A Phenomenological Derivation of the First- and Second-Order Magnetostriction and Morphic Effects for a Nickel Crystal

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In order to account for experimental results which showed that the saturation elastic constants of a single nickel crystal varied with the direction of magnetization, a phenomenological investigation has been made of the stress, strain, and magnetic relations for single nickel crystals. The variation in elastic constants is shown to be a "morphic" effect caused by the change in the crystal symmetry due to the magnetostriction effect. In the energy equation this effect is represented by additional terms which involve squares and products of both the magnetic intensities and stresses. These terms are as large as the magnetostrictive terms when the stresses are of the order of 10^{10} dynes/cm². The energy equation has been used to derive the first- and second-order magnetostrictive effect, and the resulting terms agree with Becker and Döring's empirical constants for saturation conditions. For smaller magnetic intensities the terms divide up into first- and second-order terms which vary differently with magnetic field intensity. It is shown that the morphic effects involve six measurable constants, and some of these are evaluated experimentally.

I. INTRODUCTION

CAREFUL measurements of the change in elastic constants with magnetization in single nickel crystals[1] have shown that the elastic constants for a saturated crystal depend on the direction of saturation. It is the primary purpose of this paper to show that the change in saturation elastic constants with direction of magnetization is due to what Mueller[2] has called a "morphic" effect; namely, that the distortion of the crystal by the magnetostrictive effect produces a crystal of lower symmetry which involves more than the three elastic constants of a cubic crystal. It is shown in this paper that the additional elastic constants and their variation with magnetic orientation can be obtained by adding terms to the energy equation which involve squares and products of both the magnetic intensity and the stresses. Symmetry determines the number of constants, and it is shown that there are six measurable constants which characterize the effect.

The complete energy equation has been applied to a derivation of the magnetostrictive effect, and it is shown that first-and second-order terms are involved. At saturation the terms agree with those given previously by Becker and Döring,[3] but for lower magnetic intensities the terms divide up into first- and second-order terms which vary differently with magnetic field intensity. The experimental results of Masiyama[4] for a single nickel crystal are fitted by the phenomenological formula derived here, and a slightly better fit is obtained. The evaluation of some of the first- and second-order terms is accomplished.

The tensor method is followed, since this results in a considerable economy of effort and readily allows one to change from one coordinate system to another. The tensors are all of the cartesian type, since only rectangular coordinate systems are considered. The method followed is to add terms to the energy equation of a form required to agree with the measured effects and to determine the resulting constants by means of symmetry considerations.

II. ENERGY FUNCTIONS FOR MAGNETOSTRICTIVE CRYSTALS

It is well known that the changes in dimensions, magnetic fields, and temperature for any body subject to magnetostrictive and magnetic effects can be derived from a thermodynamic function. In order to use the measured magnetostrictive constants, it is better to use the stress, intensity of magnetization, and entropy as the independent variables. To do this we introduce the elastic enthalpy H_1, defined by the equation

$$H_1 = U - T_{ij}S_{ij}, \qquad (1)$$

where U is the internal energy function and T_{ij} and S_{ij} are respectively the stresses and tensor strains, i.e., $[\frac{1}{2}(\partial u_i/\partial x_j) + \partial u_j/\partial x_i]$, where u is the displacement. The differential form of H_1 is known to be

$$dH_1 = -S_{ij}dT_{ij} + H_m dI_m + \Theta d\sigma, \qquad (2)$$

where H_m are the magnetic fields, I_m the intensities of magnetization, Θ the absolute temperature, and σ the entropy. Hence, one obtains

$$S_{ij} = -\partial H_1/\partial T_j; \quad H_m = \partial H_1/\partial I_m; \quad \Theta = \partial H_1/\partial \sigma. \quad (3)$$

Since all of the measurements have been carried out under adiabatic conditions, the entropy change is zero and H_1 can be considered to be a function of the stresses and magnetizations.

The usual method for deriving the energy terms is to add first- and second-order terms and to determine the number and types of constants by the symmetry con-

[1] These measurements were made primarily to determine the "ΔE" effect and the microeddy current effects in single nickel crystals and will be discussed in another paper, "The frequency dependence of elastic constants and losses in nickel," by Bozorth, Mason, and McSkimin. The measurements are reproduced here in order to evaluate the morphic effect.

[2] H. Mueller, Phys. Rev. **58**, 805 (1940).

[3] R. Becker and W. Döring, *Ferromagnetismus* (Verlag. Julius Springer, Berlin, 1934), page 275.

[4] Y. Masiyama, Sci. Rep. Tohoku Imp. Univ. **17**, 945 (1928).

dition. Use can be made of the fact that nickel has a center of symmetry which results in having all the odd rank tensors equal to zero. Furthermore, nickel can be described as being soft magnetically but hard elastically. Hence, we can neglect all energy terms that involve terms higher than products or squares of the stresses. With these limitations, the elastic enthalpy function H_1 can be written in the form

$$2H_1 = -[s^I_{ijkl}T_{ij}T_{kl} + R_{ijklno}I_nI_oT_{ij}T_{kl}$$
$$+ M_{ijmn}I_mI_nT_{ij} + N_{ijmnop}I_mI_nI_oI_pT_{ij}]$$
$$+ K^T_{mn}I_mI_n + K^T_{mnop}I_mI_nI_oI_p$$
$$+ K^T_{mnopqr}I_mI_nI_oI_pI_qI_r. \quad (4)$$

These equations hold for a crystal with a large number of domains when the directions of the domains are uncorrelated, for then the components of magnetization are independent. For a single domain, the magnetic intensity has a fixed value, the saturation value I_0, and only the direction of magnetism can be changed. If α_1, α_2, α_3 are the direction cosines of the magnetic intensity with respect to the crystal axes, then one has

$$I_1 = \alpha_1 I_0; \quad I_2 = \alpha_2 I_0; \quad I_3 = \alpha_3 I_0; \quad (5)$$

and there is a relation between the three direction cosines, namely,

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1. \quad (6)$$

The stresses T_{ij} and the strains S_{kl} are symmetrical tensors, and hence we can interchange the order of the subscripts. The same is true for the subscripts of the

magnetic intensities, and hence it has become customary to use a single index related to the two indices by the equations

$$11 = 1; \quad 22 = 2; \quad 33 = 3; \quad 12 = 21 = 6;$$
$$13 = 31 = 5; \quad 23 = 32 = 4. \quad (7)$$

Similarly, for the fourth-rank tensors s_{ijkl}, M_{klmn}, K_{mnop} we can interchange i and j, k and l, o and p so that only two indices are necessary for these terms. Furthermore, for the tensor $s_{ab} = s_{ijkl}$ it is proved in elastic theory that a and b can be interchanged so that there are at most 21 independent constants. This is not true for the magnetostrictive tensor $M_{ab} = M_{ijkl}$ and in general there are 36 independent terms. For the anisotropic magnetic energy tensor $K_{ab} = K_{ijkl}$ all four of the terms i, j, k, l can be interchanged, and in general there are only 15 independent terms. The number and type of these independent constants for the symmetry of nickel O_h are well known from crystal theory.[5] Since the number and type of the constants of a sixth-rank tensor do not appear to have been completely worked out, they are considered in Appendix I.

With these components known, the expression for H_1 can be written down. In this equation I is replaced by $\alpha_1 I_0$, etc., indicating that it is a component of the saturation intensity of magnetization I_0. The energy for the demagnetized case given by $\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \frac{1}{3}$ has also been subtracted from the expression so that the energy represents the difference between any state and the demagnetized state. Certain combinations of the α-values have also been made to cut the number of independent terms to a minimum. With these simplifications, we can write $-2H_1$ as

$$-2H_1 = s^I_{11}(T_1^2 + T_2^2 + T_3^2) + 2s^I_{12}(T_1T_2 + T_1T_3 + T_2T_3) + 2s^I_{44}[T_4^2 + T_5^2 + T_6^2]$$
$$+ (\alpha_1^2 - \tfrac{1}{3})I_0^2[(R_{111} - R_{112})T_1^2 + 2(R_{123} - R_{121})T_2T_3 + 4(R_{441} - R_{551})T_4^2]$$
$$+ (\alpha_2^2 - \tfrac{1}{3})I_0^2[(R_{111} - R_{112})T_2^2 + 2(R_{123} - R_{121})T_1T_3 + 4(R_{441} - R_{551})T_5^2]$$
$$+ (\alpha_3^2 - \tfrac{1}{3})I_0^2[(R_{111} - R_{112})T_3^2 + 2(R_{123} - R_{121})T_1T_2 + 4(R_{441} - R_{551})T_6^2]$$
$$+ 8\alpha_1\alpha_2 I_0^2\{T_6[R_{144}T_3 + R_{155}(T_1 + T_2)] + R_{456}T_4T_5\}$$
$$+ 8\alpha_1\alpha_3 I_0^2\{T_5[R_{144}T_2 + R_{155}(T_1 + T_3)] + R_{456}T_4T_6\}$$
$$+ 8\alpha_1\alpha_2 I_0^2\{T_4[R_{144}T_1 + R_{155}(T_2 + T_3)] + R_{456}T_5T_6\}$$
$$+ \{(M_{11} - M_{12})I_0^2 + [2(N_{112} - N_{123}) + 4(N_{551} - N_{441})]I_0^4\}[(\alpha_1^2 - \tfrac{1}{3})T_1 + (\alpha_2^2 - \tfrac{1}{3})T_2 + (\alpha_3^2 - \tfrac{1}{3})T_3]$$
$$+ \{[N_{111} - 3N_{112} + 2N_{123} + 4(N_{441} - N_{551})]I_0^4[(\alpha_1^4 + \tfrac{2}{3}s - \tfrac{1}{3})T_1 + (\alpha_2^4 + \tfrac{2}{3}s - \tfrac{1}{3})T_2 + (\alpha_3^4 + \tfrac{2}{3}s - \tfrac{1}{3})T_3]\}$$
$$+ [\tfrac{2}{3}(N_{111} - N_{123} - 4N_{551} - 2N_{441})]I_0^4(\tfrac{1}{3} - s)(T_1 + T_2 + T_3)$$
$$+ [2M_{44}I_0^2 + 4(N_{112} + N_{551})I_0^4][\alpha_2\alpha_3T_4 + \alpha_1\alpha_3T_5 + \alpha_1\alpha_2T_6]$$
$$+ 4[N_{123} + N_{441} + 2N_{456} - N_{112} - N_{551}]I_0^4[\alpha_1^2\alpha_2\alpha_3T_4 + \alpha_1\alpha_2^2\alpha_3T_5 + \alpha_1\alpha_2\alpha_3^2T_6]$$
$$- K^T_1(\tfrac{1}{3} - s) - K^T_2[(1/27) - \alpha_1^2\alpha_2^2\alpha_3^2], \quad (8)$$

[5] See, for example, W. G. Cady, *Piezoelectricity* (McGraw-Hill Book Company, Inc., New York, 1946) or W. P. Mason, *Piezoelectric Crystals* (D. Van Nostrand Company, Inc., New York, 1950) for the elastic tensors. The magnetostrictive tensor is the same as the photoelastic tensor of Pockels. For the tensor K_{ijkl} there are only two independent constants.

where

$$s = \alpha_1^2\alpha_2^2 + \alpha_1^2\alpha_3^2 + \alpha_2^2\alpha_3^2,$$

$$K^T_1 = [(-2K^T_{11}+6K^T_{12})I_0^4 + (-3K^T_{111}+15K^T_{112})I_0^6],$$

$$K^T_2 = [66K^T_{123} - 45K^T_{112} + 3K^T_{111}]I_0^6.$$

The first three terms represent the elastic energy, the next six terms represent the energy stored by morphic effects, i.e., by the change of shape of the body, the next five terms represent the energy stored by the first- and second-order magnetostrictive effects, while the last two terms represent the magnetic anisotropy terms measured at constant stress. If we neglect the morphic energy terms and set the five constants for the magnetostrictive terms equal to B_1 to B_5, the internal energy reduces to the form given by Becker and Döring.[4] From the value of the R constants found experimentally, the morphic energy terms are about equal to the magnetostrictive terms for stresses in the order of 10^{10} dynes/cm², but are smaller for smaller stresses. The anisotropy energy constants K_1 and K_2 are those measured for constant stress. As shown by Kittel,[6] these include the anisotropy energy for constant lattice separation plus the magnetostrictive energy caused by the lattice distortion. To obtain the tensor strains from this function one differentiates with respect to the corresponding stress, as shown by Eq. (3). For the engineering strain values which are twice as large as the tensor strain values for the shearing components

$$S_n = -2\partial H_1/\partial T_n; \quad n = 4, 5, 6. \tag{9}$$

Hence, the shear elastic constants s^I_{44}, and other terms which are multiplied by the stresses T_4, T_5, and T_6 are one-fourth as large as the corresponding terms which express the relationship between engineering strains and stresses.

III. MAGNETOSTRICTIVE EFFECTS IN CUBIC CRYSTALS

When a nickel crystal is magnetized, it contracts along the direction of magnetization and expands at right angles to the direction of magnetization. The measurements of Masiyama[4] are all elongations along crystallographic axes or along directions such as the [110] or [111]. The elongations or contractions for any direction z' can be obtained from the energy expression (8) by differentiation with respect to the longitudinal stress in this direction or

$$S_{33}' = -\partial H_1/\partial T_{33}'; \tag{10}$$

but the strain S_{33}' is related to the strains referred to the crystallographic axes by the tensor transformation equation

$$S_{33}' = \frac{\partial x_3'}{\partial x_i}\frac{\partial x_3'}{\partial x_j}S_{ij} = -\frac{\partial x_3'}{\partial x_i}\frac{\partial x_3'}{\partial x_j}\frac{\partial H_1}{\partial T_{ij}}, \tag{11}$$

[4] C. Kittel, Revs. Modern Phys. 21, 541 (1949).

where the partial derivatives are the direction cosines β_1, β_2, and β_3 between the z' axis and the x, y, and z axes, respectively. In terms of the one index stress and strain terms

$$S_3' = -(\beta_1^2\partial H_1/\partial T_1) - (\beta_2^2\partial H_1/\partial T_2) - (\beta_3^2\partial H_1/\partial T_3)$$
$$- (2\beta_2\beta_3\partial H_1/\partial T_4) - (2\beta_1\beta_3\partial H_1/\partial T_5)$$
$$- 2\beta_1\beta_2\partial H_1/\partial T_6. \tag{12}$$

Performing this differentiation, we have for the magnetostrictive terms

$$S_3' = h_1[\alpha_1^2\beta_1^2 + \alpha_2^2\beta_2^2 + \alpha_3^2\beta_3^2 - \tfrac{1}{3}]$$
$$+ h_2[\alpha_1\alpha_2\beta_1\beta_2 + \alpha_1\alpha_3\beta_1\beta_3 + \alpha_2\alpha_3\beta_2\beta_3]$$
$$+ h_3(\tfrac{1}{3} - s) + h_4[\alpha_1^4\beta_1^2 + \alpha_2^4\beta_2^2 + \alpha_3^4\beta_3^2 + \tfrac{2}{3}s - \tfrac{1}{3}]$$
$$+ h_5[\alpha_1^2\alpha_2\alpha_3\beta_2\beta_3 + \alpha_1\alpha_2^2\alpha_3\beta_1\beta_3 + \alpha_1\alpha_2\alpha_3^2\beta_1\beta_2], \tag{13}$$

where

$$h_1 = [\tfrac{1}{2}(M_{11}-M_{12})I_0^2 + [N_{112}-N_{123}+2N_{551}-2N_{441}]I_0^4],$$

$$h_2 = [2M_{44}I_0^2 + 4(N_{112}+N_{551})I_0^4],$$

$$h_3 = \tfrac{1}{3}[N_{111}-N_{123}-4N_{551}-2N_{441}]I_0^4, \tag{14}$$

$$h_4 = [\tfrac{1}{2}(N_{111}-3N_{112})+N_{123}+2(N_{441}-N_{551})]I_0^4,$$

$$h_5 = 4[N_{123}+N_{441}+2N_{456}-N_{112}-N_{551}]I_0^4.$$

For nickel the volume magnetostriction is zero. If we add the sum of S_1', S_2', and S_3' and set the resultant equal to zero for any direction of magnetization, $h_3 = 0$ and there is a relation between four of the constants

$$N_{111} - N_{123} - 4N_{551} - 2N_{441} = 0$$

or

$$2N_{551} = \tfrac{1}{2}(N_{111}-N_{123}) - N_{441}. \tag{15}$$

This reduces the formulas for the h values to the values given below:

$$h_1 = [\tfrac{1}{2}(M_{11}-M_{12})I_0^2 + [\tfrac{1}{2}(N_{111}-N_{112})$$
$$+ \tfrac{3}{2}(N_{112}-N_{123}) - 3N_{441}]I_0^4],$$

$$h_2 = [2M_{44}I_0^2 + (4N_{112}+N_{111}-N_{123}-2N_{441})I_0^4],$$

$$h_3 = 0, \tag{16}$$

$$h_4 = [\tfrac{3}{2}(N_{123}-N_{112})+3N_{441}]I_0^4,$$

$$h_5 = [5(N_{123}-N_{112}) - (N_{111}-N_{112})+6N_{441}+8N_{456}]I_0^4.$$

An empirical formula for the magnetostrictive constant in nickel has been given by Becker and Döring[3] which takes the same form as Eq. (13) with $h_3=0$. Equation (13) with the constants of Eq. (14) is more general for when it refers to the multidomain crystal with uncorrelated domains, it gives the magnetostriction as a function of magnetization when we replace $\alpha_1 I_0$ by I_1, etc. Hence, when one measures the magneto-
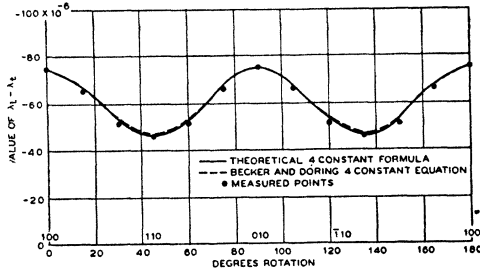
FIG. 1. Difference between longitudinal and transverse magnetostrictive constants for 001 plane.

striction starting from a fully demagnetized crystal, Eq. (13) gives the magnetostriction as a function of the magnetic intensity.

Becker and Döring have compared this formula with the measured curves of Masiyama[4] for the magnetostriction at saturation for nickel. They compare the difference between the longitudinal and transverse magnetostriction with their formula for the 001 plane, the 011 plane, and the $\overline{1}11$ plane. This difference was chosen because it is independent of the domain structure of the demagnetized state. By choosing the values

$$h_1 = -24\times10^{-6}, \quad h_2 = -94\times10^{-6}, \quad h_4 = -51\times10^{-6},$$

$$h_5 = +104\times10^{-6}, \quad h_3 = 0, \tag{17}$$

the dotted curves of Figs. 1, 2, and 3 result for these three planes.

Another comparison was made with the experimental data, and it appears that the data are fitted better with slightly different coefficients. For the $00\overline{1}$ plane the equation for $\lambda_l - \lambda_t$, the difference between the longitudinal and transverse magnetostrictive constants becomes

$$\lambda_l - \lambda_t = (h_1 + h_4)\cos^2 2\theta + \tfrac{1}{2}h_2\sin^2 2\theta. \tag{18}$$

The solid curve of Fig. 1 results when

$$h_1 + h_4 = -76\times10^{-6}, \quad \tfrac{1}{2}h_2 = -46\times10^{-6}. \tag{19}$$

For the $01\overline{1}$ plane the equation for $\lambda_l - \lambda_t$ is

$$\lambda_l - \lambda_t = \tfrac{1}{6}(6h_1 + 5h_4)[\cos^2\Psi - \sin^2\Psi][\cos^2\Psi - \sin^2\tfrac{1}{2}\Psi]$$

$$+ \tfrac{1}{4}h_2\sin^2\Psi[7\cos^2\Psi + 2\sin^2\tfrac{1}{2}\Psi]$$

$$+ \tfrac{1}{2}h_5\sin^2\Psi\cos^2\Psi, \tag{20}$$

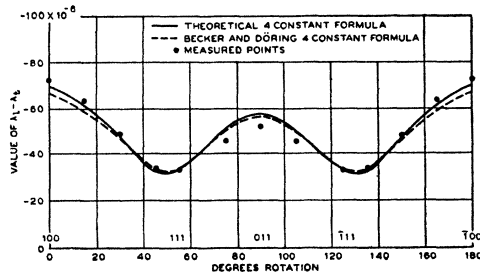FIG. 2. Difference between longitudinal and transverse magnetostrictive constants for $01\overline{1}$ plane.

where Ψ is the angle measured from the 100 direction The best fit with the measured points is obtained by setting

$$\tfrac{1}{6}(6h_1 + 5h_4) = -70\times10^{-6}, \quad \tfrac{1}{2}h_5 = +54\times10^{-6}. \tag{21}$$

The solid line of Fig. 2 results. This still does not fit the measured values completely, which indicates that higher order energy terms are not negligible; but a better agreement is obtained than with the values used by Becker and Döring. If we combine the first value of (19) with the first value of (21), we have the relation

$$h_1 = -40\times10^{-6}, \quad h_4 = -36\times10^{-6}. \tag{22}$$

For the $\overline{1}11$ plane, a calculation of $\lambda_l - \lambda_t$ shows that the difference is independent of orientation and has the value

$$\lambda_l - \lambda_t = h_1/3 + 2h_4/9 + h_2/3 + h_5/18. \tag{23}$$

No new constants are involved, but one check is obtained for the values determined previously. These add up to -46×10^{-6}, which agrees well with the measured results as shown by Fig. 3.

These measurements do not allow one to separate out all of the constants, since we cannot resolve the values of the constants of Eq. (13). The two involving both second- and fourth-order constants, h_1 and h_2,

$$h_1 = \tfrac{1}{2}(M_{11} - M_{12})I_0^2 + [\tfrac{1}{2}(N_{111} - N_{112})$$

$$+ \tfrac{3}{2}(N_{112} - N_{123}) - 3N_{441}]I_0^4, \tag{24}$$

$$h_2 = 2M_{44}I_0^2 + [4N_{112} + (N_{111} - N_{123}) - 2N_{441}]I_0^4,$$

could be resolved if the magnetostriction constants were measured in terms of the magnetic intensity, since terms in I^2 and I^4 occur. Unfortunately, however, Masiyama's measurements are presented in terms of the field strengths. Two of these measurements for the 001 plane are shown by Fig. 4. In order to obtain the magnetic intensities, one would need to know the permeabilities for the two directions as a function of field strength. The initial slopes of these two curves, however, can be used to determine approximately the ratio of the M_{44} constant to the $M_{11} - M_{12}$ constant. This follows from the fact that the initial permeability is a second-rank tensor and hence has the same value for all directions for a cubic crystal. From the initial slopes it appears that the ratio is

$$2M_{44}/(M_{11} - M_{12}) = 69/40$$

or

$$M_{44} = 0.86(M_{11} - M_{12}). \tag{25}$$

Further constants could be evaluated if shear magnetostriction measurements were available.

IV. CHANGE OF ELECTRIC CONSTANTS WITH THE DIRECTION OF MAGNETIZATION FOR A SATURATED CRYSTAL

When the crystal is magnetically saturated, it is found experimentally that for elastic waves propagated along the [110], [111], and [100] directions the

velocity depends on the direction of magnetic saturation. This can be explained if we include "morphic" effects; i.e., if we take account of the change of shape of the crystal from a cubic form due to the magnetostrictive effects in the crystal. These changes are caused by the R terms of Eq. (8).

For a plane progressive ultrasonic wave it is much more advantageous to express the stresses in terms of the strains, since only one strain occurs in an uncoupled plane progressive wave. This involves expressing the results in terms of the internal energy function U rather than the elastic enthalpy function H_1. The resulting terms can be determined by eliminating the stresses fro H_1 and replacing them by the strains. The details are discussed in Appendix II.
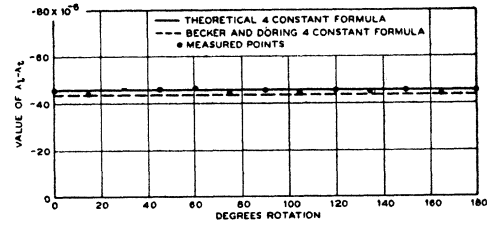

FIG. 3. Difference between longitudinal and transverse magnetostrictive constants for $\bar{1}11$ plane.

If we differentiate U with respect to S_n and neglect the constant forces generated by the magnetostrictive effect, the stress-strain relation can be expressed in the matrix form

	S_1	S_2	S_3	S_4	S_5	S_6
T_1	$c^I_{11}+\delta_{11}$	$c^I_{12}+\delta_{12}$	$c^I_{12}+\delta_{13}$	δ_{14}	δ_{15}	δ_{16}
T_2	$c^I_{12}+\delta_{12}$	$c^I_{11}+\delta_{22}$	$c^I_{12}+\delta_{23}$	δ_{24}	δ_{25}	δ_{16}
T_3	$c^I_{12}+\delta_{13}$	$c^I_{12}+\delta_{23}$	$c^I_{11}+\delta_{33}$	δ_{24}	δ_{15}	δ_{36}
T_4	δ_{14}	δ_{24}	δ_{24}	$c^I_{44}+\delta_{44}$	δ_{45}	δ_{46}
T_5	δ_{15}	δ_{25}	δ_{15}	δ_{45}	$c^I_{44}+\delta_{55}$	δ_{56}
T_6	δ_{16}	δ_{16}	δ_{36}	δ_{46}	δ_{56}	$c^I_{44}+\delta_{66}$,

where c^I_{11}, c^I_{12}, c^I_{44} are the saturated elastic constants and the δ's are the modifications caused by the morphic effects. These are given in terms of the R values in Appendix II.

All the measurements were made on a (110) section and for transmission in this direction the resulting velocities are determined by solving the determinant[7]

$$\begin{vmatrix} \frac{1}{2}(c^I_{11}+c^I_{44}+\delta_{11}+\delta_{66}+2\delta_{16})-\rho v^2; & \frac{1}{2}(c^I_{12}+c^I_{44}+\delta_{12}+\delta_{66}+2\delta_{16}); & \frac{1}{2}(\delta_{14}+\delta_{15}+\delta_{46}+\delta_{56}) \\ \frac{1}{2}(c^I_{12}+c^I_{44}+\delta_{12}+\delta_{66}+2\delta_{16}); & \frac{1}{2}(c^I_{11}+c^I_{44}+\delta_{22}+\delta_{66}+2\delta_{16})-\rho v^2; & \frac{1}{2}(\delta_{24}+\delta_{25}+\delta_{46}+\delta_{56}) \\ \frac{1}{2}(\delta_{14}+\delta_{15}+\delta_{46}+\delta_{56}); & \frac{1}{2}(\delta_{24}+\delta_{25}+\delta_{46}+\delta_{56}); & \frac{1}{2}(2c^I_{44}+\delta_{44}+\delta_{55}+2\delta_{45})-\rho v^2 \end{vmatrix} = 0. \quad (26)$$

If we neglect squares of the δ-quantities, the three solutions of this equation can be written

$$\rho v_1{}^2 = \tfrac{1}{2}(c^I_{11}+c^I_{12})+c^I_{44}+\tfrac{1}{4}(\delta_{11}+\delta_{22}+2\delta_{12})$$
$$+\delta_{66}+2\delta_{16} \quad \text{(long)},$$

$$\rho v_2{}^2 = \tfrac{1}{2}(c^I_{11}-c^I_{12})+\tfrac{1}{4}(\delta_{11}+\delta_{22}-2\delta_{12}) \quad (27)$$

(shear, particle velocity along $[1\bar{1}0]$),

$$\rho v_3{}^2 = c^I_{44}+\tfrac{1}{2}(\delta_{44}+\delta_{55}+2\delta_{45})$$

(shear, particle velocity along $[001]$).

From the last shear velocity v_3, we find that the morphic value of the shear elastic constant is

$$c^M_{44} = c^I_{44}+\tfrac{1}{2}c^I_{44}{}^2I_0{}^2[(\alpha_3{}^2-\tfrac{1}{3})(R_{441}-R_{551})$$
$$-2\alpha_1\alpha_2 R_{456}]. \quad (28)$$

This allows one to derive one relation between two of the independent constants.

For the other shear velocity v_2, one has an elastic constant

$$c^M_{66}{}' = \tfrac{1}{2}(c^I_{11}-c^I_{12})+\tfrac{1}{2}(c^I_{11}-c^I_{12})^2I_0{}^2$$

$$\times[(\alpha_3{}^2-\tfrac{1}{3})(R_{111}-R_{112}+2R_{123}-2R_{121})]. \quad (29)$$

The measurement allows one to determine the sum of two other combinations.

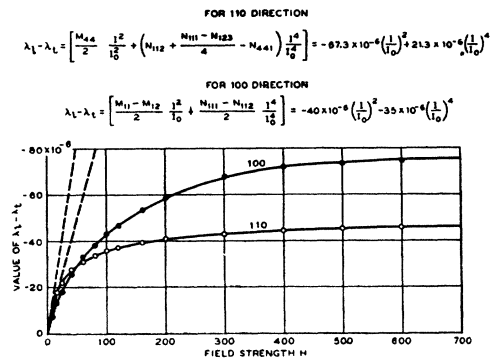Finally, for the longitudinal velocity v_1, the elastic

FIG. 4. Longitudinal minus transverse magnetostriction effect plotted as a function of field strength H for 001 plane.
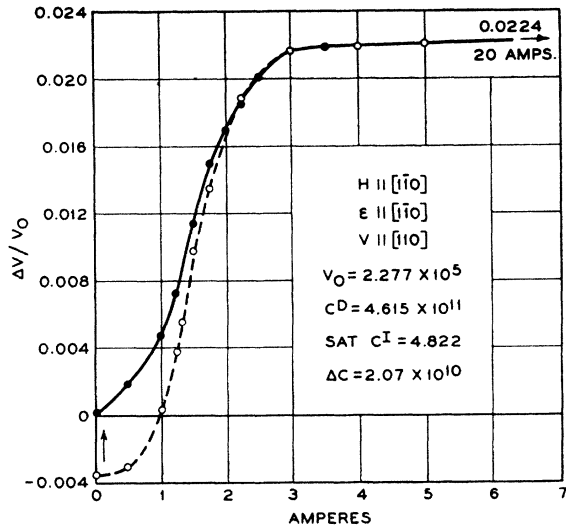
[7] Love, *Theory of Elasticity*, Fourth Edition (Cambridge University Press, London, 1928), p. 298.

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FIG. 5. Change in velocity for shear wave No. 1 as a function of magnetization (10 megacycles).

constant becomes

$$c^M{}_{11}{}' = \tfrac{1}{2}[c^I{}_{11}+c^I{}_{12}]+c^I{}_{44}+(\alpha_3{}^2-\tfrac{1}{3})I_0{}^2$$

$$\times\{[\tfrac{1}{2}(c^I{}_{11}+c^I{}_{12})^2-c^I{}_{12}{}^2](R_{111}-R_{112})$$

$$-\tfrac{1}{2}(c^I{}_{11}{}^2-c^I{}_{12}{}^2)(R_{123}-R_{121})-c^I{}_{44}{}^2(R_{441}-R_{551})\}$$

$$-2c^I{}_{44}\alpha_1\alpha_2 I_0{}^2[c^I{}_{12}R_{144}+(c^I{}_{11}+c^I{}_{12})R_{244}]. \quad (30)$$
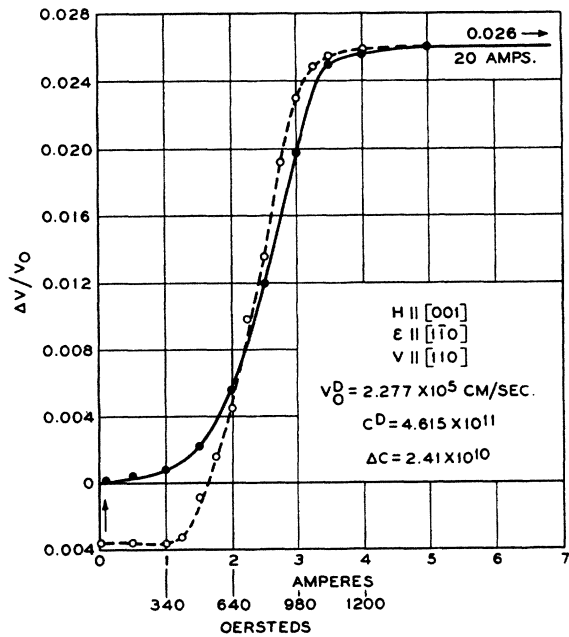

FIG. 6. Change in velocity for shear wave No. 1 as a function of magnetization (10 megacycles).

[8] These measurements were made by H. J. McSkimin by a technique described in J. Acoust. Soc. Am. 24, 413 (1950).

V. EXPERIMENTAL RESULTS

The original measurements[8] which indicated the presence of a "morphic" effect were measurements of the velocity and attenuation of ultrasonic waves in single nickel crystals. These were made primarily to determine the "ΔE" effect in single crystals and will be discussed in detail in a companion paper.[1] The values are reproduced here in order to evaluate the magnitude of the morphic effects. All the measurements were made on a (110) section, since it has been shown that three independent waves, two shear and one longitudinal, can be propagated in such a section. One shear wave (called No. 1) is generated when the particle motion is along the [1$\bar{1}$0] direction, while the other (No. 2) is generated when the particle motion is along
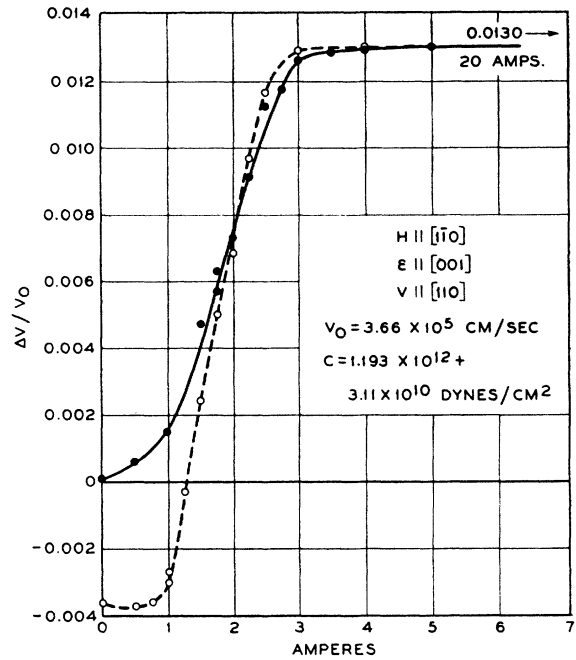

FIG. 7. Change in velocity for shear No. 2 as a function of magnetization (10 megacycles).

the [001] direction. Figure 5 shows the increase in velocity divided by the velocity for the demagnetized condition as a function of current through the magnetizing coil for the No. 1 shear when the field is in the direction of particle motion [1$\bar{1}$0]. At saturation the velocity is increased by a factor of 0.0224. The dotted curve shows the velocity under decreasing conditions. The velocity at zero field is less than that for the demagnetized case, but returns to it when the crystal is again demagnetized. Figure 6 shows a measurement for the same shear with the field parallel to the [001] direction. For this case the increase is 0.026 giving an increase in elastic constant Δc of 2.41×10^{10} dynes/cm, compared with 2.07×10^{10} dynes/cm^2 for the field in the [1$\bar{1}$0] direction. The difference between these of 3.4×10^9 dynes/cm is a morphic effect. Figures 7 and 8

show similar measurements for the No. 2 shear as a function of magnetic orientation. Here the morphic effect is 2.5×10^9 dynes/cm². Figures 9 and 10 show similar measurements for the longitudinal wave, giving a morphic effect of 0.9×10^9 dynes/cm². All these effects are gathered together in Table I, which shows also the combinations of constants involved.

The measurements give three relations between the six measurable combinations. While these are not enough to evaluate all of the constants, they do show the existence of a morphic effect. It is interesting to observe that the changes measured are in the same order of magnitude as the change in the elastic constant occasioned by a temperature expansion which produces a distortion comparable to the magnetostrictive distor-
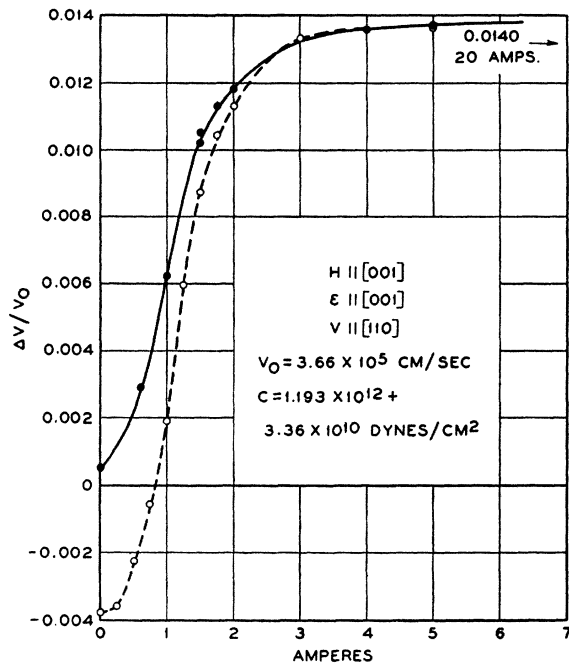

Fig. 8. Change in velocity for shear wave No. 2 as a function of magnetization (10 megacycles).

tion. Since the temperature expansion coefficient of nickel is 12 parts in 10^6 per degree C, an increase in temperature of 4°C will produce an expansion as large as the magnetostrictive effect. This increase in temperature will cause a decrease of the elastic modulus of 0.14 percent[9] or a change of 3×10^9 dynes/cm² in Young's modulus, which is intermediate between the values measured for the longitudinal and shear effects. While no direct comparison can be made between magnetostriction effects and temperature effects, since one causes a change in volume and the other does not, the fact that they produce effects of the same order of magnitude is indicative of the related nature of the effects, i.e., a separation of adjacent molecules.

[9] See G. W. Pierce, Proc. Am. Acad. Arts Sci. **63**, No. 1 (1928).
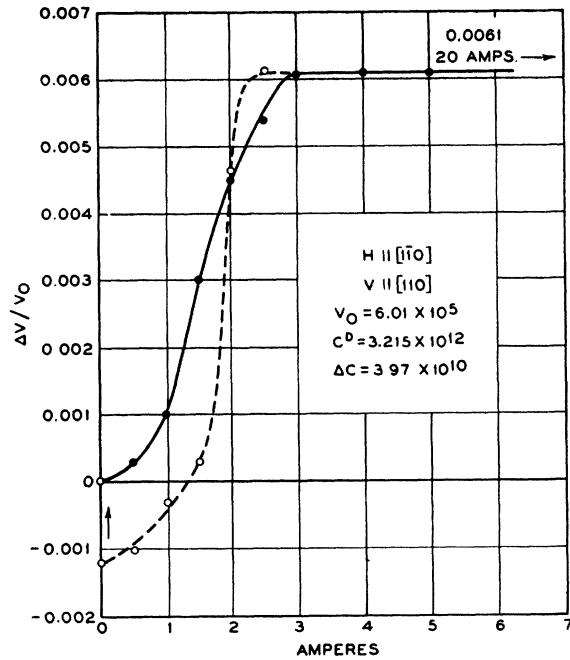

Fig. 9. Change in velocity for longitudinal wave as a function of magnetization (10 megacycles).

APPENDIX I. NUMBER OF INDEPENDENT TERMS OF A SIXTH-RANK TENSOR FOR A CUBIC CRYSTAL OF CLASS $O_h = (4/m)\bar{3} - (2/m)$

When second-order effects are taken account of in magnetostrictive effects, there are 3 sixth-rank tensors to consider, N_{ijnops}, R_{ijklno}, and K_{mnopqr} of Eqs. (4). For the most general case there are 729 terms in a sixth-rank tensor; but since i and j can be
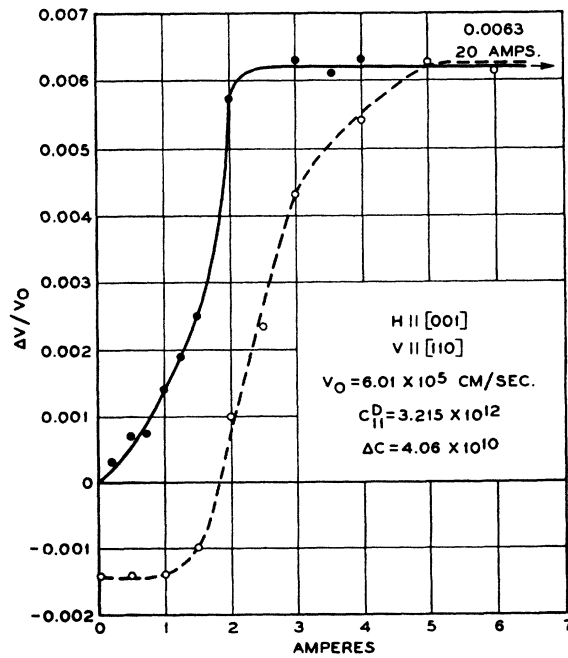

Fig. 10. Change in velocity for longitudinal wave as a function of magnetization (10 megacycles).

TABLE I. Measured velocities and elastic constants for a 110 section of a nickel crystal as a function of magnetic orientation.

Crystal No. 1 mode	Velocity demagnetized	Δc	Constants determined
Long. particle velocity along 110	6.01×10^5 cm/sec	I along 001; $\Delta c = 4.06 \times 10^{10}$ dynes/cm^2 I along 1$\bar{1}$0; $\Delta c = 3.97 \times 10^{10}$ dynes/cm^2	$c^M{}_{11}(001) - c^M{}_{11}(1\bar{1}0) = 9 \times 10^8$ dynes/cm^2 $= I_0^2[[\frac{1}{2}(c^I{}_{11}{}^2 - c^I{}_{12}{}^2) - c^I{}_{12}{}^2](R_{111} - R_{112})$ $+ \frac{1}{2}(c^I{}_{11}{}^2 - c^I{}_{12}{}^2)(R_{121} - R_{123}) + c^I{}_{44}{}^2(R_{551} - R_{441})$ $+ c^I{}_{44}(c^I{}_{12}R_{144} + (c^I{}_{11} + c^I{}_{12})R_{244})]$
Shear particle velocity along 001	3.66×10^5 cm/sec	I along 001; $\Delta c = 3.36 \times 10^{10}$ dynes/cm^2 I along 1$\bar{1}$0; $\Delta c = 3.11 \times 10^{10}$ dynes/cm^2	$c^M{}_{44}(001) - c^M{}_{44}(1\bar{1}0) = 2.5 \times 10^9$ dynes/cm^2 $= c^I{}_{44}{}^2[R_{441} - R_{551} - R_{456}]I_0^2$
Shear particle velocity along 1$\bar{1}$0	2.277×10^5 cm/sec	I along 001; $\Delta c = 2.41 \times 10^{10}$ dynes/cm^2 I along 1$\bar{1}$0; $\Delta c = 2.07 \times 10^{10}$ dynes/cm^2	$c^M{}_{66}(001) - c^M{}_{66}(1\bar{1}0) = 3.4 \times 10^9$ dynes/cm^2 $= \frac{1}{4}(c^I{}_{11} - c^I{}_{12})^2[R_{111} - R_{112} + 2(R_{123} - R_{121})]I_0^2$

interchanged in N_{ijnop} and n, o, p, and s can be interchanged without affecting the values of the terms, there are only 90 independent terms for even the most unsymmetrical crystal, a triclinic crystal. For the R tensor, i and j, k and l, n, and o, can be interchanged leaving only 216 independent terms. Furthermore, from the definition of R_{ijklno}, it is obvious that i and j can be interchanged with k and l. This reduces the number of independent constants to 126 for $R_{ijklmno}$.

Since the number of independent constants of a sixth-rank tensor[10] does not seem to have been worked out for a cubic crystal of class O_h, it is the purpose of this appendix to derive the constants. This can be done by applying the symmetry conditions for this crystal in conjunction with the transformation equations for a sixth-rank tensor

$$R_{i'j'k'l'n'o'} = \frac{\partial x_i'}{\partial x_i} \frac{\partial x_j'}{\partial x_j} \frac{\partial x_k'}{\partial x_k} \frac{\partial x_l'}{\partial x_l} \frac{\partial x_n'}{\partial x_n} \frac{\partial x_o'}{\partial x_o} R_{ijklno}, \quad (31)$$

where $\partial x_i'/\partial x_i$, $\cdots$, $\partial x_o'/\partial x_o$, the partial derivatives, are the direction cosines l_1 to n_3.

For a cubic crystal of class $T_d = \bar{4}3m$, the symmetry conditions are

$$x = -x; \quad y = -y; \quad z = -z(\bar{4}, m) \quad \text{and} \quad x = y; \quad y = z; \quad z = x(3); \quad (32)$$

that is, if the x axis is shifted 180° to the $-x$ axis, the constants

remain unchanged, etc. The symmetry O_h is similar to this except that a center of symmetry is added. This does not affect polar properties of even order, and hence for a sixth-rank tensor class O_h is equivalent to class T_d given by Eq. (32).

The simplest conditions to apply is that a 180° orientation around the x, y, and z axes results in the same elastic constants as existed without orientation. For a rotation around the z axis of 180°, the direction cosines are

$$\partial x_1'/\partial x_1 = l_1 = -1; \quad \partial x_1'/\partial x_2 = m_1 = 0; \quad \partial x_1'/\partial x_3 = n_1 = 0$$
$$\partial x_2'/\partial x_1 = l_2 = 0; \quad \partial x_2'/\partial x_2 = m_2 = -1; \quad \partial x_2'/\partial x_3 = n_2 = 0$$
$$\partial x_3'/\partial x_1 = l_3 = 0; \quad \partial x_3'/\partial x_2 = m_3 = 0; \quad \partial x_3'/\partial x_3 = n_3 = 1.$$

Applying this transformation, all terms for which 1, 2, or 3 occur an even number of times have the same sign and hence are not changed. However, if the terms 1 or 2 occur an odd number of times, the sign of the term is negative, and hence such terms must be equal to zero. If we apply the 180° transformation around x and y also, terms 1, 2, and 3 occurring an odd number of times disappear. This is equivalent to an orthorhombic crystal of class 222 or D_2. If we replace the six index symbols by three index symbols, such that

$$11 = 1, \quad 22 = 2, \quad 33 = 3, \quad 23 = 4, \quad 13 = 5, \quad 12 = 6,$$

the remaining terms are

$$(33)$$

$$\begin{vmatrix} R_{111} & R_{121} & R_{131} & 0 & 0 & 0 \\ R_{211} & R_{221} & R_{231} & 0 & 0 & 0 \\ R_{311} & R_{321} & R_{331} & 0 & 0 & 0 \\ 0 & 0 & 0 & R_{441} & 0 & 0 \\ 0 & 0 & 0 & 0 & R_{551} & 0 \\ 0 & 0 & 0 & 0 & 0 & R_{661} \end{vmatrix} \begin{vmatrix} R_{112} & R_{122} & R_{132} & 0 & 0 & 0 \\ R_{212} & R_{222} & R_{232} & 0 & 0 & 0 \\ R_{312} & R_{322} & R_{332} & 0 & 0 & 0 \\ 0 & 0 & 0 & R_{442} & 0 & 0 \\ 0 & 0 & 0 & 0 & R_{552} & 0 \\ 0 & 0 & 0 & 0 & 0 & R_{662} \end{vmatrix}$$

$$\begin{vmatrix} R_{113} & R_{123} & R_{133} & 0 & 0 & 0 \\ R_{213} & R_{223} & R_{233} & 0 & 0 & 0 \\ R_{313} & R_{323} & R_{333} & 0 & 0 & 0 \\ 0 & 0 & 0 & R_{443} & 0 & 0 \\ 0 & 0 & 0 & 0 & R_{553} & 0 \\ 0 & 0 & 0 & 0 & 0 & R_{663} \end{vmatrix} \begin{vmatrix} 0 & 0 & 0 & R_{144} & 0 & 0 \\ 0 & 0 & 0 & R_{244} & 0 & 0 \\ 0 & 0 & 0 & R_{344} & 0 & 0 \\ R_{414} & R_{424} & R_{434} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & R_{564} \\ 0 & 0 & 0 & 0 & R_{654} & 0 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 0 & 0 & 0 & R_{155} & 0 \\ 0 & 0 & 0 & 0 & R_{255} & 0 \\ 0 & 0 & 0 & 0 & R_{355} & 0 \\ 0 & 0 & 0 & 0 & 0 & R_{465} \\ R_{515} & R_{525} & R_{535} & 0 & 0 & 0 \\ 0 & 0 & 0 & R_{645} & 0 & 0 \end{vmatrix} \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & R_{166} \\ 0 & 0 & 0 & 0 & 0 & R_{266} \\ 0 & 0 & 0 & 0 & 0 & R_{366} \\ 0 & 0 & 0 & 0 & R_{456} & 0 \\ 0 & 0 & 0 & R_{546} & 0 & 0 \\ R_{616} & R_{626} & R_{636} & 0 & 0 & 0 \end{vmatrix}$$

Hence, for an orthorhombic crystal there are 60 independent terms if $R_{abc} \neq R_{bac}$. For the tensor R_{abc} of Eq. (4), since $R_{abc} = R_{bac}$, the number of independent terms is 39.

From the second condition of (32) we have in addition the

symmetry that a rotation of 90° around x, y or z results in the same elastic constants as in the unrotated crystals. These three rotations are given by the direction cosines:

$$\begin{array}{ccc} x \text{ rotation} & y \text{ rotation} & z \text{ rotation} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (34) \end{array}$$

[10] Sixth-rank tensors of the elastic type have been considered by F. Birch, Phys. Rev. 71, 809 (1947); but these are not general enough for the R_{ijklno} tensor, since only the ij and kl terms can be interchanged rather than all three sets.

Applying these symmetries to the remaining constants, there are number of relations between the constants. These are

$$R_{111}=R_{222}=R_{333},$$
$$R_{112}=R_{113}=R_{221}=R_{223}=R_{331}=R_{332},$$
$$R_{121}=R_{131}=R_{212}=R_{122}=R_{211}=R_{311}$$
$$=R_{132}=R_{313}=R_{232}=R_{322}=R_{233}=R_{323},$$
$$R_{123}=R_{132}=R_{213}=R_{312}=R_{231}=R_{321},$$
$$R_{441}=R_{552}=R_{663},$$
$$R_{551}=R_{443}=R_{442}=R_{553}=R_{661}=R_{662},$$
$$R_{144}=R_{414}=R_{255}=R_{525}=R_{366}=R_{636},$$
$$R_{155}=R_{515}=R_{166}=R_{616}=R_{244}=R_{424}$$
$$=R_{266}=R_{626}=R_{344}=R_{434}=R_{355}=R_{535},$$
$$R_{456}=R_{546}=R_{465}=R_{645}=R_{564}=R_{654}. \tag{35}$$

Hence, there are nine remaining independent constants for the R tensor. These relations are used in Eqs. (8). For the N tensor, since the last four numbers can be interchanged, there are three relations between the 9 constants and the number reduce to six. The relations are

$$N_{112}=N_{111122}=N_{111212}=N_{121}=N_{166}=N_{155};$$
$$N_{123}=N_{112323}=N_{144}. \tag{36}$$

The remaining constants are then

$$N_{111}, N_{112}, N_{123}, N_{441}, N_{551}, \text{ and } N_{456}. \tag{37}$$

This is the same number and type as in Birch's sixth-rank tensor,[10] but the order of some of the subscripts is different.

Finally, for the tensor K_{mnopqr} of Eq. (4), since all of the six subscripts can be interchanged separately, there are three more relations between the constants and there are only three inde-

pendent constants. These relations are

$$K_{112}=K_{551}; \quad K_{123}=K_{441}=K_{456}; \tag{38}$$

and there are only three independent constants,

$$K_{111}, K_{112}, \text{ and } K_{123}. \tag{39}$$

APPENDIX II. DERIVATION OF TERMS IN INTERNAL ENERGY FUNCTION

The method for transforming the expression for H_1 into the potential function U, the internal energy is one of solving for the stresses in terms of the strains and magnetic intensities and substituting for these in H_1. For the strains we have

$$S_n=-\partial H_1/\partial T_n$$

or

$$S_1=(s^I_{11}+\Delta_{11})T_1+(s^I_{12}+\Delta_{22})T_2+(s^I_{12}+\Delta_{33})T_3+\Delta_{14}T_4$$
$$+\Delta_{15}T_5+\Delta_{16}T_2+h_1(\alpha_1^2-\tfrac{1}{3})+h_4[\alpha_1^4+\tfrac{2}{3}s-\tfrac{1}{3}]+h_3(\tfrac{1}{3}-s) \tag{40}$$

$$\dots \dots \dots \dots$$

$$S_6=\Delta_{16}T_1+\Delta_{16}T_2+\Delta_{36}T_3+\Delta_{46}T_4+\Delta_{56}T_5+(s^I_{44}+\Delta_{66})T_6$$
$$+h_2\alpha_1\alpha_2+h_5\alpha_1\alpha_2\alpha_3^2,$$

where

$$\Delta_{11}=(\alpha_1^2-\tfrac{1}{3})I_0^2(R_{111}-R_{112}); \quad \dots; \quad \Delta_{66}=(\alpha_3^2-\tfrac{1}{3})I_0^2(R_{441}-R_{551}).$$

If we solve these for the stresses, we find that the stresses T_1 to T_6 can be expressed in terms of the strains S_1' to S_6' given by

$$S_1'=S_1-[h_1(\alpha_1^2-\tfrac{1}{3})+h_4(\alpha_1^4+\tfrac{2}{3}s-\tfrac{1}{3})+h_3(\tfrac{1}{3}-s)] \tag{41}$$

$$\dots \dots \dots \dots$$

$$S_6'=S_6-[h_2\alpha_1\alpha_2+h_5\alpha_1\alpha_2\alpha_3^2]$$

according to the relation

	S_1'	S_2'	S_3'	S_4'	S_5'	S_6'
T_1	$c^I_{11}+\delta_{11}$	$c^I_{12}+\delta_{12}$	$c^I_{12}+\delta_{13}$	δ_{14}	δ_{15}	δ_{16}
T_2	$c^I_{12}+\delta_{12}$	$c^I_{11}+\delta_{22}$	$c^I_{12}+\delta_{23}$	δ_{24}	δ_{25}	δ_{16}
T_3	$c^I_{13}+\delta_{13}$	$c^I_{12}+\delta_{23}$	$c^I_{11}+\delta_{33}$	δ_{24}	δ_{15}	δ_{36}
T_4	δ_{14}	δ_{24}	δ_{24}	$c^I_{44}+\delta_{44}$	δ_{45}	δ_{46}
T_5	δ_{15}	δ_{25}	δ_{15}	δ_{45}	$c^I_{44}+\delta_{55}$	δ_{56}
T_6	δ_{16}	δ_{16}	δ_{36}	δ_{46}	δ_{56}	$c^I_{44}+\delta_{66}$

$$\tag{42}$$

where any c^I_{ij} is equal to

$$c^I_{ij}=(-1)^{i+j}\Delta s^{ij}/\Delta s, \tag{43}$$

where Δs is the determinant of Eq. (40) and Δs^{ij} the minor obtained by suppressing the ith row and jth column.

Since the Δ_{ij} terms of Eq. (40) are all very small, it is permissible to neglect terms having products or powers of these small quantities. With that restriction one finds that the δ_{ij} values are

$$\delta_{14}=-2\alpha_2\alpha_3 I_0^2 c^I_{44}[c^I_{11}R_{144}+2c^I_{12}R_{244}],$$
$$\delta_{15}=\delta_{35}=-2\alpha_1\alpha_3 I_0^2 c^I_{44}[(c^I_{11}+c^I_{12})R_{244}+c^I_{12}R_{144}],$$
$$\delta_{16}=\delta_{26}=-2\alpha_1\alpha_2 I_0^2 c^I_{44}[(c^I_{11}+c^I_{12})R_{244}+c^I_{12}R_{144}],$$
$$\delta_{24}=\delta_{34}=-2\alpha_2\alpha_3 I_0^2 c^I_{44}[(c^I_{11}+c^I_{12})R_{244}+c^I_{12}R_{144}],$$
$$\delta_{25}=-2\alpha_1\alpha_3 I_0^2 c^I_{44}[c^I_{11}R_{144}+2c^I_{12}R_{244}],$$
$$\delta_{36}=-2\alpha_1\alpha_2 I_0^2 c^I_{44}[c^I_{11}R_{144}+2c^I_{12}R_{244}],$$
$$\delta_{44}=-c^I_{44}^2(\alpha_1^2-\tfrac{1}{3})I_0^2(R_{441}-R_{551}),$$
$$\delta_{55}=-c^I_{44}^2(\alpha_2^2-\tfrac{1}{3})I_0^2(R_{441}-R_{551}),$$
$$\delta_{66}=-c^I_{44}^2(\alpha_3^2-\tfrac{1}{3})I_0^2(R_{441}-R_{551}),$$
$$\delta_{45}=-c^I_{44}^2\alpha_1\alpha_2 I_0^2 R_{456}, \quad \delta_{46}=-c^I_{44}^2\alpha_1\alpha_3 I_0^2 R_{456},$$
$$\delta_{56}=-c^I_{44}^2\alpha_2\alpha_3 I_0^2 R_{456}, \tag{44}$$
$$\delta_{11}=-(c^I_{11}-c^I_{12})(\alpha_1^2-\tfrac{1}{3})I_0^2[(c^I_{11}+c^I_{12})(R_{111}-R_{112})$$
$$-2c^I_{12}(R_{123}-R_{121})],$$
$$\delta_{22}=-(c^I_{11}-c^I_{12})(\alpha_2^2-\tfrac{1}{3})I_0^2[(c^I_{11}+c^I_{12})(R_{111}-R_{112})$$
$$-2c^I_{12}(R_{123}-R_{121})],$$
$$\delta_{33}=-(c^I_{11}-c^I_{12})(\alpha_3^2-\tfrac{1}{3})I_0^2[(c^I_{11}+c^I_{12})(R_{111}-R_{112})$$
$$-2c^I_{12}(R_{123}-R_{121})],$$

$$\delta_{12}=-(c^I_{11}-c^I_{12})(\alpha_3^2-\tfrac{1}{3})I_0^2[c^I_{11}(R_{123}-R_{121})-c^I_{12}(R_{111}-R_{112})],$$
$$\delta_{13}=-(c^I_{11}-c^I_{12})(\alpha_2^2-\tfrac{1}{3})I_0^2[c^I_{11}(R_{123}-R_{121})-c^I_{12}(R_{111}-R_{112})],$$
$$\delta_{23}=-(c^I_{11}-c^I_{12})(\alpha_1^2-\tfrac{1}{3})I_0^2[c^I_{11}(R_{123}-R_{121})-c^I_{12}(R_{111}-R_{112})].$$

Inserting the values of T_1 to T_6 given by Eq. (42) in Eq. (8) for H_1, the value of the internal energy function U is given by

$$2U=c^I_{11}[S_1^2+S_2^2+S_3^2]+2c^I_{12}[S_1S_2+S_1S_3+S_2S_3]$$
$$+c^I_{44}[S_4^2+S_5^2+S_6^2]+\delta_{11}S_1^2+\delta_{22}S_2^2+\delta_{33}S_3^2+2\delta_{12}S_1S_2$$
$$+2\delta_{13}S_1S_3+2\delta_{23}S_2S_3+2\delta_{14}S_1S_4+2\delta_{15}S_1S_5+2\delta_{16}S_1S_6$$
$$+2\delta_{24}S_2S_4+2\delta_{25}S_2S_5+2\delta_{16}S_2S_6+2\delta_{24}S_3S_4+2\delta_{15}S_3S_5$$
$$+2\delta_{36}S_3S_6+\delta_{44}S_4^2+\delta_{55}S_5^2+\delta_{66}S_6^2+2\delta_{45}S_4S_5+2\delta_{46}S_4S_6$$
$$+2\delta_{56}S_5S_6-2h_1(c^I_{11}-c^I_{12})[(\alpha_1^2-\tfrac{1}{3})S_1+(\alpha_2^2-\tfrac{1}{3})S_2$$
$$+(\alpha_3^2-\tfrac{1}{3})S_3]-2h_4(c^I_{11}-c^I_{12})[(\alpha_1^4+\tfrac{2}{3}s-\tfrac{1}{3})S_1$$
$$+(\alpha_2^4+\tfrac{2}{3}s-\tfrac{1}{3})S_2+(\alpha_3^4+\tfrac{2}{3}s-\tfrac{1}{3})S_3]-2h_3(c^I_{11}+2c^I_{12})$$
$$\times[\tfrac{1}{3}-s][S_1+S_2+S_3]-2h_2c^I_{44}[\alpha_2\alpha_3S_4+\alpha_1\alpha_3S_5+\alpha_1\alpha_2S_6]$$
$$-2h_5c^I_{44}[\alpha_1^2\alpha_2\alpha_3S_4+\alpha_1\alpha_2^2\alpha_3S_5+\alpha_1\alpha_2\alpha_3^2S_6]$$
$$+2K_1^S(\tfrac{1}{3}-s)+2K_2^S(\alpha_1^2\alpha_2^2\alpha_3^2-1/27), \tag{45}$$

where K_1^S and K_2^S are the anisotropy magnetic constants measured at constant strain. These are related to the anisotropy constants measured at constant stress (as used in Eq. (8)) by the equations

$$K_1^S=K_1^T-(c^I_{11}-c^I_{12})(h_1^2+(7/3)h_1h_4+(4/9)h_4^2)$$
$$-(3/2)(c^I_{11}+2c^I_{12})h_3^2(\tfrac{1}{3}-s)+\tfrac{1}{2}c^I_{44}h_2^2, \tag{46}$$
$$K_2^S=K_2^T-(c^I_{11}-c^I_{12})(3h_1h_4+(8/3)h_4^2)+\tfrac{1}{2}c^I_{44}(6h_2h_5+h_5^2).$$