand at $x=2$ and the branch point of the logarithm at $x=4/3$ in the negative sense, and then returning to ∞. The plane is cut along that part of the real axis for which $x\geq 4/3$. Care has to be exercised in continuing $(x-x^2)^{\frac{1}{2}}$ to points on path II as well as in choosing the branch of the logarithm on the upper and lower half of this contour. The contributions near the pole $x=2$ cancel again and the final result is

$$D_2=-(1/2\eta+1)\{\tfrac{1}{2}\ln[2(\eta+1)/(4\eta+1)]+i\tan^{-1}(\eta^{\frac{1}{2}}/(2\eta+1))\},$$

which was used in Eq. (61).

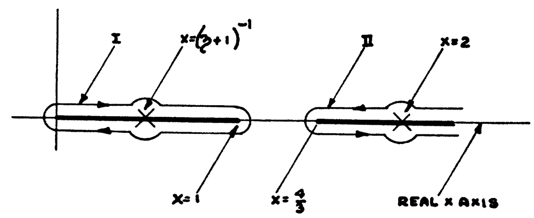

FIG. 2. Deformation of the integration path of the integral (IV.1).

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Transport Rates of the Helium II Film Over Various Surfaces*

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Liquid helium II film transport over various surfaces has been studied by a new method in which a cylindrical capacitor using liquid helium as the dielectric is employed as a depth gauge. Changes in liquid level resulting from film transport produce changes in capacitance which in turn cause frequency changes in a high frequency circuit. The details of this method are described. The film transport rates, measured to 1.25°K, were found to depend on the substrate; at 1.25°K, the highest rate observed was 51×10^{-5} cm^3/cm sec for etched copper and the lowest, 7.5×10^{-5} cm^3/cm sec for glass. The rates were also measured over iron in the magnetized and unmagnetized state and over a superconductor in the superconducting and in the normal state. No differences were noted. In the latter case the thermal conductivity of the container is abruptly changed and the absence of an effect supports the view that heat transfer plays no significant role in determining the transport rate.

INTRODUCTION

SINCE the discovery[1] of the film transport phenomenon of liquid helium II, various investigations have been made to determine the characteristics of the transport. Among the phenomena studied was the effect of the underlying surface material.[2] These studies indicated that the transport rate is independent of surface material. However, a recent examination of this property by Mendelssohn and White[3,4] and the present authors[5,6] has shown that this is not the case.

Theories of the HeII film have been advanced by Frenkel,[7] Schiff,[8] Temperley,[9] and Bijl, de Boer, and Michels.[10] The first two authors have considered helium atoms to be under the influence of gravity and of the van der waals attractive forces of the walls. This treatment indicates that film thicknesses on conducting surfaces are greater, by about a factor of two, than films on dielectrics. Film flow, according to Frenkel, should be limited by viscosity. For the case of HeII, the theoretical result becomes ambiguous owing to the presence of a zero viscosity, or superfluid component. Temperley has treated the film as an adsorbed phase in which He atoms occupy bound sites on the surface of the solid walls. The adsorbed layers farthest from the wall are assumed to occupy only a fraction of the available sites, and film flow is considered to arise from a transition of atoms from their existing sites to empty neighbors. The influence of the wall is considerably

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